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# INERTIAL MANIFOLDS FOR THE KURAMOTO-SIVASHINSKY EQUATION

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**ABSTRACT:** A new theorem is applied to the Kuramoto-Sivashinsky equation with $L$-periodic boundary conditions, proving the existence of an asymptotically complete inertial manifold attracting all initial data. Combining this result with a new estimate of the size of the globally absorbing set yields an improved estimate of the dimension, $N \sim L^{2.46}$.

## 1. Introduction

The Kuramoto-Sivashinsky equation

$$v_t + v_{xxxx} + v_{xx} + \tfrac{1}{2}v_x^2 = 0 \tag{0.1}$$

with periodic boundary conditions

$$v(x + L, t) = v(x, t)$$

has attracted much attention recently as a paradigm of finite-dimensional behaviour in a partial differential equation (see [7],[8], and [10] for numerical studies).

One way to show rigorously that such an equation can be reduced to a set of ordinary differential equations is to prove the existence of an "inertial manifold" [6]. An inertial manifold is a finite dimensional Lipschitz manifold which attracts all orbits exponentially, and is positively invariant under the flow. Restricting the equation to this manifold yields a finite dimensional dynamical system describing the asymptotic dynamics.

The analytical work of [11] culminated in an existence proof of inertial manifolds for (1) in [4], the dimension $N$ being bounded by $cL^{3.75}$. These analytical results were only valid for the case of even initial data with Neumann boundary conditions. Since the first result, many different methods of proof have been advocated ([2], [3], [5], [15]), and the dimension estimates improved (see section 5). In this paper a new theorem of the author's [13] and a new result from [1] are applied to prove the existence of an inertial manifold for the Kuramoto-Sivashinsky equation attracting all initial data, with a dimension $N \sim L^{2.46}$.

The possibility of the existence of inertial manifolds for general initial conditions follows from the results of [9] and is suggested therein; however, the results of that paper would not give the improvement in the dimension bound that is obtained here.

## 2. The functional setting of the equation

It is convenient to work with the equation for the derivative $u = v_x$,

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \tag{0.2}$$

$$u(x+L,t) = u(x,t). \tag{0.3}$$

Previous work has been for odd solutions of this equation, which integrate up to give even solutions of (1).

Defining a positive linear operator $A$ as the unique extension to $L^2$ of

$$Au = u_{xxxx} \qquad (0.4),$$

the equation can be rewritten as

$$du/dt + Au - A^{1/2}u + uu_x = 0, \tag{0.5}$$

as in [5]. $A$ has two sets of mutually orthogonal eigenfunctions

$$w_k^0 = \sin(2\pi kx/L) \qquad w_k^1 = \cos(2\pi kx/L) \tag{0.6}$$

corresponding to the eigenvalues $\Lambda_k = (2\pi k/L)^4$.

Define $P_n$ as the projector onto the first $2n$ eigenfunctions of $A$,

$$P_n u = \sum_{i=0}^{1}\sum_{j=1}^{n}(u, w_j^i)w_j^i, \tag{0.7}$$

and $Q_n$ as its orthogonal complement in $H$, $Q_n = I - P_n$. Clearly, the dimension of $P_nH$ is $2n$.

Solutions are contained in the Hilbert space $H$ which is the closure in $L^2$ of the space generated by the two sets of eigenfunctions $w_k^0$ and $w_k^1$. The closure of the same space in the Sobolev space $H^1$ is denoted by $V$, and is equal to $D(A^{1/4})$, the domain of $A^{1/4}$ in $H$.

For the norm in $H$ write $|\cdot|$ and for the inner product $(\cdot,\cdot)$. The norm in $V$ is denoted $\|v\|$, and is equal to $|v_x|$ for functions that are differentiable.

It is a standard result [15] that the equation (2) together with the boundary condition (3) generates a continuous semigroup $S(\cdot)$ so that the solution through initial condition $u_0$ at time $t$ is $u(t) = S(t)u_0$.

### 3. Absorbing sets

The first step in proving the existence of an inertial manifold to show that the equation is dissipative, i.e. that it possesses an absorbing set $B$ s.t. $S(t)W \subset B$ for $t \geq t_0(W)$ for any bounded set $W \subset H$.

Previous published results have only been valid for odd initial conditions; the result of [9], valid for all initial conditions, has only recently appeared in print but was known as early as 1980. However, it is a very coarse result, and gives an absorbing set with radius $\sim L^{72}$. The new result of [1] gives a significantly improved bound on the size of this absorbing set. It is shown that

$$\limsup_{t\to\infty} |u(\cdot,t)| \le cL^{8/5} \tag{0.8}$$

which beats the "previous best" of [12], which was $cL^{5/2}$.

The method of [5] can be used to convert this absorbing set in $H$ into an absorbing set in $V$; if $\{|u| \le \rho_0\}$ is absorbing in $H$, then $\{\|u\| \le \rho_1\}$ is absorbing in $V$, where $\rho_1 = c\rho_0^{7/5}$. When $\rho_0 = c_1 L^\alpha$ then $\rho_1 = c_2 L^{7\alpha/5}$.

