MASDOC A1
Linear Partial Differential Equations
Problems & Solutions

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Problems One.

1. Let $\{x_n\}$ be a Cauchy sequence in a normed space $(X, \|\cdot\|)$. Show that if x_n has a subsequence that converges to some $x \in X$, then $x_n \to x$.

Given $\epsilon > 0$ choose N such that $\|x_n - x_m\| < \epsilon/2$ for all $n, m \geq N$. Then choose J such that $\|x_{n_j} - x\| < \epsilon/2$ for all $j \geq J$. Then for all $n \geq n_J$,

$$\|x_n - x\| \leq \|x_n - x_{n_J}\| + \|x_{n_J} - x\| < \epsilon$$

and so $x_n \to x$.

2. Prove the polarisation identity

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + \mathrm{i}\|x + \mathrm{i}y\|^2 - \mathrm{i}\|x - \mathrm{i}y\|^2$$

in a complex inner product space.

Expand the right-hand side as inner products:

$$\begin{aligned}
(x+y, x+y) - (x-y, x-y) + \mathrm{i}(x+\mathrm{i}y, x+\mathrm{i}y) - \mathrm{i}(x-\mathrm{i}y, x-\mathrm{i}y)& \\
= \|x\| + 2 + (x, y) + (y, x) + \|y\|^2 - (\|x\|^2 - (x, y) - (y, x) + \|y\|^2)& \\
+ \mathrm{i}(\|x\|^2 + \mathrm{i}(y, x) - \mathrm{i}(x, y) + \|y\|^2) - \mathrm{i}(\|x\|^2 - \mathrm{i}(y, x) + \mathrm{i}(x, y) + \|y\|^2)& \\
= 4(x, y).&
\end{aligned}$$

**3. Suppose that $x_n \to x$ and $y_n \to y$ in H. Show that*

$$(x_n, y_n) \to (x, y).$$

4. Consider $C^0([0, 1])$. Show that neither the supremum norm

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

nor the L^1 norm

$$\|f\|_{L^1} = \int_0^1 |f(x)| \, \mathrm{d}x$$

can be derived from an inner product.

We have to find functions f and g for which the parallelogram law is violated. For the sup norm, consider $f(x) = |x - (1/2)|$ and $g(x) = (1/2) - f(x)$. Then $\|f\|_\infty = \|g\|_\infty = 1/2$, $\|f + g\|_\infty = 1$, and $\|f - g\|_\infty = 1/2$. So

$$\frac{5}{4} = \|f + g\|^2 + \|f - g\|^2 \neq 2(\|f\|^2 + \|g\|^2) = \frac{1}{2}.$$

For the L^1 norm we can use the same example, since $\|f\|_{L^1} = \|g\|_{L^1} = \|f-g\|_{L^1} = 1/4$ and $\|f+g\|_{L^1} = 1/2$.

5. Show that $U^\perp$ is always a closed linear subspace of H.

If $x_n \in U^\perp$ with $x_n \to x$ then, using question 3, for any $u \in U$

$$(x, u) = \lim_{n\to\infty} (x_n, u) = 0,$$

i.e. $x \in U^\perp$.

**6. Show that if U is a closed linear subspace of H then $(U^\perp)^\perp = U$.*

7. Show that if A is closed an $\frac{1}{2}(x+y) \in A$ whenever $x, y \in A$ then A is convex.

Take $x, y \in A$ and suppose that $\lambda x + (1-\lambda)y \in A$ for every $\lambda \in [0,1]$ such that $\lambda = a2^{-k}$, where $a \in \mathbb{N}$. Then given $\lambda = b2^{-(k+1)}$ with $b \in \mathbb{N}$ and b odd, clearly one has

$$\lambda = \frac{1}{2}(b-1)2^{-(k+1)} + \frac{1}{2}(b+1)2^{-(k+1)}.$$

It follows that $\lambda x + (1-\lambda)y \in A$ for every $\lambda = b2^{-(k+1)}$, $b \in \mathbb{N}$. Now simply use the fact that for any $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y$ can be approximated by elements of the form $\lambda_n x + (1-\lambda_n)y$, where $\lambda_n = a_n 2^{-n}$; since A is closed, it follows that $\lambda x + (1-\lambda)y \in A$.

