NODAL PARAMETRISATION OF ANALYTIC ATTRACTORS

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Abstract. Friz and Robinson showed that analytic global attractors consisting of periodic functions can be parametrised using the values of the solution at a finite number of points throughout the domain, a result applicable to the 2d Navier-Stokes equations with periodic boundary conditions. In this paper we extend the argument to cover any attractor consisting of analytic functions; in particular we are now able to treat the 2d Navier-Stokes equations with Dirichlet boundary conditions.

1. Introduction. In [9], Foias and Temam showed that a sufficient number of ‘nodal values’ determine the asymptotic behaviour of solutions of the 2d incompressible Navier-Stokes equations. They showed that if \( u(x,t) \) and \( v(x,t) \) are two solutions of

\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f(x,t) \quad \nabla \cdot u = 0 \quad x \in \Omega \subset \mathbb{R}^2,
\]

with either periodic or Dirichlet boundary conditions, there exists a distance \( \delta \) such that for any finite collection of nodes \( \{x_j\}_{j=1}^N \subset \Omega \), with

\[
\sup_{x \in \Omega} |x - x_j| < \delta,
\]

convergence at these nodes,

\[
\max_{j=1,\ldots,N} |u(x_j,t) - v(x_j,t)| \to 0 \quad \text{as} \quad t \to \infty,
\]

implies convergence throughout the domain

\[
\sup_{x \in \Omega} |u(x,t) - v(x,t)| \to 0 \quad \text{as} \quad t \to \infty.
\]

We call the \( \{x_j\} \) a set of (asymptotically) determining nodes.

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Foias and Temam conjectured that if attention is restricted to the global attractor \( \mathcal{A} \) (which can be thought of heuristically as ‘taking the limit’ in (1.2) and (1.3)) then in fact functions on the attractor should be determined by their values at a finite number of points in the domain.

In [12], Friz and Robinson proved this conjecture in the case of periodic boundary conditions, a restriction which allowed the use of Fourier expansions. Here we treat the more physically relevant case of Dirichlet boundary conditions, by proving two key technical results from [12] without recourse to such expansions.

Although our main interest is the Navier-Stokes equations, we treat the problem in a general situation, and show later that our assumptions apply to this particular example. Let \( \Omega \subset \mathbb{R}^n \) be bounded, open and connected, and assume that \( \mathcal{A} \) is a subset of \( L^2(\Omega) = [L^2(\Omega)]^m \) with fractal dimension \( d_f \) (see section 2), which consists of functions that are analytic on \( \Omega \) (in a sense which is made precise in section 3).

In the statement of the theorem “almost every” is with respect to \( n_k \)-dimensional Lebesgue measure.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n \), and let \( \mathcal{A} \) be a compact connected* subset of \( L^2(\Omega) = [L^2(\Omega)]^m \), with finite fractal dimension \( d_f(\mathcal{A}) \). Furthermore, suppose that functions on \( \mathcal{A} \) are uniformly analytic on all compact subdomains \( K \subset \Omega \), i.e., for each \( K \) there exist \( C_K \) and \( \sigma_K \) such that

\[
\sup_{x \in K} |D^\alpha u(x)| \leq C_K \alpha! \sigma_K^{-|\alpha|} \quad \text{for every multi-index } \alpha \geq 0.
\]

For \( k \) points

\[ \mathbf{x} = (x_1, \ldots, x_k), \]

define the observation map

\[ E_\mathbf{x}[u] = (u(x_1), \ldots, u(x_k)). \]

Then \( E_\mathbf{x} \) is one-to-one between \( \mathcal{A} \) and its image for almost every choice of \( k \) points in \( \Omega \), provided that \( k \geq 16 d_f(\mathcal{A}) + 1 \).

In particular, the values \( (u(x_1), \ldots, u(x_k)) \) provide a parametrisation of \( \mathcal{A} \) which is continuous from \( \mathbb{R}^{mk} \) into \( L^2(\Omega) \).

Note that we do not obtain any information on the smoothness of the parametrisation, so that it is still an open question whether the map \( E_\mathbf{x}[u] \to u \) is Hölder continuous from \( E_\mathbf{x}[\mathcal{A}] \) into \( L^2(\Omega) \) (cf. section 4.2).

We also improve the estimate from [12] of the number of points required in the periodic case (from \( k \geq 16 d_f(\mathcal{A}) + 1 \) to \( k \geq 16 d_f(\mathcal{A}) + 1 \), removing the dependence on \( n \), and this bound agrees with that obtained for Dirichlet boundary conditions (a formal statement is given as theorem 6.2).

For more discussion of the physical interpretation of this result, see [12] and [26].

2. **The global attractor.** For (1.1) with Dirichlet boundary conditions, it is a classical result (see Constantin and Foias [3], Robinson [27], or Temam [29]) that for an initial condition \( u_0 \in H \), where

\[ H \subset L^2(\Omega) = [L^2(\Omega)]^2 \]

*That \( \mathcal{A} \) is connected is only used to deal with the case \( d_f(\mathcal{A}) \leq \frac{1}{4} \), see comments after the proof of theorem 6.1.
(the exact definition will not be important in what follows), there exists a unique solution $u(t; u_0)$ which belongs to

$$L^2(0, T; H^1(\Omega)) \cap C^0([0, T]; H).$$

It follows that when $f(t) = f$, independent of time, the solutions can be used to define a semigroup on $H$, by setting $S(t)u_0 = u(t; u_0)$.

