Problems One.

1. Let \( \{x_n\} \) be a Cauchy sequence in a normed space \((X, \|\cdot\|)\). Show that if \( x_n \) has a subsequence that converges to some \( x \in X \), then \( x_n \to x \).

Given \( \epsilon > 0 \) choose \( N \) such that \( \|x_n - x_m\| < \epsilon/2 \) for all \( n, m \geq N \). Then choose \( J \) such that \( \|x_n_j - x\| < \epsilon/2 \) for all \( j \geq J \). Then for all \( n \geq n_J \),

\[
\|x_n - x\| \leq \|x_n - x_{n_J}\| + \|x_{n_J} - x\| < \epsilon
\]

and so \( x_n \to x \).

2. Prove the polarisation identity

\[
4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2
\]

in a complex inner product space.

Expand the right-hand side as inner products:

\[
(x + y, x + y) - (x - y, x - y) + i(x + iy, x + iy) - i(x - iy, x - iy)
\]

\[
= \|x\|^2 + 2(x, y) + \|y\|^2 - (\|x\|^2 - (x, y) - (y, x) + \|y\|^2)
\]

\[
+ i(\|x\|^2 + i(y, x) - i(x, y) + \|y\|^2) - i(\|x\|^2 - i(y, x) + i(x, y) + \|y\|^2)
\]

\[
= 4(x, y).
\]

*3. Suppose that \( x_n \to x \) and \( y_n \to y \) in \( H \). Show that

\[
(x_n, y_n) \to (x, y).
\]

4. Consider \( C^0([0, 1]) \). Show that neither the supremum norm

\[
\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)|
\]

nor the \( L^1 \) norm

\[
\|f\|_{L^1} = \int_0^1 |f(x)| \, dx
\]

can be derived from an inner product.

We have to find functions \( f \) and \( g \) for which the parallelogram law is violated. For the sup norm, consider \( f(x) = |x - (1/2)| \) and \( g(x) = (1/2) - f(x) \). Then \( \|f\|_\infty = \|g\|_\infty = 1/2 \), \( \|f + g\|_\infty = 1 \), and \( \|f - g\|_\infty = 1/2 \). So

\[
\frac{5}{4} = \|f + g\|^2 + \|f - g\|^2 \neq 2(\|f\|^2 + \|g\|^2) = \frac{1}{2}.
\]
For the $L^1$ norm we can use the same example, since $\|f\|_{L^1} = \|g\|_{L^1} = \|f - g\|_{L^1} = 1/4$ and $\|f + g\|_{L^1} = 1/2$.

5. Show that $U^\perp$ is always a closed linear subspace of $H$.

If $x_n \in U^\perp$ with $x_n \to x$ then, using question 3, for any $u \in U$

$$(x, u) = \lim_{n \to \infty} (x_n, u) = 0,$$

i.e. $x \in U^\perp$.

*6. Show that if $U$ is a closed linear subspace of $H$ then $(U^\perp)^\perp = U$.

7. Show that if $A$ is closed an $\frac{1}{2}(x + y) \in A$ whenever $x, y \in A$ then $A$ is convex.

Take $x, y \in A$ and suppose that $\lambda x + (1 - \lambda)y \in A$ for every $\lambda \in [0, 1]$ such that $\lambda = a 2^{-k}$, where $a \in \mathbb{N}$. Then given $\lambda = b 2^{-(k+1)}$ with $b \in \mathbb{N}$ and $b$ odd, clearly one has

$$\lambda = \frac{1}{2} (b - 1) 2^{-(k+1)} + \frac{1}{2} (b + 1) 2^{-(k+1)}.$$

It follows that $\lambda x + (1 - \lambda)y \in A$ for every $\lambda = b 2^{-(k+1)}$, $b \in \mathbb{N}$. Now simply use the fact that for any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y$ can be approximated by elements of the form $\lambda_n x + (1 - \lambda_n)y$, where $\lambda_n = a_n 2^{-n}$; since $A$ is closed, it follows that $\lambda x + (1 - \lambda)y \in A$.

8. A set $\{e_j\}_{j=1}^n$ is orthonormal if

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Show that an orthonormal set is linearly independent.

If $\sum_{j=1}^n \alpha_j e_j = 0$ then taking the inner product with $e_k$ ($k = 1, \ldots, n$) shows that $\alpha_k = 0$. Since this is true for each $k$, the set is linearly independent.

9. (In this question life is much easier if you take $H$ to be real.) The space $E$ spanned by $\{e_j\}_{j=1}^n$ consists of all elements of $H$ of the form

$$\sum_{j=1}^n \alpha_j e_j \quad \alpha_j \in \mathbb{K}.$$
Show that the closest point to \( x \) in \( E \) is given by

\[
\hat{x} = \sum_{j=1}^{n} (x, e_j) e_j.
\]

(You should try to do this (i) directly and (ii) using the fact that \( x - \hat{x} \in E^\perp \) - (ii) will be much easier.)

(i) Directly: consider

\[
\|x - \sum_{j=1}^{n} \alpha_j e_j\|^2 = (x - \sum_{j=1}^{n} \alpha_j e_j, x - \sum_{k=1}^{n} \alpha_k e_k) 
\]

\[
= \|x\|^2 - 2(x, \sum_{j=1}^{n} \alpha_j e_j) + \sum_{j=1}^{n} |\alpha_j|^2 
\]

\[
= \|x\|^2 - \sum_{j=1}^{n} |(x, e_j)|^2 + \sum_{j=1}^{n} |(x, e_j) - 2\alpha_j (x, e_j) + |\alpha_j|^2 
\]

\[
= \|x\|^2 - \sum_{j=1}^{n} |(x, e_j)|^2 + \sum_{j=1}^{n} [(x, e_j) - \alpha_j]^2. 
\]

This expression is clearly minimised when \( \alpha_j = (x, e_j) \).

(ii) If \( \hat{x} = \sum_{j=1}^{n} \alpha_j e_j \) and \( x - \hat{x} \in E^\perp \) then for each \( k = 1, \ldots, n \) we must have

\[
(x - \sum_{j=1}^{n} \alpha_j e_j, e_k) = (x, e_k) - \alpha_k = 0,
\]

i.e. \( \alpha_k = (x, e_k) \).

10. Let \( X \) and \( Y \) be two normed spaces. Show that

\[
\|L\|_{\mathcal{L}(X,Y)} = \sup_{x \in X: \|x\|_X = 1} \|Lx\|_Y.
\]

Denote \( M \) the right-hand side by \( M \), and note that for any \( x \in X \),

\[
\|Lx\|_Y = \left\| \frac{x}{\|x\|_X} \right\|_Y \|x\|_X \leq M \|x\|_X,
\]

so that \( \|L\|_{\mathcal{L}(X,Y)} \leq M \). Since \( \|Lx\|_Y \leq \|L\|_{\mathcal{L}(X,Y)} \|x\|_X \) for any \( x \in X \), it is clear that \( M \leq \|L\|_{\mathcal{L}(X,Y)} \).
11. Show that the map \( T : \ell^2 \to \ell^2 \) defined by

\[
T \mathbf{x} = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)
\]

is bounded, but that \( R(T) \), the range of \( T \), is not closed. (Recall that \( R(T) = T(\ell^2) = \{ y \in \ell^2 : y = Tx \text{ for some } x \in \ell^2 \} \).)