8. A set $\{e_j\}_{j=1}^n$ is orthonormal *if*

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Show that an orthonormal set is linearly independent.

If $\sum_{j=1}^n \alpha_j e_j = 0$ then taking the inner product with e_k $(k = 1, \ldots, n)$ shows that $\alpha_k = 0$. Since this is true for each k, the set is linearly independent.

9. (In this question life is much easier if you take H to be real.) The space E spanned by $\{e_j\}_{j=1}^n$ consists of all elements of H of the form

$$\sum_{j=1}^n \alpha_j e_j \qquad \alpha_j \in \mathbb{K}.$$

Show that the closest point to x in E is given by

$$\hat{x} = \sum_{j=1}^{n} (x, e_j)e_j.$$

(You should try to do this (i) directly and (ii) using the fact that $x - \hat{x} \in E^\perp$ – (ii) will be much easier.)

(i) Directly: consider

$$\begin{aligned}\|x - \sum_{j=1}^{n} \alpha_j e_j\|^2 &= (x - \sum_{j=1}^{n} \alpha_j e_j, x - \sum_{k=1}^{n} \alpha_k e_k) \\ &= \|x\|^2 - 2(x, \sum_{j=1}^{n} \alpha_j e_j) + \sum_{j=1}^{n} |\alpha_j|^2 \\ &= \|x\|^2 - \sum_{j=1}^{n} |(x, e_j)|^2 + \sum_{j=1}^{n} |(x, e_j)|^2 - 2\alpha_j (x, e_j) + |\alpha_j|^2 \\ &= \|x\|^2 - \sum_{j=1}^{n} |(x, e_j)|^2 + \sum_{j=1}^{n} [(x, e_j) - \alpha_j]^2.\end{aligned}$$

This expression is clearly minimised when $\alpha_j = (x, e_j)$.

(ii) If $\hat{x} = \sum_{j=1}^{n} \alpha_j e_j$ and $x - \hat{x} \in E^\perp$ then for each $k = 1, \ldots, n$ we must have

$$(x - \sum_{j=1}^{n} \alpha_j e_j, e_k) = (x, e_k) - \alpha_k = 0,$$

i.e. $\alpha_k = (x, e_k)$.

10. Let X and Y be two normed spaces. Show that

$$\|L\|_{\mathscr{L}(X,Y)} = \sup_{x \in X : \|x\|_X = 1} \|Lx\|_Y.$$

Denote M the right-hand side by M, and note that for any $x \in X$,

$$\|Lx\|_Y = \left\| L \frac{x}{\|x\|_X} \right\|_Y \|x\|_X \leq M \|x\|_X,$$

so that $\|L\|_{\mathscr{L}(X,Y)} \leq M$. Since $\|Lx\|_Y \leq \|L\|_{\mathscr{L}(X,Y)} \|x\|_X$ for any $x \in X$, it is clear that $M \leq \|L\|_{\mathscr{L}(X,Y)}$.

11. Show that the map $T : \ell^2 \to \ell^2$ defined by

$$T\underline{x} = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$$

is bounded, but that $R(T)$, the range of T, is not closed. [Recall that $R(T) =$ "$T(\ell^2)$"$= \{\underline{y} \in \ell^2 : \ \underline{y} = T\underline{x}$ for some $x \in \ell^2\}$.