For $f \in H$ one can show that there exists a compact absorbing set (in fact an absorbing set in $H^1(\Omega)$), and then apply the general theory of global attractors (as developed in the books by Babin and Vishik [1], Hale [15], Ladyzhenskaya [22], or Temam [30]), to prove the existence of a set $A$ which is

(i) compact in $L^2(\Omega)$,

(ii) invariant for the semigroup $S(t)$, so that

$$S(t)A = A \quad \text{for all} \quad t \in \mathbb{R},$$

and

(iii) attracts all orbits, uniformly over bounded sets of initial conditions, i.e.,

$$\sup_{u_0 \in B} \text{dist}(S(t)u_0, A) \to 0 \quad \text{as} \quad t \to \infty.$$

Property (i) guarantees that the attractor is ‘small’ (for example, there are non-compact invariant subsets for the 2d Navier-Stokes equations with periodic boundary conditions [5]), (ii) that it makes sense to talk about “the dynamics restricted to $A$”, and (iii) that we can expect $A$ to determine all the asymptotic behaviour of the system. In the context of the Navier-Stokes equations, the attractor can be thought of as describing all the phenomena associated with ‘fully-developed’ flows.

3. Measures of dimension. One of the main assumptions behind our theorem is that the global attractor $A$ is a finite-dimensional subset of $L^2(\Omega)$. This is true for many equations, and in particular for the 2d incompressible Navier-Stokes equations (see Constantin and Foias [2], and section 7). As in [12], we will make use of both of the common definitions of dimension, which we now briefly recall.

Note that, in fact, the three definitions below all depend on the space in which they are calculated. For example, a priori there is no reason why $d_f(A, L^2(K))$ should be the same as $d_f(A, H^1(K))$. Although we do not make the dependence on the space explicit in this section, it will be important in those that follow.

3.1. The Hausdorff dimension. The Hausdorff dimension is based on the definition of the $d$-dimensional Hausdorff measure, which for integer values of $d$ is proportional to $d$-dimensional Lebesgue measure (theorem 1.12 in Falconer [7]).

To find the $d$-dimensional Hausdorff measure of a set $X$, cover $X$ by balls $B(x_i, r_i)$ with radii $r_i \leq \epsilon$, and define

$$\mu(X, d, \epsilon) = \inf \left\{ \sum_i r_i^d : r_i \leq \epsilon \text{ and } X \subseteq \bigcup_i B(x_i, r_i) \right\}.$$  

The $d$-dimensional Hausdorff measure of $X$, $\mathcal{H}^d(X)$, is given by

$$\mathcal{H}^d(X) = \lim_{\epsilon \to 0} \mu(X, d, \epsilon),$$

and the Hausdorff dimension of $X$, $d_H(X)$, is defined as

$$d_H(X) = \inf \{d \geq 0 : \mathcal{H}^d(X) = 0\} = \inf \{d \geq 0 : \mathcal{H}^d(X) < \infty\}$$

we set $d_H(X) = \infty$ if one of these sets (and hence the other) is empty.)
It is central to our argument that $d_H$ is stable under countable unions,
\begin{equation}
  d_H \left( \bigcup_{j=1}^{\infty} X_j \right) = \sup_j d_H(X_j).
\end{equation}

In the periodic case we used a well-known property which bounds $d_H(f(X))$ in terms of $d_H(X)$ when $f$ is a Hölder function:
\begin{equation}
  d_H(f(X)) \leq \frac{d_H(X)}{\theta}
\end{equation}
when
\[ |f(x) - f(y)| \leq C|x - y|^\theta. \]

However, to obtain the result of this paper we will need the following more careful estimate.

**Lemma 3.1.** Let $(X,d_X)$ be a metric space, with $d_H(X) \leq n$, and let $f : X \to \mathbb{R}^m$ satisfy
\[ |f(x_1) - f(x_2)| \leq C_0 d_X(x_1,x_2)^\theta, \]
with $C_0 > 0$ and $\theta \in [0,1]$. Then the Hausdorff dimension of the graph
\[ G = \{(x,f(x)) : x \in X \} \subset X \times \mathbb{R}^m \]
is less than or equal to $n + (1-\theta)m$.

In the proof we write
\[ d_Z((x_1,a_1),(x_2,a_2)) = d_X(x_1,x_2) + |a_1 - a_2|, \quad (x_1,a_1),(x_2,a_2) \in Z \]
for the metric on $Z = X \times \mathbb{R}^m$.

**Proof.** Let $\epsilon > 0$. Since $d_H(X) \leq n$, we can cover $X$ by a collection $\{B_{r_i}(x_i)\}_{i \in I}$ of balls with centers $x_i \in X$ and radii $r_i > 0$ such that
\[ \sum_{i \in I} r_i^{n+\epsilon} < \infty. \]

Note that $G$ is then covered by $\{B_{r_i}(x_i) \times B_{C_0 r_i^\theta}(f(x_i))\}_{i \in I}$. Since $B_{C_0 r_i^\theta}(f(x_i))$ is a subset of $\mathbb{R}^m$, we can cover it by $m_i$ balls $B_{r_i}(y_{ij})$, where $m_i \leq C_1(C_0 r_i^\theta - 1)^m + 1$ with $C_1$ depending only on $m$. We therefore obtain
\[ G \subseteq \bigcup_{i \in I} \bigcup_{j=1}^{m_i} B_{2r_i}((f(x_i),y_{ij})). \]

Since
\begin{align*}
  \sum_{i \in I} \sum_{j=1}^{m_i} (2r_i)^{n+(1-\theta)m+\epsilon} & = 2^{n+(1-\theta)m+\epsilon} \sum_{i \in I} m_i r_i^{n+(1-\theta)m+\epsilon} \\
  & \leq C_1 2^{n+(1-\theta)m+\epsilon} \sum_{i \in I} r_i^{n+\epsilon} + \sum_{i \in I} r_i^{n+(1-\theta)m+\epsilon} < \infty,
\end{align*}
we have $\dim_H(G) \leq n + (1-\theta)m + \epsilon$, and since $\epsilon > 0$ is arbitrary, the lemma follows. \qed
3.2. **The fractal dimension.** We define the fractal dimension of \( X \), \( d_f(X) \), by

\[
d_f(X) = \limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{\log(1/\epsilon)},
\]

where \( N(X, \epsilon) \) is the minimum number of balls of radius \( \epsilon \) necessary to cover \( X \).