Clearly

\[
\| T \mathbf{x} \|_{\ell^2}^2 = \sum_{j=1}^{\infty} \left| \frac{x_j}{j} \right|^2 \leq \sum_{j=1}^{\infty} |x_j|^2 = \| \mathbf{x} \|_{\ell^2}^2,
\]

and so \( T \) is bounded. However, consider \( x_n = (1, 1, 1, 1, 0, 0, \ldots) \), i.e. \( n \) 1s followed by zeros. Then \( x_n \in \ell^2 \), and so \( T x_n = (1, 1/2, \ldots, 1/n, 0, 0, 0, \ldots) \). But while \( T x_n \to \mathbf{z} \) with \( z_j = 1/j \), there is no \( \mathbf{x} \in \ell^2 \) with \( T \mathbf{x} = \mathbf{z} \), since \((1,1,1,\ldots) \notin \ell^2 \).

*12. Suppose that \( A : H \to H \) is a bounded linear operator. Show that there exists a unique \( A^* : H \to H \), also a bounded linear operator, such that

\[
(Au, v) = (u, A^*v) \quad \text{for every } u, v \in H.
\]

The operator \( A^* \) is known as the (Hilbert) adjoint of \( A \).

13. Prove the contraction mapping theorem in a Hilbert space: if \( f : H \to H \) is a map such that

\[
\| f(u) - f(v) \| \leq \theta \| u - v \| \quad \text{for all } u, v \in H
\]

for some \( \theta < 1 \) then there exists a unique \( u \in H \) such that \( f(u) = u \).

Note that

\[
\left\| f^{n+1}(x) - f^n(x) \right\| \leq \theta \left\| f^n(x) - f^{n-1}(x) \right\|
\]

\[
\leq \theta^n \| f(x) - x \|,
\]

and so if \( m > n \)

\[
\left\| f^m(x) - f^n(x) \right\| \leq \sum_{j=n}^{m-1} \theta^j \| f(x) - x \|
\]

\[
\leq \left( \sum_{j=n}^{\infty} \theta^j \right) \| f(x) - x \|
\]

\[
= \frac{\theta^n}{1 - \theta} \| f(x) - x \|.
\]
It follows that \( \{f^n(x)\} \) is Cauchy; since \( H \) is complete, \( f^n(x) \to x^* \) as \( n \to \infty \). Then
\[
f(x^*) = f \left( \lim_{n \to \infty} f^n(x) \right) = \lim_{n \to \infty} f^{n+1}(x) = x^*,
\]
so \( f(x^*) = x^* \) as required. If \( x^* \) and \( y^* \) are both fixed points of \( f \) then
\[
\|x^* - y^*\| = \|f(x^*) - f(y^*)\| \leq \theta \|x^* - y^*\|
\]
which is impossible unless \( x^* = y^* \).

Problems Two.

*1. Show that \( \ell^p, 1 \leq p < \infty \), is complete.

2. Suppose that \( (X, \| \cdot \|) \) is a complete normed space, and that \( A \) is a subspace of \( X \). Show that the closure of \( A \) in \( X \), equipped with the norm \( \| \cdot \| \), is a complete normed space. [One has \( x \in \overline{A} \) iff \( x = \lim_{n \to} a_n \), with \( a_n \in A \).]

We need \( A \) to be a subspace of \( X \) (a subset will not do). Take a Cauchy sequence \( \{x_n\} \in A \). Then since this is also a Cauchy sequence in \( X \) it has converges; since \( A \) is closed the limit must lie in \( A \), and hence \( A \) is complete.

3. Show that \( c_0 \) (the subspace of \( \ell^\infty \) consisting of all sequences that converge to zero) equipped with the \( \ell^\infty \) norm is separable.

We show that the set of all sequences with only finitely many non-zero components, all of which are rational, is dense (this set is countable). Given \( x \in c^0 \), there exists an \( N \) such that \( |x_j| < \epsilon/2 \) for every \( j \geq N \). To approximate \( x \) to within \( \epsilon \) we therefore only need approximate the first \( N \) components to within \( \epsilon/2 \). Clearly we can do this using rationals (choose \( q_j \in \mathbb{Q} \) such that \( |q_j - x_j| < \epsilon/2N \) for \( j = 1, \ldots, N \)).

4. Show that the product of a finite number of separable spaces is separable; and that a closed linear subspace of a separable space is separable.

If \( \{x_j\} \) is a countable dense subset of \( X \) and \( \{y_j\} \) is a countable dense subset of \( Y \) then clearly the set \( \{(x_j, y_k)\} \) is a countable dense subset of \( X \times Y \), and the result as stated follows by induction.

Now, if \( M \) is a closed linear subspace of \( X \), start with a countable dense subset \( \{x_j\} \) of \( X \). Fix \( \epsilon > 0 \), and for each \( x_j \) for which \( B(x_j, \epsilon/2) \cap M \neq 0 \), choose an element \( m_j \in M \). Then any \( m \in M \) lies in one of the \( B(x_j, \epsilon/2) \).
since \( m_j \) does too, it follows that \( m \in B(m_j, \epsilon) \). Apply this construction for \( \epsilon = 2^{-k} \) for each \( k \in \mathbb{N} \); the union of the resulting \( m_j \)s provides a countable dense subset of \( M \).

5. Suppose that \( \{f_n\} \in C^0_0(\Omega) \), and that \( \{f_n\} \) is Cauchy in the sup norm. Show that \( f_n \to f \), where \( f \in C^0_0(\bar{\Omega}) \) with \( f = 0 \) on \( \partial \Omega \). Show further that any such \( f \) can be obtained as the uniform limit of functions in \( C^0_0(\Omega) \). \([This shows that \( C^0_0(\bar{\Omega}) \) – continuous functions on \( \bar{\Omega} \) that are zero on the boundary – is the completion of \( C^0_0(\Omega) \) wrt the supremum norm.\]

First, clearly \( f_n \in C^0(\bar{\Omega}) \), and since \( f_n \to f \) uniformly on \( \Omega \), \( f \in C^0(\bar{\Omega}) \). Furthermore if \( x \in \partial \Omega \) then \( f_n(x) = 0 \) for every \( n \), whence \( f = 0 \) on \( \partial \Omega \).