Clearly

$$\|T\underline{x}\|_{\ell^2}^2 = \sum_{j=1}^{\infty} \frac{|x_j|^2}{j^2} \le \sum_{j=1}^{\infty} |x_j|^2 = \|\underline{x}\|_{\ell^2}^2,$$

and so T is bounded. However, consider $\underline{x_n} = (1, 1, \ldots, 1, 0, 0, \ldots)$, i.e. n 1s followed by zeros. Then $\underline{x_n} \in \ell^2$, and so is $T\underline{x_n} = (1, 1/2, \ldots, 1/n, 0, 0, 0, \ldots)$. But while $T\underline{x_n} \to \underline{z}$ with $z_j = 1/j$, there is no $\underline{x} \in \ell^2$ with $T\underline{x} = \underline{z}$, since $(1, 1, 1, 1, \cdots) \notin \ell^2$.

**12. Suppose that $A : H \to H$ is a bounded linear operator. Show that there exists a unique $A^* : H \to H$, also a bounded linear operator, such that*

$$(Au, v) = (u, A^*v) \qquad \textit{for every} \quad u, v \in H.$$

The operator A^* is known as the (Hilbert) adjoint of A.

13. Prove the contraction mapping theorem in a Hilbert space: if $f : H \to H$ is a map such that

$$\|f(u) - f(v)\| \le \theta \|u - v\| \qquad \textit{for all} \qquad u, v \in H$$

for some $\theta < 1$ then there exists a unique $u \in H$ such that $f(u) = u$.

Note that

$$\begin{aligned}\|f^{n+1}(x) - f^n(x)\| &\le \theta \|f^n(x) - f^{n-1}(x)\| \\ &\le \theta^n \|f(x) - x\|,\end{aligned}$$

and so if $m > n$

$$\begin{aligned}\|f^m(x) - f^n(x)\| &\le \sum_{j=n}^{m-1} \theta^j \|f(x) - x\| \\ &\le \left(\sum_{j=n}^{\infty} \theta^j\right) \|f(x) - x\| \\ &= \frac{\theta^n}{1-\theta} \|f(x) - x\|.\end{aligned}$$

It follows that $\{f^n(x)\}$ is Cauchy; since H is complete, $f^n(x) \to x^*$ as $n \to \infty$. Then

$$f(x^*) = f\left(\lim_{n\to\infty} f^n(x)\right) = \lim_{n\to\infty} f^{n+1}(x) = x^*,$$

so $f(x^*) = x^*$ as required. If x^* and y^* are both fixed points of f then

$$\|x^* - y^*\| = \|f(x^*) - f(y^*)\| \le \theta\|x^* - y^*\|$$

which is impossible unless $x^* = y^*$.

Problems Two.

*****1. *Show that ℓ^p, $1 \le p < \infty$, is complete.*

2. *Suppose that $(X, \|\cdot\|)$ is a complete normed space, and that A is a subspace of X. Show that the closure of A in X, equipped with the norm $\|\cdot\|$, is a complete normed space. [One has $x \in \overline{A}$ iff $x = \lim_{n\to} a_n$, with $a_n \in A$.]*

We need A to be a subspace of X (a subset will not do). Take a Cauchy sequence $\{x_n\} \in A$. Then since this is also a Cauchy sequence in X it has converges; since A is closed the limit must lie in A, and hence A is complete.

3. *Show that c_0 (the subspace of ℓ^∞ consisting of all sequences that converge to zero) equipped with the ℓ^∞ norm is separable.*

We show that the set of all sequences with only finitely many non-zero components, all of which are rational, is dense (this set is countable). Given $\underline{x} \in c^0$, there exists an N such that $|x_j| < \epsilon/2$ for every $j \ge N$. To approximate $\underline{x}$ to within ϵ we therefore only need approximate the first N components to within $\epsilon/2$. Clearly we can do this using rationals (choose $q_j \in \mathbb{Q}$ such that $|q_j - x_j| < \epsilon/2N$ for $j = 1, \ldots, N$).

