1. Show that
\[ \|A\|_{B(X,Y)} = \sup_{\|x\| \leq 1} \|Ax\|_Y = \sup_{\|x\| = 1} \|Ax\|_Y \]

and
\[ \|A\|_{B(X,Y)} = \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X} \].

2. Consider the map \( T : \ell^2 \to \ell^2 \) defined by
\[ T\mathbf{x} = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots). \]

Show that \( T \) is bounded, but that the range of \( T \) is not closed.

3. If \( X = C^0([a,b]) \), for any \( \phi \in X \) define \( T_\phi : X \to \mathbb{R} \) by
\[ T_\phi(f) = \int_a^b \phi(t)f(t) \, dt \quad \text{for all } f \in X. \]

Show that \( T_\phi \in X^* \) with \( \|T_\phi\|_* \leq \int_a^b |\phi(t)| \, dt \). By choosing an appropriate sequence of functions \( f_n \in X \) for which \( |T_\phi(f_n)|/\|f_n\|_\infty \to \int_a^b |\phi(t)| \, dt \) show that in fact
\[ \|T_\phi\|_* = \int_a^b |\phi(t)| \, dt. \]

4. Let \( K \in L^2((a,b) \times (a,b)) \), i.e.
\[ \int_a^b \int_a^b |K(t,s)|^2 \, dt \, ds < \infty. \]

Show that the adjoint of the integral operator \( T : L^2(a,b) \to L^2(a,b) \) defined as
\[ (Tf)(t) = \int_t^b K(t,s)f(s) \, ds \]

is given by
\[ (T^*f)(t) = \int_t^b K(s,t)f(s) \, ds. \]
5. If $T \in B(H, H)$, show that 

$$\text{Ker}(T^*) = (\text{range}(T))^\perp.$$ 

Deduce that if $\text{range}(T - \lambda I) \neq H$ then $\bar{\lambda}$ is an eigenvalue of $T$. (Recall that $X^\perp = \overline{X}^\perp$, where $\overline{X}$ is the closure of $X$ in $H$.)

6. We show that $T \in B(H, H)$ is compact iff $T^*$ is compact.

(i) Show that if $T \in B(H, H)$ is compact then $TT^*$ is compact.

(ii) Show that if $\{x_n\}$ is a bounded sequence in $H$ such that $\{TT^*x_n\}$ is Cauchy then $\{T^*x_n\}$ is also Cauchy. [Hint: first show that $\|T^*x\|^2 \leq \|TT^*x\|\|x\|$]

(iii) Deduce that $T^*$ is compact.

7. Let $H$ be a Hilbert space, and $\{x_n\}$ a sequence in $H$ such that for some $x \in H$, 

$$(x_n, y) \to (x, y) \quad \text{for every } y \in H.$$ 

We say that $x_n$ converges ‘weakly’ to $x$ in $H$ and write $x_n \rightharpoonup x$.

(i) Show that weak limits are unique (i.e. if $x_n \to x$ and $x_n \to y$ then $x = y$);

(ii) It is clear that if $x_n \to x$ then $x_n \to x$, but the converse is not true: Show that if $\{e_n\}$ is a countable orthonormal basis for $H$ then $e_n \rightharpoonup 0$ (clearly $e_n \not\to 0$ since $\|e_n\| = 1$ for every $n$).

We now want to show that if $x_n \to x$ and $T \in B(H, H)$ is a compact operator then in fact $Tx_n \to Tx$ (without extracting a subsequence):

(iii) Use the Riesz Representation Theorem to show that for any $T \in B(H, H)$, $x_n \to x$ implies that $Tx_n \to Tx$, i.e. for any $z \in H$,

$$(Tx_n, z) \to (Tx, z).$$ 

[You may find it useful to look again at the proof of the existence of the adjoint operator.]

(iv) Arguing by contradiction, show that $Tx_n \to Tx$. [You will need to use the fact that if $\xi_n \to \xi$ then $\xi_n \rightharpoonup \xi$, and the uniqueness of weak limits from part (i).]
8. Suppose that \( \{\phi_j(x)\}_{j=1}^{\infty} \) is an orthonormal basis for \( L^2(a, b) \). Let \( K \in L^2((a, b) \times (a, b)) \).

(i) For almost every \( x \in (a, b) \), \( K(x, \cdot) \in L^2(a, b) \). Deduce that one can write

\[
K(x, y) = \sum_{i=1}^{\infty} k_i(x) \phi_i(y)
\]

where \( k_i \in L^2(a, b) \).