Now, suppose that \( f \in C^0(\bar{\Omega}) \) with \( f = 0 \) on \( \partial \Omega \). Let

\[
f_{\epsilon}(x) = \begin{cases} f(x) & |f(x)| > 2\epsilon \\ 0 & |f(x)| < \epsilon \\ f(x) \left[ \frac{|f(x)|}{\epsilon} - 1 \right] & \epsilon < |f(x)| < 2\epsilon. \end{cases}
\]

Then \( f_{\epsilon} \) converges uniformly to \( f \) on \( \Omega \), and since \( f \) is uniformly continuous on \( \Omega \), given any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(x)| < \epsilon \) whenever \( \text{dist}(x, \partial \Omega) < \delta \), so that \( f_{\epsilon} \) has compact support.

6. The Stone–Weierstrass Theorem states that if \( K \) is a compact space and \( A \) is an algebra of real-valued continuous functions on \( K \) such that for all \( x, y \in K \) with \( x \neq y \), there exists an \( f \in A \) such that \( f(x) \neq f(y) \), and for every \( x \in K \) there exists an \( f \in A \) such that \( f(x) \neq 0 \), then \( A \) is dense in \( C(K; \mathbb{R}) \). \( [An algebra is a set on which one can define multiplication and addition.] \) Deduce that the space of polynomials is dense in \( C^k(\bar{\Omega}) \) for any \( k \geq 0 \), where \( \Omega \) is a bounded subset of \( \mathbb{R}^n \).

The space of polynomials is an algebra on \( \bar{\Omega} \). Given \( x, y \in \bar{\Omega} \) with \( x \neq y \), they must have at least one component, say the \( j \)th, distinct. So one can consider the polynomial \( x_j \). For every \( x \in \bar{\Omega} \) we can consider the polynomial \( f \equiv 1 \). The conditions of the theorem are then satisfied.

7. Show by induction that \( C^k(\bar{\Omega}) \) is complete wrt the \( C^k \) norm.

We give the proof for \( k = 1 \), since the notation becomes significantly more involved moving from \( C^k \) to \( C^{k+1} \). Since \( C^0(\bar{\Omega}) \) is complete, we know that \( f_n \to f \) uniformly on \( \bar{\Omega} \), and that \( \partial_j f_n \to g_j \) for some \( g_j \in C^0(\bar{\Omega}) \) uniformly on \( \bar{\Omega} \). We just have to show that \( g_j = \partial_j f \).
Since \( f_n \in C^1 \), we know that
\[
f_n(x + he_j) = f_n(x) + \int_0^h \partial_j f_n(x + re_j) dr.
\]
Since the integrand is continuous and converges uniformly, we easily deduce that
\[
f(x + he_j) = f(x) + \int_0^h g_j(x + re_j) dr,
\]
which implies that \( g_j = \partial_j f \) as required.

8*. Show that if \( f \in L^p \), \( g \in L^q \), and \( h \in L^r \), where
\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,
\]
then \( fgh \in L^1 \) with
\[
\int |fgh| \leq \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.
\]

9. Suppose that \( p < r < q \) and
\[
\frac{1}{r} = \frac{\alpha}{p} + \frac{1 - \alpha}{q}.
\]
Show that if \( f \in L^p \) and \( f \in L^q \) then
\[
\|f\|_{L^r} \leq \|f\|_{L^p}\|f\|_{L^q}^{1-\alpha}.
\]
Write
\[
\int |f|^r = \int |f|^{r\alpha}|f|^{r(1-\alpha)}
\]
and use Hölder’s inequality with exponents \( p/r\alpha \) and \( q/r(1-\alpha) \), noting that
\[
\frac{r\alpha}{p} + \frac{r(1-\alpha)}{q} = 1.
\]
Then
\[
\int |f|^r = \left( \int |f|^{p} \right)^{\alpha r/p} \left( \int |f|^q \right)^{(1-\alpha)r/p},
\]
which gives the required inequality.

10. Show that \( L^\infty \) is complete.
If \( \{f_n\} \) is Cauchy in \( L^\infty \) then there exists a measure zero subset \( A \) of \( \Omega \) such that
\[
|f_n(x)| \leq \|f_n\|_\infty \quad \text{and} \quad |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty
\]
for every \( x \notin A \) and every \( n, m \in \mathbb{N} \). It follows that \( f_n \) converges uniformly on \( \Omega \setminus A \) to a bounded function \( f \); setting \( f(x) = 0 \) for every \( x \in A \) we obtain a function \( f \in L^\infty(\Omega) \) such that \( f_n \to f \) in \( L^\infty(\Omega) \).

11. Show that \( W^{k,p} \) is separable for \( 1 \leq p < \infty \).

The map \( W^{k,p} \to \prod_{0 \leq |\alpha| \leq k} L^p \) given by
\[
u \mapsto (\partial^\alpha u)_{|\alpha| \leq k}
\]
is an isometry of \( W^{k,p} \) onto a closed subspace of \( \prod_{0 \leq |\alpha| \leq k} L^p \). The space \( \prod_{0 \leq |\alpha| \leq k} L^p \) is a finite product of separable spaces, and so separable; so are all its closed subspaces.

12. Suppose that \( f \in W^{k,p} \) and that \( |\alpha| + |\beta| \leq k \). Show that \( \partial^\alpha(\partial^\beta f) = \partial^{\alpha+\beta} f \).

Take \( \varphi \in C^\infty_c(\Omega) \); then \( \partial^\alpha \varphi \in C^\infty_c(\Omega) \), so we can write
\[
\int (\partial^\beta f) \partial^\alpha \varphi = (-1)^{|\beta|} \int f \partial^{\alpha+\beta} \varphi
= (-1)^{|\beta|} (-1)^{|\alpha+\beta|} \int (\partial^{\alpha+\beta} f) \varphi
= (-1)^{|\alpha|} \int (\partial^{\alpha+\beta} f) \varphi,
\]
whence \( \partial^{\alpha+\beta} f = \partial^\alpha(\partial^\beta f) \), by definition.

*13. Show that the function \( |x|^{-s} \in W^{1,p}(B(0,1)) \), where \( B(0,1) \) is the unit ball in \( \mathbb{R}^n \), iff \( s < (n-p)/p \). Deduce that if \( r_k \) is a countable dense subset of \( B(0,1) \), and \( s < (n-p)/p \) then the function
\[
f(x) = \sum_{j=1}^{\infty} 2^{-j} |x - r_j|^{-s}
\]
is in \( W^{1,p}(B(0,1)) \), but is unbounded on any open subset of \( B(0,1) \).

14. Use separation of variables to find a non-zero solution of the equation
\[
\sum_{0 \leq |\alpha| \leq 1} (-1)^{|\alpha|} \partial^{2\alpha} f = 0
\]
for a bounded open set \( \Omega \subset [-1, 1]^2 \).

First write the equation explicitly,

\[-f_{xx} - f_{yy} + f = 0.\]

Set \( f(x, y) = X(x)Y(y) \) and then

\[-X_{xx}Y - XY_{xx} + XY = 0 \Rightarrow \frac{X_{xx}}{X} + \frac{Y_{yy}}{Y} = 1,\]

Put

\[X_{xx} = \lambda X \quad \text{and} \quad Y_{yy} = (1 - \lambda)Y;\]

with \( \lambda = 1/2 \) we can take \( X(x) = e^{x/\sqrt{2}} \) and \( Y(y) = e^{y/\sqrt{2}} \); then \( f(x, y) = e^{(x+y)/\sqrt{2}} \) is a non-zero solution of the equation.