This is a stronger definition than the Hausdorff dimension, in that we always have \( d_H(X) \leq d_f(X) \). Although the stability under countable unions no longer holds, we still have (3.5) (for proofs and more on the fractal dimension see Eden *et al.* [6] or Robinson [27]).

3.3. **The thickness exponent.** Defined in a similar manner to the fractal dimension, the thickness exponent of a set \( X \) is given by

\[
\tau(X) = \limsup_{\epsilon \to 0} \frac{\log d(X, \epsilon)}{\log(1/\epsilon)},
\]

where \( d(X, \epsilon) \) is the dimension of the smallest linear space \( V \) such that every element of \( X \) lies within \( \epsilon \) of \( V \).

The thickness exponent was introduced by Hunt and Kaloshin [17] in the context of an embedding theorem for sets of finite fractal dimension (we recall a particular case of their result in the next section). The relationship between the thickness exponent and the smoothness of functions in \( X \) is discussed in [11].

4. **Analyticity and its consequences.** In this section we define certain classes of analytic functions, and give various consequences of analyticity related to smooth parametrisations and possible sets of zeros.

4.1. **Analyticity classes.** A real analytic function \( u(x) \) on a compact domain \( K \subset \mathbb{R}^n \) is characterised by the existence of constants \( C \) and \( \sigma \) such that

\[
\sup_{x \in K} |D^\alpha u(x)| \leq C \sigma^{-|\alpha|}, \quad (4.6)
\]

for every multi-index \( \alpha \geq 0 \) (see John [18], Renardy and Rogers [25], or Rudin [28]).

For periodic functions this implies that one can define certain classes of real analytic functions (called Gevrey classes) as the domains of various exponentiated versions of the Laplace operator. This approach was first used by Foias and Temam [10] to prove the analyticity of solutions of the 2d Navier-Stokes equations with respect to both the space and time variables, and was adopted in [12]. If \( A \) is the negative Laplacian on the periodic cube \( Q \), one can define

\[
G_\sigma(Q) = D(e^{\sigma A^{1/2}}), \quad (4.7)
\]

the domain of \( e^{\sigma A^{1/2}} \) in \( L^2(Q) \). This enables the use of Fourier series, and greatly simplifies the problem.

For more general domains and different boundary conditions we cannot use exactly the same approach. Instead we work directly with (4.6), and define the analyticity class \( A_\sigma(K) \) as all those \( u \in C^\infty(K) \) such that

\[
\|u\|_{A_\sigma(K)} = \sup_{x \in K} \sup_{\alpha \geq 0} \frac{|D^\alpha u(x)| \sigma^{|\alpha|}}{\alpha!} < \infty. \quad (4.8)
\]

Not only are such functions real analytic in \( K \) with a uniform radius of analyticity \( \sigma/\sqrt{n} \), but we can also bound the supremum norm of \( u \) in a small complex
neighbourhood of \( K \) in terms of the \( A_\sigma \) norm (cf. lemma 2.1 in [12]). In what follows we denote by \( B(x, \rho) \) the \( \rho \)-ball in \( \mathbb{C}^n \) with centre \( x \),

\[
B(x, \rho) = \{ z \in \mathbb{C}^n : |z - x| < \rho \},
\]

and, in something of an abuse of notation,

\[
B(W, \rho) = \bigcup_{x \in W} B(x, \rho).
\]

That a function \( u \) is real analytic in \( D \subset \mathbb{R}^n \) with uniform analyticity radius \( r > 0 \) means that \( u \in C^\infty(D) \) and can be extended to an analytic function on \( B(D, r) \).

**Lemma 4.1.** Suppose that \( u \in A_\sigma(K) \). Then \( u \) is real analytic in \( K \) with uniform radius of analyticity \( \sigma/\sqrt{n} \), and

\[
\|u\|_{L^\infty(B(x, \sigma/2\sqrt{n}))} \leq 2^n \|u\|_{A_\sigma(K)}, \quad x \in K.
\]

**Proof.** Let \( x_0 = (x_{01}, \ldots, x_{0n}) \in K \) be arbitrary. Then

\[
\sup_{\alpha \geq 0} \frac{|D^\alpha u(x_0)|_{|\alpha|}}{\alpha!} = M_{x_0} < \infty
\]

and thus the terms in the series

\[
\sum_{\alpha \geq 0} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha
\]

are uniformly bounded for every \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) such that \(|x_j - x_{0j}| \leq \sigma \) for \( j = 1, \ldots, n \). Therefore, following the argument in Hörmander [16, p. 34], the series

\[
\sum_{\alpha \geq 0} \frac{D^\alpha u(x_0)}{\alpha!} (z - x_0)^\alpha
\]

converges uniformly on compact subsets of the polydisk

\[
\Delta(x_0, \sigma) = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j - x_j| < \sigma \text{ for } j = 1, \ldots, n \}
\]

to an analytic function which agrees with \( u \) in the sense of \( C^\infty(K) \) (i.e., in a small neighbourhood of \( K \)). Denoting also this extension by \( u \), we have, as in [16, p. 34], \(|u(z)| \leq 2^n M_{x_0} \) for \( z \in \Delta(x_0, \sigma/2) \).

\[\Box\]

4.2. **Parametrising finite-dimensional sets of analytic functions.** Foias and Olson [8] showed that a finite-dimensional subset of a Hilbert space has a Hölder continuous parametrisation. In our proof we use a related result, due to Hunt and Kaloshin [17], which also works for subsets of Banach spaces and gives a bound on the Hölder exponent.