(ii) Now, expand \( k_i(x) \) using the basis \( \{\phi_j(x)\} \) and deduce that \( \{\phi_i(y) \phi_j(x)\} \) is a basis for \( L^2((a, b) \times (a, b)) \).

9. Show that if \( \{e_j\}_{j=1}^{n} \) is an orthonormal set in \( H \) and \( T \in B(H, H) \) is defined by

\[
Tx = \sum_{j=1}^{n} \lambda_j(x, e_j)e_j
\]

with \( |\lambda_j| < 1 \) for all \( j \) then \( (I - T)^{-1} = I + S \), where

\[
Sx = \sum_{j=1}^{n} \frac{\lambda_j}{1 - \lambda_j}(x, e_j)e_j.
\]

10. Consider the integral equation (which we would like to solve for \( x(t) \) given \( f \) and \( K \)),

\[
x(t) = f(t) + \alpha \int_{a}^{b} K(t, s)x(s) \, ds. \tag{*}
\]

If we define

\[
(Tx)(t) = \int_{a}^{b} K(t, s)x(s) \, ds
\]

this can be rewritten as \( x = f + \alpha Tx \).

(i) Express the solution of (*) in terms of a series involving powers of \( T \). For what values of \( \alpha \) can this series be guaranteed to converge?

(ii) Show that

\[
(T^n x)(t) = \int_{a}^{b} K_n(t, s)x(s) \, ds,
\]
where $K_n$ is defined inductively via
\[
K_n(t, s) = \int_a^b K(t, r)K_{n-1}(r, s) \, dr.
\]

11. Show that if $H$ is a Hilbert space, $\{e_j\}$ an orthonormal set in $H$, $\lambda_j \in \mathbb{R}$ with $\lambda_j \to 0$, and $T : H \to H$ is defined by
\[
Tu = \sum_{j=1}^{\infty} \lambda_j (u, e_j) e_j,
\]
then $T$ is compact and self-adjoint. (This is a partial converse to the Hilbert-Schmidt Theorem.)

12. Suppose that $\{e_j(x)\}$ is an orthonormal set in $L^2(a, b)$ (real-valued), and that
\[
K(x, y) = \sum_{j=1}^{\infty} \lambda_j e_j(x)e_j(y) \quad \text{with} \quad \sum_{j=1}^{\infty} |\lambda_j|^2 < \infty.
\]
Show that the $\{e_j(x)\}$ are eigenvectors of
\[
(Tu)(x) = \int_a^b K(x, y)u(y) \, dy
\]
with corresponding eigenvalues $\lambda_j$ and that there are no other eigenvectors corresponding to non-zero eigenvalues. Find the eigenvalues and eigenvectors when
(i) $(a, b) = (-\pi, \pi)$ and $K(t, s) = \cos(t - s)$ [hint: use double angle formulae];
(ii) $(a, b) = (-1, 1)$ and $K(t, s) = 1 - 3(t - s)^2 + 9t^2s^2$ [hint: write $K$ in terms of orthonormal polynomials].

13. Show that the solution of $-d^2u/dx^2 = f$ is given by
\[
u(x) = \int_0^1 G(x, y)f(y) \, dy
\]
where
\[
G(x, y) = \begin{cases} 
  x(1 - y) & 0 \leq x < y \\
  y(1 - x) & y \leq x \leq 1.
\end{cases}
\]
We now show that the spectrum of a self-adjoint operator in $B(H,H)$ is real. To do this we make use of the fact that in general the spectrum can be split into three disjoint pieces: the ‘point’ spectrum or set of eigenvalues

$$\sigma_p(T) = \{ \lambda \in \mathbb{C} : Tx = \lambda x \text{ for some non-zero } x \in H \},$$

the ‘continuous spectrum’

$$\sigma_c(T) = \{ \lambda \in \mathbb{C} : (T-\lambda I)^{-1} \text{ is defined on a dense subset of } H \text{ but is not bounded} \},$$

and the ‘residual spectrum’ or ‘compression spectrum’

$$\sigma_r(T) = \{ \lambda \in \mathbb{C} : (T-\lambda I)^{-1} \text{ exists but is not defined on a dense subset of } H \}.$$ (Replacing the requirement that $(T-\lambda I)^{-1}$ is defined on all of $H$ by it only having to be defined on a dense subset is reasonable, since a bounded linear operator defined on a dense subset of $H$ has a unique extension to a bounded linear operator defined on the whole of $H$.)