**Problems Three.**

We say that a sequence \( \{\phi_n\} \in C_c^\infty(\mathbb{R}^n) \) converges to \( \phi \in C_c^\infty(\mathbb{R}^n) \) if there exists a compact set \( K \) such that \( \text{supp}(\phi_n) \subset K \) for all \( n \), and \( \partial^\alpha \phi_n \to \partial^\alpha \phi \) uniformly on \( K \) for all \( \phi \). A **distribution** is a bounded linear function \( f \) on \( C_c^\infty(\mathbb{R}^n) \) such that whenever \( \phi_n \to \phi \),

\[f(\phi_n) \to f(\phi).\]

We write \( \mathcal{D}'(\mathbb{R}^n) \) for the set of all distributions on \( \mathbb{R}^n \).

1. **Show that if** \( f \in \mathcal{D}'(\mathbb{R}^n) \) **then** \( \partial^\alpha f \) **defined by**

\[[\partial^\alpha f](\phi) = (-1)^{|\alpha|} f(\partial^\alpha \phi)\]

**is also a distribution.**

Suppose that \( \phi_n \in C_c^\infty \) converges to \( \phi \); then \( \partial^\alpha \phi_n \) converges to \( \partial^\alpha \phi \) in \( C_c^\infty \), and so

\[[\partial^\alpha f](\phi_n) = (-1)^{|\alpha|} f(\partial^\alpha \phi_n) \to (-1)^{|\alpha|} f(\partial^\alpha \phi)\]

since \( f \) is a distribution; but by definition the limit is \( [\partial^\alpha f](\phi) \), and so \( \partial^\alpha f \) is a distribution.

2. **Any** \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) **gives rise to a distribution via the definition**

\[L_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x) \, dx.\]
Show that if \( g \) is the weak derivative \( \partial_j f \), then \( \partial_j L f = L g \), i.e. ‘the distribution derivative agrees with the weak derivative when it exists’.

The equality of distributions \( \partial_j L f = L g \) means that for every \( \phi \in C_c^\infty(\Omega) \) we have

\[
(\partial_j L f)(\phi) = -L f(\partial_j \phi) = -\int f(\partial_j \phi) = \int g \phi = L g(\phi).
\]

The bracketed expression says precisely that \( \partial_j f = g \) (we know that weak derivatives are unique).

3. Show that the approximating sequence used in Theorem 2.32 also converges uniformly on \( \bar{\Omega} \) if \( f \in H^k(\Omega) \cap C^0(\bar{\Omega}) \) (use Proposition 2.9).

Note that \( f_n = g_n \mid_\Omega \), where the \( g_n \) are obtained by mollifying a function \( g \in H^k_0(\Omega^\ast) \) that is an extension of \( f \); it follows from Proposition 2.9 that the \( g_n \) converge to \( g \) uniformly on compact subsets of \( \Omega^\ast \), and in particular on \( \bar{\Omega} \). Since \( g = f \) on \( \bar{\Omega} \), it follows that \( f_n \) converges uniformly to \( f \) on \( \bar{\Omega} \), as claimed.

*4. Show that the \( H^k(\mathbb{R}^n) \) norm and the norm

\[
\left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}
\]

are equivalent. [Recall that \( (\partial^\alpha f) = (i\xi)^\alpha \hat{f} \) and that \( \|\hat{f}\|_{L^2} = \|f\|_{L^2} \).]

5. Show that

\[
\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^k} \, d\xi < \infty
\]

if and only if \( k > n/2 \).

We have

\[
\int_{|x| \leq R} \frac{1}{(1 + |\xi|^2)^k} \, d\xi = c \int_0^R \frac{1}{(1 + r^2)^k} \, r^{n-1} \, dr
\]

and since

\[
c' + \frac{1}{2} \int_1^R \frac{1}{(1 + r^2)^k} \, r^{n-1} \, dr < \int_0^R \frac{r^{n-1}}{(1 + r^2)^k} \, dr < c' + \int_1^R \frac{r^{n-1} - 2k}{(1 + r^2)^k} \, dr
\]

the integral is finite iff \( n - 1 - 2k < -1 \), i.e. iff \( k > n/2 \).
6. Show that the unbounded function
\[ f(x) = \log \log \left( 1 + \frac{1}{|x|} \right) \]
is an element of \( H^1(B(0, 1)) \), where \( B(0, 1) \) is the unit ball in \( \mathbb{R}^2 \).

First note that
\[ \int_B \left[ \log \log \left( 1 + \frac{1}{|x|} \right) \right]^2 \, dx \, dy = \int_0^{2\pi} \int_0^1 r \log \log(1 + 1/r) \, dr \, d\theta \]
is finite, since the integrand is bounded.

Away from \( x = 0 \) the derivative of \( f \) is
\[ \partial_i f(x) = g_i(x) = \frac{1}{\log(1 + 1/|x|)} \frac{x_i}{|x|^2(1 + |x|)}, \]
whence
\[ \int_B |g|^2 \, dx \, dy = 2\pi \int_0^1 \frac{1}{\log(1 + 1/r^2)} \frac{1}{r(1 + r)^2} \, dr. \]

With the substitution \( u = 1/r \) the integral becomes
\[ \int_1^\infty \frac{1}{u + (1/u) \log(1 + u^2)} \, du < \int_1^\infty \frac{1}{u(\log u)^2} \, du < \infty. \]

It remains to check that \( g \) really is the weak derivative of \( f \) throughout \( B(0, 1) \): to this end take \( \varphi \in C_c^\infty(B(0, 1)) \), and then away from the singularity at \( x = 0 \) we have
\[ \int_{B(0, 1)\setminus B(0, \epsilon)} f \partial_i \varphi = -\int_{B(0, 1)\setminus B(0, \epsilon)} g_i \varphi + \int_{|x| = \epsilon} f \varphi n_i \, dS, \]
where \( n_i \) is the outward normal on \( \partial B(0, \epsilon) \). The boundary term can be bounded according to
\[ \left| \int_{|x| = \epsilon} f \varphi n_i \, dS \right| \leq \|\varphi\|_{\infty} \int_{|x| = \epsilon} \log \log(1 + \epsilon^{-1}) \, dS \leq C\epsilon \log \log(1 + \epsilon^{-1}), \]
which tends to zero as \( \epsilon \to 0 \), showing that \( g_i \) is indeed the weak derivative \( \partial_i f \).