Their theorem guarantees that if \( X \) is a finite-dimensional subspace of some Banach space \( B \) then there are many linear maps from a Banach space \( B \) into \( \mathbb{R}^D \) (for some \( D \)) which are one-to-one between \( X \) and its image and have Hölder continuous inverse. Since \( X \) is compact this provides a continuous parametrisation of \( X \) using \( D \) coordinates. When \( B = A_\sigma(K) \) for some \( \sigma \) and compact \( K \), this parametrisation has some particularly nice properties, which are given in the following theorem.
Theorem 4.2. Let $X$ be a compact subset of $A_\sigma(K)$, with $d = d_f(X, A_\sigma(K)) < \infty$, and thickness exponent $\tau = \tau(X, A_\sigma(K))$. Then, provided that $D$ is an integer with $D > 2d$, and

$$\theta < \frac{D - 2d}{D(1 + \tau)},$$

(4.10)

there exists a parametrisation of $X$,

$$u(x; \epsilon) \quad \text{with} \quad \epsilon \in E \subset \mathbb{R}^D,$$

such that $D^\alpha u(x; \epsilon)$ is analytic in $x$ and Hölder continuous in $\epsilon$ with Hölder exponent $\theta$, for all multi-indices $\alpha \geq 0$.

Proof. The proof is essentially contained in [12], although the result is not written there explicitly in this form. The embedding theorem of Hunt and Kaloshin [17] guarantees the existence of a map $L$ from $A_\sigma(K)$ into $\mathbb{R}^D$ which is one-to-one between $X$ and its image, and whose inverse is Hölder continuous with exponent $\theta$. This provides a parametrisation of $X$ using $D$ coordinates, which is Hölder continuous from $E = LX \subset \mathbb{R}^D$ into $A_\sigma(K)$. Lemma 4.1 shows that this parametrisation is Hölder continuous into $L^\infty(B(K, \sigma/2\sqrt{n}))$, and the Cauchy integral formula,

$$u(z; \epsilon) = \frac{1}{(2\pi i)^n} \int_C \frac{u(y; \epsilon)}{(y_1 - z_1) \cdots (y_n - z_n)} \, dy$$

(4.11)

(where $C$ is an integral over a circular contour $|y_j - z_j| = r_j$ for each $j$, see Hörmander [16, p. 4]) then gives the same property for all the derivatives when $x \in K$. \hfill $\Box$

4.3. Zero sets of analytic functions. The properties of this parametrisation allow us to apply the following result, which gives some useful information about the zero set of $u(x; \epsilon)$ (for a proof, based on that of Yamazato [31], see [12]).

Theorem 4.3. Suppose that the function

$$v(x; \epsilon) : \mathbb{R}^n \times E \to \mathbb{R} \quad \text{with} \quad E \subset \mathbb{R}^D,$$

is such that $D^\alpha v(x; \epsilon)$ is analytic in $x$ and Hölder continuous in $\epsilon$ (with Hölder exponent $\theta$), for all multiindices $\alpha \geq 0$. Suppose in addition that $v(\cdot; \epsilon)$ is not identically zero for any $\epsilon \in E$. Then the zero set of $v(x; \epsilon)$, viewed as a subset of $\mathbb{R}^n \times E \subset \mathbb{R}^n \times \mathbb{R}^D$, is contained in a countable union of manifolds of the form

$$(x', x_j(x'; \epsilon); \epsilon),$$

where $x' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$, and $x_j$ is a Hölder continuous function of its arguments (with exponent $\theta$).

If $u(x; \epsilon) = (v_1(x; \epsilon), \ldots, v_m(x; \epsilon))$, and one of the components $v_j(x; \epsilon)$ satisfies the conditions of the theorem then the zeros of $u$ must be contained in a countable collection of manifolds of the same form. Note also that if $u(x; \epsilon) : \mathbb{R}^n \times E \to \mathbb{R}^m$ is analytic in $x$ and Hölder continuous in $\epsilon$, then for any choice of $d \in \mathbb{R}^m$ (with $d \neq 0$) the function

$$v(x; \epsilon) = u(x; \epsilon) \cdot d$$

(4.12)

satisfies the assumptions of the theorem.
5. Fractal dimension and thickness of $A$ in an analyticity class. We now prove the fundamental technical results which allow us to adopt the arguments from [12]. There it was shown that if $X$ is compact subset of $L^2(Q)$ with $d_f(X, L^2(Q))$ finite, and $X$ is uniformly bounded in $G$$(Q)$, we have
\[ d_f(X, G) \leq \frac{d_f(X, L^2(Q))}{1 - \varphi} \tag{5.13} \]
and
\[ \tau(X, G) = 0, \tag{5.14} \]
for all $0 \leq \varphi < 1$.

In this section we show something similar when $X$ is a bounded subset of $A_\sigma(K)$, although the result appears a little more complicated. First, we need some auxiliary lemmas.

5.1. Auxiliary lemmas. We first show (essentially) that on bounded subsets of $A_\sigma(K)$, the restriction map from $L^2(\Omega)$ into $A_\varphi(B(x, \varphi))$ is Hölder continuous, for $x \in K$ and a suitable choice of $\varphi$.

For the proof we will need the following lemma (a slight modification of lemma 3.3 in Kukavica [21]). We denote by $B(x, \rho)$ the intersection of $B(x, \rho)$ with $\mathbb{R}^n$,
\[ B(x, \rho) = \{ z \in \mathbb{R}^n : |z - x| \leq \rho \}, \]
and abuse the notation as before to write
\[ B(W, \rho) = \bigcup_{x \in W} B(x, \rho). \]

**Lemma 5.1.** Assume that $f$ is holomorphic in $B(x, R) \subset \mathbb{C}^n$ where $x \in \mathbb{R}^n$ and $R > 0$. For any $\epsilon \in (0, 1)$ there exists $\rho_0 \in (0, 1)$ depending only on $\epsilon$ and $n$ such that
\[ \|f\|_{L^\infty(B(x, \rho))} \leq \frac{C(n)R^{(1+\epsilon)n/2}}{\varphi^n} \|f\|_{L^2(B(x, R))} \|f\|_{L^\infty(B(x, R))} \tag{5.15} \]
for all $\rho \in (0, \rho_0 R)$.