4. *Show that the product of a finite number of separable spaces is separable; and that a closed linear subspace of a separable space is separable.*

If $\{x_j\}$ is a countable dense subset of X and $\{y_j\}$ is a countable dense subset of Y then clearly the set $\{(x_j, y_k\}$ is a countable dense subset of $X \times Y$, and the result as stated follows by induction.

Now, if M is a closed linear subspace of X, start with a countable dense subset $\{x_j\}$ of X. Fix $\epsilon > 0$, and for each x_j for which $B(x_j, \epsilon/2) \cap M \neq 0$, choose an element $m_j \in M$. Then any $m \in M$ lies in one of the $B(x_j, \epsilon/2)$;

since m_j does too, it follows that $m \in B(m_j, \epsilon)$. Apply this construction for $\epsilon = 2^{-k}$ for each $k \in \mathbb{N}$; the union of the resulting m_js provides a countable dense subset of M.

5. Suppose that $\{f_n\} \in C_c^0(\Omega)$, and that $\{f_n\}$ is Cauchy in the sup norm. Show that $f_n \to f$, where $f \in C^0(\bar{\Omega})$ with $f = 0$ on $\partial\Omega$. Show further that any such f can be obtained as the uniform limit of functions in $C_c^0(\Omega)$. [This shows that $C_0^0(\bar{\Omega})$ – continuous functions on $\bar{\Omega}$ that are zero on the boundary – is the completion *of $C_c^0(\Omega)$ wrt the supremum norm.]*

First, clearly $f_n \in C^0(\bar{\Omega})$, and since $f_n \to f$ uniformly on Ω, $f \in C^0(\bar{\Omega})$. Furthermore if $x \in \partial\Omega$ then $f_n(x) = 0$ for every n, whence $f = 0$ on $\partial\Omega$.

Now, suppose that $f \in C^0(\bar{\Omega})$ with $f = 0$ on $\partial\Omega$. Let

$$f_\epsilon(x) = \begin{cases} f(x) & |f(x)| > 2\epsilon \\ 0 & |f(x)| < \epsilon \\ f(x)\left[\frac{|f(x)|}{\epsilon} - 1\right] & \epsilon < |f(x)| < 2\epsilon. \end{cases}$$

Then f_ϵ converges uniformly to f on Ω, and since f is uniformly continuous on Ω, given any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x)| < \epsilon$ whenever $\mathrm{dist}(x, \partial\Omega) < \delta$, so that f_ϵ has compact support.

6. The Stone–Weierstrass Theorem states that if K is a compact space and A is an algebra of real-valued continuous functions on K such that for all $x, y \in K$ with $x \neq y$, there exists an $f \in A$ such that $f(x) \neq f(y)$, and for every $x \in K$ there exists an $f \in A$ such that $f(x) \neq 0$, then A is dense in $C(K; \mathbb{R})$. [An algebra is a set on which one can define multiplication and addition.] Deduce that the space of polynomials is dense in $C^k(\bar{\Omega})$ for any $k \geq 0$, where Ω is a bounded subset of $\mathbb{R}^n$.

The space of polynomials is an algebra on $\bar{\Omega}$. Given $x, y \in \bar{\Omega}$ with $x \neq y$, they must have at least one component, say the jth, distinct. So one can consider the polynomial x_j. For every $x \in \bar{\Omega}$ we can consider the polynomial $f \equiv 1$. The conditions of the theorem are then satisfied.

7. Show by induction that $C^k(\bar{\Omega})$ is complete wrt the C^k norm.

We give the proof for $k = 1$, since the notation becomes significantly more involved moving from C^k to C^{k+1}. Since $C^0(\bar{\Omega})$ is complete, we know that $f_n \to f$ uniformly on $\bar{\Omega}$, and that $\partial_j f_n \to g_j$ for some $g_j \in C^0(\bar{\Omega})$ uniformly on $\bar{\Omega}$. We just have to show that $g_j = \partial_j f$.