*7. Show that if \( f \in H^k(\mathbb{R}^n) \) the for any \( 0 < \alpha < \min(k - n/2, 1) \) there exists a constant \( C \) such that
\[ |f(x) - f(y)| \leq C\|f\|_{H^k} |x - y|^\alpha. \]
8. In the proof of the Arzelà-Ascoli Theorem, it is relatively straightforward to find a subsequence such that $f_{n_j}(x_k)$ converges for each $k \in \mathbb{N}$, where the $\{x_k\}$ are points in $K$ such that for every $x \in K$ and $\delta > 0$, there exists an $N$ such that

$$|x - x_k| < \delta \quad \text{for some } k \in \{1, \ldots, N\}.$$ (Can you prove the existence of such a collection of points? And of such a subsequence?) Given this, use the equicontinuity of the $\{f_n\}$ to show that $f_{n_j}$ converges uniformly on $K$.

We give the full proof. For each $n \in \mathbb{N}$ we have

$$K \subset \bigcup_{x \in K} B(x, 2^{-n}).$$

since $K$ is compact, a finite number of these balls still cover $K$, giving a finite set of points such that any $k \in K$ lies within $2^{-n}$ of one of these points. Taking the sequence of these points created with ever larger choices of $n$ gives the set $\{x_k\}$ in the question.

We now perform the famous ‘diagonal argument’. Since $f_n(x_1)$ is a bounded sequence of real numbers, we can take a subsequence of $\{f_n\}$, call it $f_{1,n}$, such that $f_{1,n}(x_1)$ converges. Now observe that $f_{1,n}(x_2)$ is a bounded sequence of real numbers, so we can take a subsequence of $f_{1,n}$, call it $f_{2,n}$, such that $f_{2,n}(x_2)$ converges. Note that $f_{2,n}(x_1)$ is a subsequence of $f_{1,n}(x_1)$; since $f_{1,n}(x_1)$ converges, $f_{2,n}(x_2)$ still converges. Now observe that $f_{2,n}(x_3)$ is a bounded sequence of real numbers... We continue inductively, obtaining nested subsequences $f_{k,n}$ such that

$$f_{k,n}(x_j) \text{ converges as } n \to \infty \text{ for every } j = 1, \ldots, k.$$ The ‘diagonal subsequence’ $f^*_k = f_{k,k}$ is such that $f^*_j(x_j)$ converges for every $j \in \mathbb{N}$.

We can now answer the question as posed. Choose $\epsilon > 0$; since the $f_n$ are equicontinuous, there exists a $\delta > 0$ such that for every $n \in \mathbb{N}$

$$|x - y| < \delta \quad \Rightarrow \quad |f^*_n(x) - f^*_n(y)| < \epsilon.$$ We know that there exists an $M$ such that for every $x \in K$ there is an $x_i$ with $i \leq M$ such that $|x - x_i| < \delta$. Now take $N$ large enough that

$$|f^*_n(x_i) - f^*_m(x_i)| < \epsilon/3 \quad \text{for all } m, n \geq N \text{ and } i = 1, \ldots, M.$$
Then for any $x \in K$ choose an $x_i$ such that $i \leq M$ and $|x - x_i| < \delta$; it follows that

$$|f_n^*(x) - f_m^*(x)|$$

$$\leq |f_n(x) - f_n^*(x_i)| + |f_n^*(x_i) - f_m^*(x_j)| + |f_m^*(x_j) - f_m^*(x)|$$

$$\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

which shows that $\{f_n^*\}$ is Cauchy in the sup norm and hence uniformly convergent on $K$.

9. Suppose that $f, g, h \in H^1(\mathbb{R}^3)$. Show that

$$\left| \int fg \nabla h \, dx \right| \leq c \|f\|_{L^3} \|g\|_{H^1} \|h\|_{H^1}.$$ 

Note that if $f, g, h \in H^1(\mathbb{R}^3)$ then $f, g, h \in L^p(\mathbb{R}^3)$ for any $2 \leq p \leq 6$. So first we use the result of question 8 on Problems 2 with exponents $p = 3$, $q = 6$, and $r = 2$, to give

$$\left| \int fg \nabla h \, dx \right| \leq \|f\|_{L^3} \|g\|_{L^6} \|\nabla h\|_{L^2}.$$

Now use the fact that $\|g\|_{L^6} \leq c \|g\|_{H^1}$ and that $\|\nabla h\|_{L^2} \leq \|h\|_{H^1}$ to give the required inequality.

*10. Reinterpret Poisson’s equation

$$-\Delta u = f \quad u|_{\partial \Omega} = 0$$

as an abstract variational problem

$$(u, \phi)_{H_0^1} = f(\phi) \quad \text{for all } \phi \in H_0^1(\Omega)$$

with $f \in H^{-1}$, and show that given $f \in H^{-1}$ the equation has a unique solution $u \in H_0^1$.

Problems Four.

1. Let $V$ be the subspace of $H^1(\Omega)$ consisting of functions with zero integral over $\Omega$,

$$V = \{f \in H^1(\Omega) : \int_\Omega f = 0 \}.$$ 

Arguing by contradiction, use the fact that $H^1(\Omega)$ is compactly embedded in
$L^2(\Omega)$ to show that there is a constant $C > 0$ such that
$$\|f\|_{L^2} \leq C\|\nabla f\|_{L^2} \quad \text{for all } f \in V.$$  
(You may assume that if $\nabla f = 0$ almost everywhere then $f$ is constant.)

Suppose that the result is not true; then there exists a sequence $f_n \in V$ such that
$$\|f_n\|_{L^2} \geq n\|\nabla f_n\|_{L^2}.$$  
Replacing $f_n$ by $f_n/\|f_n\|_{L^2}$ we obtain a sequence with $\|f_n\|_{L^2} = 1$ such that
$$\|\nabla f_n\|_{L^2} \leq n^{-1}.$$  
In particular $f_n$ is a bounded sequence in $H^1$, and so has a subsequence that converges in $L^2$ to some $g \in H^1$ with $\|g\|_{L^2} = 1$ and $\int g = 0$ (since $V$ is closed in the $L^2$ norm).

However, for any $\phi \in C^\infty_c(\Omega)$ we have
$$\int_\Omega g \partial_j \phi \, dx = \lim_{n \to \infty} \int_\Omega f_n \partial_j \phi \, dx = -\lim_{n \to \infty} \int_\Omega (\partial_j f_n) \phi \, dx = 0$$
since $f_n \to g$ in $L^2$ and $\|\nabla f_n\|_{L^2} \leq 1/n$. It follows that $g$ is constant on $\Omega$, contradiction $\int g = 0$ and $\|g\|_{L^2} = 1$.

2. Examine the argument used to prove Theorem 3.2 and convince yourself that the following is also true: If $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ satisfies $Lu = f$, then for every $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists a constant $c_{\Omega',\Omega''}$ such that
$$\|u\|_{H^2(\Omega')} \leq c_{\Omega'}(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega')}).$$

*3. Suppose that $u \in H^1_0(\Omega)$ is a solution of
$$(\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in H^1_0(\Omega).$$  
Show that if $f \in H^k(\Omega)$ then $u \in H^{k+2}(\Omega')$ for any $\Omega' \subset\subset \Omega$. [Take $w \in C^\infty_c(\Omega)$, which is dense in $H^k_0(\Omega)$, and consider the test function $v = (-1)^{|\alpha|}\partial^\alpha w$, where $|\alpha| = k$. You could do the same analysis with a more general $B(u, v)$ as in the notes, but it takes longer to write down.]