**Proof.** Without loss of generality, $x = 0$. For $z = (z_1, \ldots, z_n) \in B(0, R)$, let
\[ F(z_1, \ldots, z_n) = \int_{z_1}^{\zeta_1} \cdots \int_{0}^{\zeta_n} f(\zeta_1, \ldots, \zeta_n) d\zeta_1 \cdots d\zeta_n \tag{5.16} \]
where the integrals are taken over the straight paths connecting 0 and $z_j$, for $j = 1, \ldots, n$. As in [21], we can find $r_0 \in (0, 1)$ depending only $\epsilon$ and $n$ such that
\[ \|F\|_{L^\infty(B(0, r))} \leq \|F\|_{L^\infty(B(0, R))} \|F\|_{L^\infty(B(0, r))} \tag{5.17} \]
provided $r \in (0, r_0 R)$. Note that
\[ f = \frac{\partial^n F}{\partial z_1 \cdots \partial z_n} \]
theorem 2.2.3 in Hörmander [16] now implies that
\[ \|f\|_{L^\infty(B(0, r/2))} \leq \frac{C(n)}{r^n} \|F\|_{L^\infty(B(0, r))}. \]
Also, by (5.16),
\[ \|F\|_{L^\infty(B(0, R))} \leq R^n \|F\|_{L^\infty(B(0, R))} \]
and, using the Cauchy-Schwarz inequality,
\[ \|F\|_{L^\infty(B(0, R))} \leq R^{n/2} \|F\|_{L^2(B(0, R))}. \]
Collecting the last three inequalities and using (5.17) the result follows, with \( \rho_0 = r_0/2 \).

We use the result of lemma 5.1 in the form of the following corollary.

**Corollary 5.2.** Let the assumptions of theorem 1.1 hold, and let \( K \) be a compact subset of \( \Omega \) such that
\[
\{ u_0 | K : u_0 \in \mathcal{A} \}
\]
is bounded in \( A_\sigma(K) \). Let \( \epsilon \in (0, 1) \), and \( x \in K \). Then there exists a \( \varphi \in (0, 1) \), which depends only on \( \epsilon \) and \( n \), such that
\[
\| u \|_{A_\sigma(B_\rho(x, \varphi \sigma))} \leq \frac{C'(n)}{\varphi^{n \frac{2}{\sigma^2}(1-\epsilon)}} \| u \|_{L^\infty(\Omega)}^{1-\epsilon} \| u \|_{A_\sigma(K)}^{\epsilon}.
\] (5.18)

It is understood that \( A_\sigma(B_\rho(x, \varphi \sigma)) = A_\sigma(B_\rho(x, \varphi \sigma)) \).

**Proof.** Take \( \varphi < \rho_0/4/\sqrt{n} \) (where \( \rho_0 \) is as in lemma 5.1). The bound on the left-hand side follows using the Cauchy integral formula (4.11): for any \( y \in B_\rho(x, \varphi \sigma) \),
\[
|D^\alpha u(y)| = (2\pi)^{-n} \int_{C} \frac{\alpha! u(\zeta)}{(\zeta - y)^{1+\alpha}} d\zeta
\]
\[
\leq \frac{\alpha! \| u \|_{L^\infty(B(y, \varphi \sigma))}}{(\varphi \sigma)^{\alpha}},
\]
and so
\[
\frac{|D^\alpha u(y)| (\varphi \sigma)^\alpha}{\alpha!} \leq \| u \|_{L^\infty(B(y, \varphi \sigma))}.
\] (5.19)

This implies that
\[
\| u \|_{A_\sigma(B_\rho(x, \varphi \sigma))} \leq \| u \|_{L^\infty(B(x, 2 \rho \sigma))}.
\]
Now apply lemma 5.1, with \( R = \sigma/2\sqrt{n} \) and \( r = 2 \varphi \sigma \leq \rho_0 R \), to obtain
\[
\| u \|_{A_\sigma(B_\rho(x, \varphi \sigma))} \leq \frac{C'(n)}{\varphi^{n \frac{2}{\sigma^2}(1-\epsilon)}} \| u \|_{L^2(B_\rho(x, \sigma/2\sqrt{n}))}^{1-\epsilon} \| u \|_{L^\infty(B(x, \sigma/2\sqrt{n}))}^{\epsilon}.
\]
To obtain (5.18) the bound on the \( L^2 \) term is trivial, since \( B_\rho(x, \sigma/2\sqrt{n}) \subset \Omega \), and the final term can be bounded using lemma 4.1.

**5.2. Dimension and thickness bounds.** We now use the above lemmas to prove the equivalent of (5.13) and (5.14) in the Dirichlet case.