It is therefore possible to "prepare" the equation by truncating the term $(-A^{1/2}u + uu_x)$ so that it is zero for $\|u\| \ge 6\rho_1$, and still preserve all the asymptotic dynamics. Defining $\theta(r)$ by

$$\theta(r) = \max(0, \min(1, 2(3-r))),$$

the prepared Kuramoto-Sivashinsky equation is

$$du/dt + Au + \theta(\|u\|/\rho_1)(-A^{1/2}u + uu_x) = 0. \tag{0.9}$$

Importantly, an inertial manifold for (9) implies an inertial manifold for (2), as is shown in [5] (proposition 2.3).

## 4. The strong squeezing property and inertial manifolds

The formal definition of an inertial manifold is [6]

**Definition 1** *An inertial manifold $\mathcal{M}$ in $X$ is a finite-dimensional Lipschitz manifold, which is positively invariant and attracts all orbits exponentially, that is*

$$\mathrm{dist}_X(S(t)u_0, \mathcal{M}) \le C(W)e^{-kt} \qquad \textit{for all } u_0 \in W,$$

*where $W$ is a bounded set in $X$.*

Often of central importance in the proof of the existence of inertial manifolds is the strong squeezing property [5]. This is a cone invariance property coupled with an exponential decay for solutions outside the cone;

**Definition 2** *The strong squeezing property holds in a space $X$ if for some $l$,$n$ the cone*

$$\mathcal{C}_l^n = \{w(t) \equiv u_1(t) - u_2(t) : \|Q_n w(t)\|_X \leq l \|P_n w(t)\|_X\} \tag{0.10}$$

*is strictly invariant under the flow, that is if $u_1$ and $u_2$ are two solutions of equation (1) and $u_1(0) - u_2(0) \in \mathcal{C}_l^n$, then $u_1(t) - u_2(t) \in \operatorname{int} \mathcal{C}_l^n$ for all $t \geq 0$, and furthermore, if $w(t_0) \notin \mathcal{C}_l^n$ then*

$$\|Q_n w(t)\|_X \leq e^{-kt} \|Q_n w(0)\|_X \tag{0.11}$$

*for some $k > 0$ and all $0 \leq t \leq t_0$. $\|u\|_X$ is the norm of $u$ in $X$.*

It is unsurprising that this property is often important in existence proofs due to the following theorem [13].

**Theorem 3** *The strong squeezing property in $X$ ensures the existence of an inertial manifold in $X$. Furthermore the inertial manifold is asymptotically complete, in other words for all initial conditions $u_0 \in W \subset X$ there exists a trajectory $\bar{u}(t)$ lying on $\mathcal{M}$ such that*

$$\|S(t)u_0 - \bar{u}(t)\|_X \leq C(W) e^{-kt}.$$

The inertial manifold is given as the graph of a Lipschitz function from $P_n X$ into $Q_n X$.

**5. Inertial manifolds for the KSE**

The existence of inertial manifolds for the Kuramoto-Sivashinsky equation, and a comparison of dimension estimates, is complicated by the fact that such manifolds can be shown to exist in a variety of spaces. The clearest example of this is in [15], where a similar analysis shows that existence of a manifold in both $H$ and $V$, with $N_H \sim L^{\alpha+2}$ and $N_V \sim L^{7\alpha/5+2}$. The manifold in $V$ is more regular, but the estimate of the dimension of the manifold in $H$ is smaller. Indeed, the apparent "improvement" from the estimate $L^{7/2}$ in [5] to $L^3$ in [3] arises since the former is for an inertial manifold in $V$ and the latter for a manifold in $H$; the estimate $L^3$ is already contained in [5], as will now be shown.

The analysis of [5] proceeds via the strong squeezing property in both $H$ and $V$ and a fixed point argument (which is standard [4],[5],[6],[15]). The important results can be summarised as

**Theorem 4** [5, theorems 3.2 & corollary 3.7] *For the prepared equation (9) there exist*

$$N_H \sim L^{3\alpha/5+3/2} \qquad \textit{and} \qquad N_V \sim L^{7\alpha/10+7/4}$$

*such that for $N > N_H$ the strong squeezing property holds in $H$, and for $N > N_V$ the strong squeezing property holds in $V$.*

The expressions for $N_H$ and $N_V$ are obtained by following the analysis of [5] with $\rho_0 \sim L^\alpha$. Using theorem 3, and the comments in section 3, an immediate corollary is

**Corollary 5** *The Kuramoto-Sivashinsky equation has inertial manifolds in both $H$ and $V$, with $N_H \sim L^{3\alpha/5+3/2}$ and $N_V \sim L^{7\alpha/10+7/4}$. The inertial manifolds are asymptotically complete.*

This result is an improvement on previous results in three ways. Its proof (theorem 2 of [13]) makes clear the central position of the strong squeezing property, and the manifold is shown to be asymptotically complete. Although this follows from properties in [5] and could be shown by adapting a method in [3], this is the first time that is has been explicitly proven. Furthermore the explicit dependence of the dimension on the size of the absorbing set enables new estimates to be easily incorporated. Using the new result of [1], the asymptotic expressions

$$N_H \sim L^{2.46} \qquad N_V \sim L^{2.89} \tag{0.12}$$

are obtained.