14. We already know that if $T \in B(H,H)$ is self-adjoint then all the eigenvalues are real. So we show first that any $\lambda \in \sigma_c(T)$ must be real, and then that $\sigma_r(T)$ must be empty.

(i) A linear operator $T \in B(H,H)$ is bounded below if there exists an $\alpha > 0$ such that

$$\|Tx\|^2 \geq \alpha \|x\|^2$$

for all $x \in H$. Show that if $T$ is bounded below then its inverse $T^{-1}$ (defined on range($T$)) is bounded. [To show that its inverse exists show that Ker($T$) = \{0\}.]

(ii) Show that if $\lambda \in \mathbb{C}$ with non-zero imaginary part, then $T - \lambda I$ is bounded below (recall that $T$ is self-adjoint).

(iii) Deduce that if $(T - \lambda I)$ has a densely defined inverse then $\lambda \in \text{R}(T)$, so $\sigma_c(T) \subset \mathbb{R}$.

And we now show that $\sigma_r(T)$ is empty by showing that the assumption that $\lambda \in \sigma_r(T)$ leads to a contradiction

(iv) If $\lambda \in \sigma_r(T)$ then the range of $(T - \lambda I)$ is not dense in $H$. Use the result of question 5 to show that this implies that $\lambda$ is an eigenvalue of $T$. Deduce that the residual spectrum is empty, and hence that the spectrum is real.
The final question is really moving beyond the scope of this course to Functional Analysis II, but you should be in a position to do it nevertheless.

*15. Recall that $\ell^p := \ell^p(\mathbb{R})$ is the space of all real sequences $\underline{x} = (x_1, x_2, \ldots)$ that are $p$th power summable, equipped with the norm $\|\underline{x}\|_p$ defined by

$$
\|\underline{x}\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p},
$$

and that we proved there Hölder’s inequality: if $\underline{x} \in \ell^p$ and $\underline{y} \in \ell^q$ with $p^{-1} + q^{-1} = 1$ ($1 < p, q < \infty$) then

$$
\sum_{j=1}^{\infty} |x_jy_j| \leq \|\underline{x}\|_p \|\underline{y}\|_q.
$$

(i) Show that if $\underline{y} \in \ell^q$ then the mapping $l : \ell^p \to \mathbb{R}$

$$
\underline{x} \mapsto \sum_{j=1}^{\infty} x_jy_j
$$

is an element of $(\ell^p)^*$, and that

$$
\|l\|_{p*} \leq \|\underline{y}\|_q,
$$

where $\|\cdot\|_{p*}$ denotes the norm in $(\ell^p)^*$.

(ii) Show that if $l \in (\ell^p)^*$ then there exists a real sequence $\underline{y}$ such that

$$
l(\underline{x}) = \sum_{j=1}^{\infty} x_jy_j \quad \text{(*)}
$$

for every $\underline{x} \in \ell^p$. [Hint: the elements

$$
\underline{e}_j = (0, \cdots, 0, 1_{\text{jth place}}, 0, \cdots)
$$

form a basis for $\ell^p$.]
(iii) Given the sequence $y$ from (ii) such that (*) holds, consider the succession of sequences $x^{(n)} \in \ell^p$ given by

$$x^{(n)}_j = \begin{cases} 
    \frac{|y_j|^q / y_j}{y_j} & j \leq n \text{ and } y_j \neq 0 \\
    0 & j > n \text{ or } y_j = 0.
\end{cases}$$

Show that

$$\|x^{(n)}\|_p = \left( \sum_{j=1}^{n} |y_j|^q \right)^{1/p} \quad \text{and} \quad l(x^{(n)}) = \sum_{j=1}^{n} |y_j|^q,$$

(remember that $p^{-1} + q^{-1} = 1$), and hence show that

$$\left( \sum_{j=1}^{m} |y_j|^q \right)^{1/q} \leq \|l\|_{p^*}$$

for every $m$, from which it follows that $y \in \ell^q$ with $\|y\|_q \leq \|l\|_{p^*}$.

Combining (i) and (iii) shows that in both cases in fact $\|y\|_q = \|l\|_{p^*}$.

We have show that $(\ell^p)^* \text{ is isometric to } \ell^q$, i.e. that $\ell^q$ essentially ‘is’ the dual of $\ell^p$. This agrees with the Riesz Representation Theorem, which guarantees that the dual of $\ell^2$ (which is a Hilbert space) should be isometric to $\ell^2$, precisely via the sort of ‘inner product construction’ in (*)).