For a collection \( x = (x_1, \ldots, x_k) \) of \( k \) points in \( \Omega \), and
\[
r < \min_{j=1, \ldots, k} \text{dist}(x_j, \partial \Omega),
\]
we will denote by \( \mathcal{A}_x \) the image of \( \mathcal{A} \) under the restriction map
\[
u_0 \mapsto (u_0|_{B_\rho(x_1, r)}, \ldots, u_0|_{B_\rho(x_k, r)}).
\] (5.20)

We also write
\[
\mathcal{A}_x(x, r) = \mathcal{A}_x(B_\rho(x_1, r)) \times \ldots \times \mathcal{A}_x(B_\rho(x_k, r)).
\] (5.21)

**Proposition 5.3.** Let the assumptions of theorem 1.1 hold, and let \( K \) be a compact subset of \( \Omega \) such that
\[
\{ u_0 | K : u_0 \in \mathcal{A} \}
\]
is bounded in $A_{\sigma}(K)$. Now choose $\epsilon \in (0, 1)$, and let $\mathbf{x}$ be a choice of $k$ points within $K$. Then there exists a $\varphi$, $0 < \varphi < 1$, which depends only on $\epsilon$ and $n$, such that
\[
\|f(\mathcal{A}_K, A_{\varphi}(\mathbf{x}, \varphi\sigma)) \| \leq \|f(A_{\varphi}(\mathbf{x}, \varphi\sigma)) \| = \|f(\mathcal{A}_K, A_{\varphi}(\mathbf{x}, \varphi\sigma)) \|,
\]
and
\[
\tau(\mathcal{A}_K, A_{\varphi}(\mathbf{x}, \varphi\sigma)) = 0.
\]

Proof. It follows from the inequality in (5.18) that the restriction map in (5.20) is H"older on $A$ with exponent $1 - \epsilon$, which gives (5.22), using the fractal dimension analog of (3.5).

In order to bound the thickness exponent we first find a linear space which provides a good approximation on each ball $B_R(a, \varphi\sigma)$.

Let $l \in \mathbb{N}$ and consider the space
\[
P_l = \left\{ \sum |a| \leq l \right\}.
\]

Note that $P_l$ is linear and that $\dim P_l = m(l+n)$. Now let $u_0 \in A_{\varphi}$ be arbitrary, and set
\[
p_l(x) = \sum |\alpha| \leq l \frac{D\alpha u_0(a)}{\alpha!} (x-a)^\alpha \in P_l.
\]

We need to estimate the norm of the error made when approximating $u_0$ by $p_l$ in $A_{\varphi}(B_R(a, \varphi\sigma))$. The proof of lemma 4.1 shows that every function $u$ in $A_{\varphi}(K)$ can be extended to a holomorphic function, still denoted by $u$, in the polydisk $\Delta(a, \sigma)$. If $z \in \Delta(a, \sigma/2)$, then
\[
|u_0(z) - p_l(z)| \leq \|u_0\|_{A_{\varphi}(K)} \sum |\alpha| \geq l+1 \frac{1}{2^{l+1}} = \|u_0\|_{A_{\varphi}(K)} \sum_{j=0}^{\infty} \left( \begin{array}{c} j + n - 1 \\ n - 1 \end{array} \right) \frac{1}{2^j}
\]
from where
\[
|u_0(z) - p_l(z)| \leq \|u_0\|_{A_{\varphi}(K)} \frac{(l+2)^n}{2^{l+1}} \sum_{j=0}^{\infty} \left( \begin{array}{c} j + n - 1 \\ n - 1 \end{array} \right) \frac{1}{2^j}
\]
(since $2^n = (\sum_{j=0}^{\infty} 2^{-j})^n$). By the Cauchy formula (cf. (5.19)), if $x \in \Delta(a, \sigma/4) \cap \mathbb{R}^n$, then
\[
|D\alpha (u_0 - p_l)(x)| (\sigma/4)^n \leq \frac{(l+2)^n}{2^{l+1-n}} \|u_0\|_{A_{\varphi}(K)}
\]
and therefore
\[
\|u_0 - p_l\|_{A_{\varphi}(B_k(a, \sigma/4\sqrt{m}))} \leq \frac{(l+2)^n}{2^{l+1-n}} \|u_0\|_{A_{\varphi}(K)}.
\]
In particular, since $\varphi < 1/4\sqrt{m}$, we have
\[
\|u_0 - p_l\|_{A_{\varphi}(B_k(a, \varphi\sigma))} \leq \frac{(l+2)^n}{2^{l+1-n}} \|u_0\|_{A_{\varphi}(K)}.
\]
Now, using this approach, approximate $u \in \mathcal{A}_x$ in every ball by a polynomial, and call these polynomials $(p^{(1)}, p^{(2)}, \ldots, p^{(k)})$. Then the distance from the image of $u_0$ under the restriction map $(5.20)$ to $(p^{(1)}, p^{(2)}, \ldots, p^{(k)})$ is less than or equal to $(k(l + 2)^{n - 1}/2^{l+1-n})\|u_0\|_{A_x(K)}$. We thus obtain

$$d(A_x, \epsilon_l) \leq mk\left(\frac{l + n}{n}\right) \leq mk(l + n),$$

where

$$\epsilon_l = \frac{k(l + 2)^{n-1}}{2^{l+1-n}} M_0, \quad l \in \mathbb{N}$$

and $M_0 = \sup_{u_0 \in \mathcal{A}_x} \|u_0\|_{A_x(K)}$. Noting that $\epsilon_l + 1 \leq \epsilon_l / 2$, it follows from the result of exercise 13.2 in Robinson [27] that

$$\tau(A_x) = \limsup_{l \to \infty} \frac{\log d(A_x, \epsilon_l)}{\log(1/\epsilon_l)}.$$

For every $l \in \mathbb{N}$, we have

$$\frac{\log d(A_x, \epsilon_l)}{\log(1/\epsilon_l)} \leq \frac{\log m + \log k + n \log(l + n)}{-\log k - (n - 1) \log(l + 2) + (l + 1 - n) \log 2 - \log M_0}.$$

The right hand side converges to 0 as $l \to \infty$, and we deduce $\tau(A_x) = 0$. 

6. **Nodal parametrisation.** We now use the above results to adapt the argument from [12] to the Dirichlet case.