Since $f_n \in C^1$, we know that

$$f_n(x + he_j) = f_n(x) + \int_0^h \partial_j f_n(x + re_j)\,\mathrm{d}r.$$

Since the integrand is continuous and converges uniformly, we easily deduce that

$$f(x + he_j) = f(x) + \int_0^h g_j(x + re_j)\,\mathrm{d}r,$$

which implies that $g_j = \partial_j f$ as required.

8. Show that if $f \in L^p$, $g \in L^q$, and $h \in L^r$, where*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

then $fgh \in L^1$ with

$$\int |fgh| \le \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.$$

9. Suppose that $p < r < q$ and

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

Show that if $f \in L^p$ and $f \in L^q$ then

$$\|f\|_{L^r} \le \|f\|_{L^p}^{\alpha}\|f\|_{L^q}^{1-\alpha}.$$

Write

$$\int |f|^r = \int |f|^{r\alpha}|f|^{r(1-\alpha)}$$

and us Hölder's inequality with exponents $p/r\alpha$ and $q/r(1-\alpha)$, noting that

$$\frac{r\alpha}{p} + \frac{r(1-\alpha)}{q} = 1.$$

Then

$$\int |f|^r = \left(\int |f|^p\right)^{\alpha r/p}\left(\int |f|^q\right)^{(1-\alpha)r/p},$$

which gives the required inequality.

10. Show that L^∞ is complete.

If $\{f_n\}$ is Cauchy in L^∞ then there exists a measure zero subset A of Ω such that

$$|f_n(x)| \le \|f_n\|_\infty \qquad \text{and} \qquad |f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$$

for every $x \notin A$ and every $n, m \in \mathbb{N}$. It follows that f_n converges uniformly on $\Omega \setminus A$ to a bounded function f; setting $f(x) = 0$ for every $x \in A$ we obtain a function $f \in L^\infty(\Omega)$ such that $f_n \to f$ in $L^\infty(\Omega)$.

11. Show that $W^{k,p}$ is separable for $1 \le p < \infty$.

The map $W^{k,p} \to \prod_{0\le|\alpha|\le k} L^p$ given by

$$u \mapsto (\partial^\alpha u)_{|\alpha|\le k}$$

is an isometry of $W^{k,p}$ onto a closed subspace of $\prod_{0\le|\alpha|\le k} L^p$. The space $\prod_{0\le|\alpha|\le k} L^p$ is a finite product of separable spaces, and so separable; so are all its closed subspaces.

12. Suppose that $f \in W^{k,p}$ and that $|\alpha| + |\beta| \le k$. Show that $\partial^\alpha(\partial^\beta f) = \partial^{\alpha+\beta} f$.

Take $\varphi \in C_c^\infty(\Omega)$; then $\partial^\alpha \varphi \in C_c^\infty(\Omega)$, so we can write

$$\begin{aligned}\int (\partial^\beta f)\, \partial^\alpha \varphi &= (-1)^{|\beta|} \int f\, \partial^{\alpha+\beta} \varphi \\ &= (-1)^{|\beta|}(-1)^{|\alpha+\beta|} \int (\partial^{\alpha+\beta} f)\, \varphi \\ &= (-1)^{|\alpha|} \int (\partial^{\alpha+\beta} f)\, \varphi,\end{aligned}$$

whence $\partial^{\alpha+\beta} f = \partial^\alpha(\partial^\beta f)$, by definition.

**13. Show that the function $|x|^{-s} \in W^{1,p}(B(0,1))$, where $B(0,1)$ is the unit ball in $\mathbb{R}^n$, iff $s < (n-p)/p$. Deduce that if r_k is a countable dense subset of $B(0,1)$, and $s < (n-p)/p$ then the function*

$$f(x) = \sum_{j=1}^\infty 2^{-j} |x - r_j|^{-s}$$

is in $W^{1,p}(B(0,1))$, but is unbounded on any open subset of $B(0,1)$.