*4. Poisson’s equation with Neumann boundary conditions is: find $u$ such that
$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad \nabla u \cdot n|_{\partial \Omega} = 0,$$
(where \( n \) is the outward normal). By taking the inner product with a test function \( \varphi \in C^\infty(\Omega) \) derive the weak form of the equation: find \( u \in H^1(\Omega) \) such that

\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} fv \, dx \quad \text{for all} \quad v \in H^1(\Omega).
\]

*5. Show that

\[-\Delta u + \lambda u = f \quad \text{in} \quad \Omega \quad \text{and} \quad \nabla u \cdot n|_{\partial \Omega} = 0,
\]

has a unique weak solution \( u \in H^1(\Omega) \) (i.e. solution of the weak form of the equation) \( u \in H^1(\Omega) \) for every \( \lambda > 0 \).

6. Show that

\[-\Delta u = f \quad \nabla u \cdot n|_{\partial \Omega} = 0
\]

has a solution \( u \in H^1(\Omega) \) iff \( \int f = 0 \). [Hint: use question 1]

If \( (\nabla u, \nabla v) = (f, v) \) for every \( v \in H^1(\Omega) \) then one can choose \( v = 1 \) on \( \Omega \) to show that one must have \( \int f = 0 \). In the previous exercise we needed an extra \( \lambda \| u \|^2 \) with \( \lambda > 0 \) to show that \( u \) is coercive; in lectures we used a Poincaré inequality for the case of Dirichlet boundary conditions, and then we didn’t need this additional factor - we can recover something similar here.

Note that if \( \int f = 0 \) then

\[(f, v) = \left( f, v - \int v \right),
\]

since \( \int v \) is a constant and \( (f, 1) = 0 \). Also

\[(\nabla u, \nabla v) = \left( \nabla u, \nabla (v - \int v) \right)
\]

since \( \nabla c = 0 \) for any constant \( c \). The weak form of the equation is therefore equivalent to

\[(\nabla u, \nabla v) = (f, v) \quad \text{for all} \quad v \in V,
\]

where \( V \) is the space from question 1. If we now work in \( V \) rather than \( H^1 \) then we have a Poincaré inequality \( \| u \|_{L^2} \leq \| \nabla u \|_{L^2} \), and we can immediately deduce that for \( u \in V \) we have

\[B(u, u) \geq \| \nabla u \|_{L^2}^2 \geq c\| u \|_{H^1}^2 \]

and \( B \) is coercive. It follows that the equation has a unique solution in \( V \),
and hence at least one solution in $H^1$ (and in fact if $u$ is a solution then $u - c$ is a solution for any $c \in \mathbb{R}$).

7. Show that any compact operator is bounded. (Consider the image of the closed unit ball.)

If $T : X \to Y$ is compact then $TB_X(0,1)$ is compact. Any compact set is bounded, so $TB_X(0,1) \subset B_Y(0,M)$ for some $M$; it follows immediately that $\|T\|_{\mathcal{L}(X,Y)} \leq M$.

8. Check that the Gram–Schmidt process produces an orthonormal set.

Suppose that $\tilde{e}_1, \ldots, \tilde{e}_n$ are orthonormal. Given $e_{n+1}$, we construct

$$\hat{e}_{n+1} = e_{n+1} - \sum_{j=1}^{n} (e_{n+1}, \tilde{e}_j) \tilde{e}_j$$

and

$$\tilde{e}_{n+1} = \frac{\hat{e}_{n+1}}{\|\hat{e}_{n+1}\|}.$$ 

Clearly $\|\tilde{e}_{n+1}\| = 1$; we only have to check that $\tilde{e}_{n+1}$ is orthogonal to $\tilde{e}_1, \ldots, \tilde{e}_n$. Fix $k \in \{1, \ldots, n\}$; then

$$(\hat{e}_{n+1}, \tilde{e}_k) = \frac{1}{\|\tilde{e}_{n+1}\|} \left( e_{n+1} - \sum_{j=1}^{n} (e_{n+1}, \tilde{e}_j) \tilde{e}_j, \tilde{e}_k \right)$$

$$= \frac{1}{\|\tilde{e}_{n+1}\|} [(e_{n+1}, \tilde{e}_k) - (e_{n+1}, \tilde{e}_k)] = 0,$$

since $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ are orthonormal.

9. Show that if $T \in \mathcal{L}(H,H)$ is invertible (i.e. the equation $Tx = y$ has a unique solution for every $y \in H$) then $\text{Ker}(T) = \{0\}$. Show that the reverse implication does not hold.

If $Tx = y$ has a unique solution for any $y \in H$ but $\text{Ker}(T)$ contains a non-zero element $z$ then $T(x + \alpha z) = T x = y$ for any $\alpha \in \mathbb{R}$. So invertibility of $T$ implies that $\text{Ker}(T) = \{0\}$. The problem with the converse is that while $T$ can be such that $Tx = y$ has a unique solution for every $y \in T(H)$, it may be the case that $T(H) \neq H$ even if $\text{Ker}(T) = \{0\}$. One example is the right-shift operator on $\ell^2$:

$$\sigma(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots).$$

Clearly $\text{Ker}(\sigma) = \{0\}$, but $\sigma$ is not onto.

10. Suppose that $f \in L^2(\Omega)$ is such that $(f, v) = 0$ for every $v \in H^1_0(\Omega)$. Show that $f = 0$. 


We know that $C^\infty_c(\Omega)$ is dense in $L^2(\Omega)$; and that $C^\infty_c(\Omega) \subset H^1_0(\Omega)$. So given any $v \in L^2(\Omega)$, write $v = \lim_{n \to \infty} v_n$, where $v_n \in H^1_0(\Omega)$ and the series converges in $L^2(\Omega)$. Then $(f, v) = 0$ for every $v \in L^2(\Omega)$, in particular for $v = f_1$ which shows that $\|f\|^2_{L^2} = 0$ and hence $f = 0$ (in $L^2$, i.e. almost everywhere).

11. Show that two eigenfunctions $u$ and $v$ of our elliptic problem with distinct eigenvalues are orthogonal in $H^1_0(\Omega)$ (and not just in $L^2(\Omega)$).

This is in the notes. ??? JUST LAPLACIAN

12. Take $f \in H^{-1}$ and consider the equation

$$B(u, v) = f(v) \quad \text{for every } v \in H^1_0,$$

where $B : H^1_0 \times H^1_0 \to \mathbb{R}$ is bilinear and bounded ($|B(u, v)| \leq \alpha \|u\|_{H^1} \|v\|_{H^1}$). Define a linear operator $A : H^1_0 \to H^{-1}$ such that this equation can be written as

$$Au = f,$$

an equation in $H^{-1}$. How does this relate to the argument we used in the proof of the Lax–Milgram Lemma?