## 6. Some comments on lower bounds

It is remarked in [4] that a lower bound on the number of Fourier modes spanning the attractor $N_F$ obeys $N_F \geq cL^2$. Since the form of the inertial manifold sought here (and throughout the literature) is dependent on the Fourier (eigenfunction) expansion, its dimension can be no smaller than $cL^2$. It may however be possible to decrease this dimension by searching for an inertial manifold that cannot be expressed as a graph.

That the dimension of the inertial manifold is bounded below in this way implies further results. In particular, the two estimates

$$N_F \geq cL^2 \qquad N_H \leq c_1 L^{3\alpha/5+3/2}$$

are not compatible unless $\alpha \geq 5/6$. This gives a lower bound on the "best possible" result for the size of the absorbing set.

Another possible approach to obtaining lower bounds on the dimension of the inertial manifold is to restrict attention to asymptotically complete manifolds. The counterexample in [14] shows that unless the rate of attraction $\mu$ towards the manifold is greater than the rate of backwards separation $\beta$ of trajectories on the manifold, it need not be asymptotically complete. Tight estimates of $\mu_{max}$ and $\beta_{min}$, satisfying

$$\mu \leq \mu_{max} \qquad \beta \geq \beta_{min}$$

could be used to obtain lower bounds from the condition $\mu_{max} > \beta_{min}$.

## 7. Conclusion

Solutions of the Kuramoto-Sivashinksy equation soon mimic the behaviour of a finite system of ordinary differential equations, as is shown in [7], [8], and [10]. This is due to the asymptotic completeness property, proved here explicitly for the first time. The dimension of the manifold in $H$, $N_H \sim L^{2.46}$ is slowly being lowered towards its optimal value of $cL^2$. Short of new analytical methods to prove the existence of inertial manifolds, improvements will come from further refining of the bound on the size of the absorbing set.

A future paper will investigate the bifurcation structure of these manifolds as $L$ increases.

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## References


[1] P. Collet, J.-P. Eckmann, H. Epstein and J. Stubbe, A Global attracting set for the Kuramoto-Sivashinsky equation, Comm. Math. Phys. 152 (1993) 203.

[2] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations (Springer, New York, 1988).

[3] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, Spectral barriers and inertial manifolds for dissipative partial differential equations, J. Dyn. Diff. Eq. 1 (1989) 45.

[4] C. Foias, B. Nicolaenko, G.R. Sell and R. Temam, Variétés inertielles pour l'équation de Kuramoto Sivashinski, C. R. Acad. Sci. Paris I 301 (1985) 285.

[5] C. Foias, B. Nicolaenko, G.R. Sell and R. Temam, Inertial manifolds for the Kuramoto Sivashinsky equation and an estimate of their lowest dimension, J. Math. Pure. Appl. 67 (1988) 197.

[6] C. Foias, G.R. Sell and R. Temam, Inertial manifolds for nonlinear evolution equations, J. Diff. Eq. 73 (1988) 309.

[7] J.M. Hyman and B. Nicolaenko, The Kuramoto Sivashinsky equation: a bridge between PDEs and dynamical systems, Physica D 18 (1986) 113.

[8] J.M. Hyman, B. Nicolaenko and S. Zaleski, Order and complexity in the Kuramoto Sivashinsky model of weakly turbulent interfaces, Physica D 23 (1986) 265.

[9] J.S. Il'yashenko, Global analysis of the phase portrait for the Kuramoto Sivashinsky equation, J. Dyn. Diff. Eq. 4 (1992) 585.

[10] I.G. Kevrekidis, B. Nicolaenko and J.C. Scovel, Back in the saddle again: a computer assisted study of the Kuramoto Sivashinsky equation, SIAM J. Appl. Math. 50 (1990) 760.

[11] B. Nicolaenko, B. Scheurer and R. Temam, Quelques propriétés des attracteurs pour l'équation de Kuramoto Sivashinski, C. R. Acad. Sci. Paris I 298 (1984) 23.

[12] B. Nicolaenko, B. Scheurer and R. Temam, Some global dynamical properties of the Kuramoto Sivashinsky equation: nonlinear stability and attractors, Physica D 16 (1985) 155.

[13] J.C. Robinson, Inertial manifolds and the cone condition, Dyn. Sys. Appl. (in press)

[14] J.C. Robinson, The asymptotic completeness of inertial manifolds, submitted to J. Math. Anal. Appl.

[15] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics (Springer, New York, 1988).