**Theorem 6.1.** Let $\Omega \subset \mathbb{R}^n$, and let $\mathcal{A}$ be a compact connected subset of $L^2(\Omega) = [L^2(\Omega)]^m$, with finite fractal dimension $d_f(\mathcal{A})$. Furthermore, suppose that functions on $\mathcal{A}$ are uniformly analytic on all compact subdomains $K \subset \subset \Omega$, i.e., for each $K$ there exist $C_K$ and $\sigma_K$ such that

$$\sup_{x \in K} |D^\alpha u(x)| \leq C_K \sigma_K^{-|\alpha|}$$

for every multi-index $\alpha \geq 0$.

For $k$ points

$$x = (x_1, \ldots, x_k),$$

define the observation map

$$E_x[u] = (u(x_1), \ldots, u(x_k)).$$

Then $E_x$ is one-to-one between $\mathcal{A}$ and its image for almost every choice of $k$ points in $\Omega$, provided that $k \geq 16d_f(\mathcal{A}) + 1$.

In particular, the values $(u(x_1), \ldots, u(x_k))$ provide a parametrisation of $\mathcal{A}$ which is continuous from $\mathbb{R}^{mk}$ into $L^2(\Omega)$.

We assume always that $\sigma_K < \text{dist}(K, \partial\Omega)$. The proof is based on that in [12], with some important alterations.

**Proof.** First let $K$ be a compact subset of $\Omega$. Then, by assumption $\mathcal{A}$ is a subset of $A_x(K)$ for some $\sigma > 0$. Choose $y_1, \ldots, y_k \in K$, and let $\epsilon' \in (0, 1)$ be arbitrary.

It follows from proposition 5.3 that the dimension of $A_y$ measured in $A_{\varphi \sigma}(y, \varphi \sigma)$ is comparable to $d_f(\mathcal{A}, L^2(\Omega))$, and that its thickness exponent is zero. Note that if two analytic functions agree on the balls $B_{\mathcal{R}}(y_j, \varphi \sigma)$ then they must agree throughout $K$ (and in fact throughout $\Omega$).

Suppose that $(x_1, \ldots, x_k)$ is a choice of nodes in

$$B_{\mathcal{R}}(y_1, \varphi \sigma) \times \ldots \times B_{\mathcal{R}}(y_k, \varphi \sigma)$$
for which the one-to-one property fails. This implies that there exist two functions $u_1, u_2 \in \mathcal{A}$, with $u_1 \neq u_2$ but

$$u_1(x_j) = u_2(x_j), \quad j = 1, \ldots, k.$$ 

Using theorem 4.2 to find a "nice" parametrisation of $\mathcal{A}_k$ in terms of $D$ coordinates $\epsilon \in \tilde{E} \subset R^D$, $u(x; \epsilon)$, this means that the function

$$w(x; \epsilon_1, \epsilon_2) = u(x; \epsilon_1) - u(x; \epsilon_2),$$ 

must satisfy

$$w(x_j; \epsilon) = 0 \quad \text{for all} \quad j = 1, \ldots, k,$$

for some $\epsilon = (\epsilon_1, \epsilon_2) \in \mathcal{E} = \{(\epsilon_1, \epsilon_2) \in E \times E : \epsilon_1 \neq \epsilon_2\}$.

Theorem 4.3 guarantees that the zeros of $w$, considered as a subset of each $B_{\mathbb{R}}(y_j, \varphi_\sigma) \times \mathcal{E}$, are contained in a countable collection of sets, each of which is the graph of a Hölder function with exponent $\theta$,

$$(x', x_j(x'; \epsilon); \epsilon),$$

where $x' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. It follows that collections of $k$ such zeros, considered as a subset of the product space

$$B_{\mathbb{R}}(y_1, \varphi_\sigma) \times \ldots \times B_{\mathbb{R}}(y_k, \varphi_\sigma) \times \mathcal{E},$$

are contained in the product of $k$ such manifolds. Since the coordinate $\epsilon$ is common to each of these, they are the graphs of Hölder continuous functions from a subset of $\mathbb{R}^{2D + k(n - 1)}$ into $\mathbb{R}^k$. Lemma 3.1 shows that each of these sets has Hausdorff dimension at most

$$2D + k(n - 1) + k(1 - \theta),$$

and using (3.4) the same goes for the whole countable collection.

The projection of this collection onto

$$B_{\mathbb{R}}(y_1, \varphi_\sigma) \times \ldots \times B_{\mathbb{R}}(y_k, \varphi_\sigma)$$

enjoys the same bound on its dimension (3.5). We thus need to choose $D$ so that

$$2D + k(n - 1) + k(1 - \theta) < nk,$$

i.e., $k > 2D/\theta$; on the other hand, $\theta > 0$ has to be chosen so that

$$\theta < 1 - \frac{2d_f(A)}{D(1 - \epsilon')}.$$ 

Let $\epsilon' \in (0, (D - 2d_f(A))/D)$. Take any $\epsilon'' > 0$ such that $1 - 2d_f(A)/D(1 - \epsilon') - \epsilon'' > 0$, and let $\theta = 1 - 2d_f(A)/D(1 - \epsilon') - \epsilon''$. We need to choose $k \in \mathbb{N}$ such that

$$k > \frac{2D}{1 - 2d_f(A)/D(1 - \epsilon') - \epsilon'}. $$

Since $\epsilon'$ and $\epsilon''$ can be arbitrarily close to 0, we notice that we can pick any $k \in \mathbb{N}$ such that

$$k > \frac{2D^2}{D - 2d_f(A)}.$$

Ideally we would pick $D = 4d_f(A)$, which gives the minimum value of the right-hand side, but $D$ must be an integer. However, we can pick the integer $D$ satisfying $4d_f(A) - \frac{1}{2} \leq D < 4d_f(A) + \frac{1}{2}$, and observe that if $D = 4d_f(A) + \frac{1}{2}$ then

$$\frac{2D^2}{D - 2d_f(A)} < 16d_f(A) + 1.$$
while the same inequality holds for $D = 4d_f(A) - \frac{1}{2}$ provided that $d_f(A) > \frac{1}{2}$.