Note that for each fixed $u \in H^1_0$ the map $g : H^1_0 \to \mathbb{R}$

$$g(v) = B(u, v)$$

is linear and bounded ($|g(v)| \leq [\alpha \|u\|_{H^1}] \|v\|_{H^1}$) it follows that $g \in H^{-1}(\Omega)$. We define $Au = g$; then $A$ is linear and since

$$\|Au\|_{H^{-1}} = \|g\|_{H^{-1}} \leq \alpha \|u\|_{H^1}$$

it follows that $A$ is bounded with $\|A\|_{\mathcal{L}(H^1_0, H^{-1})} \leq \alpha$. We can therefore write the equation as $Au = f$.

In the proof of the Lax–Milgram Lemma we instead viewed the equation in $H^1_0$ using the Riesz Representation Theorem; every element of $H^{-1}$ can be derived from an element of $H^1_0$ via an inner product, so any inequality in $H^*$ can be viewed as a corresponding equality in $H$ (one switches to and fro via the ‘Riesz map’ $u \mapsto (\cdot, u)$ and its inverse).

Problems Five.

*1. We say that $f : (0, T) \to X$ is integrable if the function $(L, f(t))$ :
(0,T) → R is integrable for every L ∈ X*, and there exists a y ∈ X such that

$$\langle L, y \rangle = \int_0^T \langle L, f(t) \rangle \ dt \quad \text{for all} \quad L ∈ X^*;$$

in this case we define

$$\int_0^T f(t) \ dt = y.$$

Show that if X is reflexive and

$$\int_0^T \| f(t) \|_X \ dt < \infty$$

then such a y exists and is well-defined.

2. Show that if f : (0,T) → X is integrable (as in the previous exercise) then

$$\left\| \int_0^T f(t) \ dt \right\|_X \leq \int_0^T \| f(t) \|_X \ dt.$$

Lemma 6.2 shows that given any y ∈ X there exists an L ∈ X* such that \(\| L \|_{X^*} = 1\) and \(\langle L, y \rangle = \| y \|_X\). Using the definition of the integral from the previous exercise it follows that

$$\| y \|_X = \langle L, y \rangle = \int_0^T \langle L, f(t) \rangle \ dt \leq \int_0^T \| f(t) \|_X \ dt$$

since \(\| L \|_{X^*} = 1\).

3. Show that if X is complete and reflexive then \(L^2(0,T;X)\) is complete. [Hint: take a sequence \(\{f_n\} \in L^2(0,T;X)\), and for each \(L \in X^*\) show that \(\langle L, f_n(t) \rangle \in L^2(0,T)\) is Cauchy in \(L^2\). A Cauchy sequence has a subsequence that converges almost everywhere; follow an argument similar to that of question 1, i.e. working in \(X^{**}\), to obtain a candidate limit \(f \in L^2(0,T;X)\); finally show that \(f_n \to f\).]

First, observe that \(\langle L, f_n(t) \rangle \in L^2(0,T)\), since

$$\int_0^T |\langle L, f_n(t) \rangle|^2 \ dt \leq \int_0^T \| L \|_{X^*}^2 \| f_n(t) \|_X^2 \ dt = \| L \|_{X^*}^2 \int_0^T \| f_n(t) \|_X^2 \ dt.$$  

The same argument shows that \(\langle L, f_n(t) \rangle\) is a Cauchy sequence in \(L^2(0,T)\).
By completeness of $L^2(0,T)$, it follows that there is a function $f_L(\cdot) \in L^2(0,T)$ such that

$$\langle L, f_n \rangle \to f_L$$

in $L^2(0,T)$. It follows (Corollary 2.15) that there exists a subsequence $f_{n_j}$ such that $\langle L, f_{n_j}(t) \rangle$ converges to $f_L(t)$ for almost every $t$. At these $t$ define a linear map $I(t) : X^* \to \mathbb{R}$ by

$$\langle I(t), L \rangle = f_L(t) = \lim_{j \to \infty} \langle L, f_{n_j}(t) \rangle.$$

Then $I \in L^2(0,T; X^{**})$, since for almost every $t$ we have

$$| \langle I(t), L \rangle | = | \lim_{j \to \infty} | \langle L, f_{n_j}(t) \rangle | | \leq \| L \| \| f_{n_j}(t) \|_X,$$

i.e.

$$\| I(t) \|_{X^{**}} \leq \lim_{j \to \infty} \| f_{n_j}(t) \|_X,$$

and so

$$\int_0^T \| I(t) \|^2_{X^{**}} \, dt \leq \int_0^T \lim_{j \to \infty} \| f_{n_j}(t) \|^2_X \, dt \leq M$$

since $\{f_n\}$ is Cauchy in $L^2(0,T;X)$. Since $X$ is reflexive, we can find an $f(t) \in X$ such that $\langle I(t), L \rangle = \langle L, f(t) \rangle$. Since $\| f(t) \|_X = \| I(t) \|_{X^{**}}$, it follows that $f \in L^2(0,T;X)$. Since $\langle I(t), L \rangle = f_L(t)$ by definition, it follows that

$$\langle L, f_n \rangle \to \langle L, f \rangle$$

in $L^2(0,T)$, i.e. $\langle L, f_n - f \rangle \to 0$ in $L^2(0,T)$.

I can’t finish the proof at the moment...

4. Show that if $X$ is a Hilbert space then $L^2(0,T;X)$ is a Hilbert space.

We have completeness from question 3. The norm on $L^2(0,T;X)$ can be derived from the inner product

$$(f,g) = \int_0^T (f(t), g(t))_X \, dt,$$

which shows that $L^2(0,T;X)$ is a Hilbert space. (It is easy to check that this is an inner product.)

5. Show that if $f \in C^0([0,T])$ and $\dot{f} \in L^1(0,T)$ is the weak time derivative
of $f$, then for any $\varphi \in C^1([0,T]) \cap C^\infty(0,T)$

$$
\int_0^T \dot{f}(t)\varphi(t) \, dt = - \int_0^T f(t)\dot{\varphi}(t) \, dt + f(T)\varphi(T) - f(0)\varphi(0).
$$

(This is what lies behind the argument about the initial condition being attained when using weak convergence methods; cf. our Lemma 5.2.)

First note that if $f$ has weak derivative $\dot{f}$ and $g \in C^1([0,T])$ then $fg$ has weak derivative $f\dot{g} + \dot{f}g$, since for any $\varphi \in C^\infty_c(0,T)$

$$
\int (f\dot{g} + \dot{f}g)\varphi = \int (f\dot{g})\varphi + \dot{f}(g\varphi)
= \int (f\dot{g})\varphi - f(\dot{g}\varphi) - fg\dot{\varphi} = - \int fg\dot{\varphi},
$$

where we have used the fact that $g\varphi \in C^\infty_c(0,T)$ (one could remove the requirement that $\varphi \in C^\infty$ by approximation). We can therefore apply the argument of Lemma 5.2 with $X = \mathbb{R}$ (and using $L^1$ rather than $L^2$) to $fg$, since now $fg \in L^1(0,T)$ and the weak derivative of $fg$ is also in $L^1(0,T)$, to deduce that the above equality holds as stated.