Thus, for $d_f(A) > \frac{1}{2}$, $k \geq 16d_f(A) + 1$ suffices.

To deal with the case $d_f(A) \leq \frac{1}{2}$ (and in fact any value strictly less than 1), we simply observe that $\mathcal{A}$ must be a single point (and so any map is $1 - 1$ between $\mathcal{A}$ and its image). This follows since $d_H(\mathcal{A}) \leq d_f(A) < 1$, so that we can use lemma 3.2 from Falconer [7], which guarantees that a connected set with finite one-dimensional Hausdorff measure is arcwise connected. Since $\mathcal{H}^1(\mathcal{A}) = 0$, it follows that $\mathcal{A}$ must be a single point.

Since we must have $\mathcal{H}^{nk}(\text{bad choices}) = 0$, and $\mathcal{H}^{nk}$ is proportional to $nk$-dimensional Lebesgue measure (theorem 1.12 in Falconer [7]), it follows that almost every choice of $n$ nodes in the product space makes $E_n$ one-to-one as claimed.

Since $\Omega$ can be covered by a finite collection of such balls, almost every choice of $n$ nodes in $\mathcal{K}$ makes $E_n$ one-to-one.

Since $\Omega^k$ can be covered by countably many subsets of the form $K^k$, where $K$ is a compact subset of $\Omega$, it follows that almost every choice of $(x_1, \ldots, x_k) \in \Omega^k$ is instantaneously determining.

Finally, to show that these observations parametrise the attractor, observe that for any choice of points $x$ the map $E_n$ is continuous from $\mathcal{A}$, considered as a subset of $L^2(\Omega)$, into $\mathbb{R}^{nk}$: this follows from (5.18). Now, since $\mathcal{A}$ is a compact subset of $L^2(\Omega)$ and $E_n$ is $1 - 1$ between $\mathcal{A}$ and its image, a standard topological result then guarantees that $E_n$ is one-to-one.

Note that the connectedness of $\mathcal{A}$ is only used to deal with the case $d_f(A) \leq \frac{1}{2}$. If $\mathcal{A}$ is not connected then for $d_f(A) = 0$ we can take $D = 1$ and so $k = 3$, and for $0 < d_f(A) \leq \frac{1}{2}$ we may choose $D \in \mathbb{N}$, such that $4d_f(A) \leq D < 4d_f(A) + 1$. Then

$$2D + 4d_f(A) + \frac{8d_f(A)^2}{D - 2d_f(A)} < 8d_f(A) + 2 + 4d_f(A) + 4d_f(A) = 16d_f(A) + 2$$

which shows that we may choose any $k \in \mathbb{N}$ such that $k \geq 16d_f(A) + 2$.

6.1. The periodic case. We now state, without proof, the improved bound which follows from lemma 3.1 in the periodic case. The adjustments to the argument in [12] are straightforward, and follow the proof above.

**Theorem 6.2.** Suppose that $\mathcal{A}$ is a compact connected subset of $L^2(Q)$ which has finite fractal dimension and is uniformly bounded in $G_\sigma(Q)$, for some $\sigma > 0$. Then, provided that $k \geq 16d_f(A) + 1$, almost every choice $x$ of $k$ nodes in $Q$ makes the map $E_n$ one-to-one between $\mathcal{A}$ and its image.

This result improves our previous estimate $k \geq 16nd(\mathcal{A}) + 1$, and in particular removes the dependence on $n$.

7. Application to the 2d Navier-Stokes equations. We now check that the standing assumptions of theorem 6.1 hold for the attractor of the 2d Navier-Stokes equations.

The attractor is indeed a finite-dimensional set, as was mentioned briefly in section 3. Constantin et al. [4] have given the best bounds on its dimension in terms of the dimensionless Grashof number $G$ defined by

$$G = \frac{\|\Omega\|}{\nu^2} \|f\|^2_{L^2(\Omega)}.$$  

namely

$$d_f(A) \leq cG.$$
There are various results on analyticity of the solutions of the Navier-Stokes equations for the Dirichlet case (e.g. Komatsu [20] and Masuda [23, 24]): in particular, the arguments in Kahane [19] show that if $f$ is analytic in $\Omega$ then solutions on the global attractor are analytic in the appropriate sense. (Note that, following Grujić & Kukavica [14], it should be possible to prove this result in a more transparent manner.)

It follows that all the theorems of section 6 apply to this important physical example, providing a rigorous theoretical justification for attempts to reconstruct the dynamics of fluid flows from a finite number of experimental observations (see [26] for a more lengthy discussion).

7.1. Observations in one fixed direction. Note that for this example the two components of the velocity are not independent, since the incompressibility ($\nabla \cdot u = 0$) along with the boundary conditions allows us to deduce one component if we know the other.

Since the proof of theorem 6.1 relies on theorem 4.3, it follows from the comments near (4.12) that in fact the flow field in any given direction $d \in \mathbb{R}^2 \setminus \{0\}$ can be determined by measurements solely in that direction. Since this then fixes the flow field in the orthogonal direction, in this particular case the map $E_x$ can be replaced by the measurements of a single fixed component of the velocity at each point,

$$E_{x,d}[u] = (u(x_1) \cdot d, \ldots, u(x_k) \cdot d).$$

7.2. Asymptotically determining nodes. Finally, noting that there in fact exists an absorbing set in $H^2(\Omega)$, it follows from standard interpolation inequalities that $A$ attracts in the $L^\infty(\Omega)$ norm as well as the $L^2(\Omega)$ norm. Corollary 7.1 in [12] therefore applies, and show that a set of ‘instantaneously determining nodes’ for the 2d Navier-Stokes equations is also ‘asymptotically determining’ in the sense of Foias and Temam [9] (see the introduction).

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