6. Suppose that $f \in L^2$ and that $g \in H^1_0$. Show that $f \in H^{-1}$, in the sense that

$$
F(g) := (f, g)_{L^2}
$$
defines an element $F \in H^{-1}$. (This is what it means that say that $F \in H^{-1}$ is also in $L^2$.)

We have

$$
|F(g)| = |(f, g)_{L^2}| \leq \|f\|_{L^2} \|g\|_{L^2} \leq \|f\|_{L^2} \|g\|_{H^1},
$$

so $F$ is a bounded linear functional on $H^1_0$.

7. Given $f \in H^{-1}$, extend $f$ to a linear functional $F \in (H^1)^*$, such that $F(x) = f(x)$ for all $x \in H^1_0$ and $\|F\|_{(H^1)^*} = \|f\|_{H^{-1}}$. [Hint: use the Riesz Representation Theorem. This allows one to extend the proof of Lemma 5.3 to the case $f \in L^2(0,T;H^1)$, $\dot{f} \in L^2(0,T;H^{-1})$, which is useful for higher regularity of solutions.]

Given $f \in H^{-1}$, the Riesz Representation Theorem guarantees that there exists a $u \in H^1_0$ such that

$$
f(v) = (u, v)_{H^1} \quad \text{for all} \quad v \in H^1_0 \quad \text{and} \quad \|f\|_{H^{-1}} = \|u\|_{H^1}.
$$
If we define a linear map $F : H^1 \to \mathbb{R}$ by

$$F(v) = (u, v)_{H^1}$$

with the $u$ above the clearly this agrees with $f$ for $v \in H^1_0$; it is linear, and it is bounded on $H^1$ since

$$|F(v)| \leq \|u\|_{H^1} \|v\|_{H^1}.$$ 

It follows that $\|F\|_{(H^1)^*} \leq \|f\|_{H^{-1}}$, and since $H^1_0 \subset H^1$, we have equality here.

8. Suppose that $f \in L^2(0, T; \mathbb{H}^1_0)$ and that $\dot{f} \in L^2(0, T; H^{-1})$. If $u_0 \in H^1_0 \cap H^2$ show that $u \in C^1([0, T]; H^1)$. [Hint: consider the equation for $v = \dot{u}$, and use the Sobolev embedding $H^2(0, T; H^{-1}) \subset C^1([0, T]; H^{-1})$ (we have not explicitly proved such a result). Lemma 5.3 implies that $f \in C^0([0, T]; L^2)$. Consider the equation for $v = \dot{u}$, which is

$$\dot{v} - \Delta v = \dot{f}(t), \quad v(0) = \Delta u_0 + f(0).$$

This is an equation for $v$ in our standard parabolic form, since our assumptions mean that $\dot{f} \in L^2(0, T; H^{-1})$ and $v(0) \in L^2$. It follows that the equation has a unique solution $v \in L^2(0, T; H^1)$ and $\dot{v} \in L^2(0, T; H^{-1})$, i.e. $v \in H^1(0, T; H^{-1})$. Since the solutions are unique, we must have $v = \dot{u}$, and hence $\dot{u} \in L^2(0, T; H^1) \cap H^1(0, T; H^{-1})$. It follows that $u$ itself is an element of $H^2(0, T; H^{-1})$, and hence of $C^1([0, T]; H^{-1})$.

9. Prove the Hahn-Banach theorem in a Hilbert space. [Show first that $f$ extends to the closure of $U$ if $U$ is not closed; then consider $U$ and $(U)^\perp$.]

If $f$ is a bounded linear functional on $U$ and $x \in U$, one can write $x = \lim_{n \to \infty} x_n$, where $x_n \in U$. One can define

$$f(x) = \lim_{n \to \infty} f(x_n).$$

The limit exists since $f$ is bounded and linear, and is well-defined. Clearly

$$|f(x)| \leq \lim_{n \to \infty} \|f\| \|x_n\| \leq \|f\| \|x\|$$

so the norm of $f$ is not increased. So assume that $U$ is closed. In this case one can decompose any $x \in H$ as $x = u + v$, where $v \in U^\perp$. For any $x \in H$ define

$$F(x) = f(u).$$
Then $F$ is an extension of $f$, and $|F(x)| = |f(u)| \leq \|f\|_u \leq \|f\|_x$ since $\|u\| \leq \|x\|$.

*10. Suppose that $x$ and $y$ are linearly independent elements of $X$. Show that there exists a constant $c$ such that

$$|\alpha x + \beta y| \leq \|\alpha x + \beta y\|$$

for every $\alpha, \beta \in \mathbb{R}$. [If you can prove this you can prove the Hahn–Banach Theorem... more or less.]

*11. It is relatively easy to show that any element $\phi$ of the sequence space $\ell^\infty$ gives rise to a linear functional on $\ell^1$ via

$$L_\phi(x) = \sum_{k=1}^{\infty} \phi_k x_k$$

with $\|L_\phi\|_{(\ell^1)^*} = \|\phi\|_{\ell^\infty}$. Show that any $L \in (\ell^1)^*$ can be written as $L_\phi$ for some $\phi \in \ell^\infty$, and hence that $(\ell^1)^* = \ell^\infty$. [Hint: let $\{e_k\}_{k=1}^{\infty}$ be the basis for $\ell^1$ where $e_k$ consists of all zeros apart from a 1 in the $k^{th}$ position and write $x = \sum_{k=1}^{\infty} x_k e_k$. What is $L_\sum e_k$?]

12. Show that weak convergence in $X^*$ implies weak-* convergence; and that if $X^*$ is reflexive then weak-* convergence implies weak convergence.

In general we know that any element of $X$ gives rise to an element of $X^{**}$ via

$$G_x(f) = f(x) \quad \text{for all} \quad f \in X^*.$$ 

So if $f_n \rightharpoonup f$ in $X^*$, i.e. $G(f_n) \to G(f)$ for every $G \in (X^*)^* = X^{**}$, then in particular for any $x \in X$ this holds for $G = G_x$, so that

$$G_x(f_n) = f_n(x) \to G_x(f) = f(x) \quad \text{for all} \quad x \in X,$$

i.e. $f_n \rightharpoonup f$ in $X^*$.

If $X$ is reflexive then any element $G \in X^{**}$ can be written as $G_x$ for some $x \in X$; so given $G \in X^{**}$ we have

$$G(f_n) = G_x(f_n) = f_n(x) \to f(x) = G_x(f) = G(f),$$

which is weak convergence in $X^*$. 