14. Use separation of variables to find a non-zero solution of the equation

$$\sum_{0\le|\alpha|\le 1} (-1)^{|\alpha|} \partial^{2\alpha} f = 0$$

for a bounded open set $\Omega \subset [-1,1]^2$.

First write the equation explicitly,

$$-f_{xx} - f_{yy} + f = 0.$$

Set $f(x,y) = X(x)Y(y)$ and then

$$-X_{xx}Y - XY_{xx} + XY = 0 \qquad \Rightarrow \qquad \frac{X_{xx}}{X} + \frac{Y_{yy}}{Y} = 1,$$

Put

$$X_{xx} = \lambda X \qquad \text{and} \qquad Y_{yy} = (1-\lambda)Y;$$

with $\lambda = 1/2$ we can take $X(x) = \mathrm{e}^{x/\sqrt{2}}$ and $Y(y) = \mathrm{e}^{y/\sqrt{2}}$; then $f(x,y) = \mathrm{e}^{(x+y)/\sqrt{2}}$ is a non-zero solution of the equation.

Problems Three.

We say that a sequence $\{\phi_n\} \in C_c^\infty(\mathbb{R}^n)$ converges to $\phi \in C_c^\infty(\mathbb{R}^n)$ if there exists a compact set K such that $\mathrm{supp}(\phi_n) \subset K$ for all n, and $\partial^\alpha \phi_n \to \partial^\alpha \phi$ uniformly on K for all ϕ. A *distribution* is a bounded linear function f on $C_c^\infty(\mathbb{R}^n)$ such that whenever $\phi_n \to \phi$,

$$f(\phi_n) \to f(\phi).$$

We write $\mathscr{D}'(\mathbb{R}^n)$ for the set of all distributions on $\mathbb{R}^n$.

1. Show that if $f \in \mathscr{D}'(\mathbb{R}^n)$ *then* $\partial^\alpha f$ *defined by*

$$[\partial^\alpha f](\phi) = (-1)^{|\alpha|} f(\partial^\alpha \phi)$$

is also a distribution.

Suppose that $\phi_n \in C_c^\infty$ converges to ϕ; then $\partial^\alpha \phi_n$ converges to $\partial^\alpha \phi$ in C_c^∞, and so

$$[\partial^\alpha f](\phi_n) = (-1)^{|\alpha|} f(\partial^\alpha \phi_n) \to (-1)^{|\alpha|} f(\partial^\alpha \phi)$$

since f is a distribution; but by definition the limit is $[\partial^\alpha f](\phi)$, and so $\partial^\alpha f$ is a distribution.

2. Any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *gives rise to a distribution via the definition*

$$L_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)\,\mathrm{d}x.$$

Show that if g is the weak derivative $\partial_j f$, then $\partial_j L_f = L_g$, i.e. 'the distribution derivative agrees with the weak derivative when it exists'.

The equality of distributions $\partial_j L_f = L_g$ means that for every $\phi \in C_c^\infty(\Omega)$ we have

$$(\partial_j L_f)(\phi) = -L_f(\partial_j \phi) = \underbrace{-\int f(\partial_j \phi) = \int g\phi} = L_g(\phi).$$

The bracketed expression says precisely that $\partial_j f = g$ (we know that weak derivatives are unique).

3. Show that the approximating sequence used in Theorem 2.32 also converges uniformly on $\bar{\Omega}$ if $f \in H^k(\Omega) \cap C^0(\bar{\Omega})$ (use Proposition 2.9).

Note that $f_n = g_n|_\Omega$, where the g_n are obtained by mollifying a function $g \in H_0^k(\Omega^*)$ that is an extension of f; it follows from Proposition 2.9 that the g_n converge to g uniformly on compact subsets of Ω^*, and in particular on $\bar{\Omega}$. Since $g = f$ on $\bar{\Omega}$, it follows that f_n converges uniformly to f on $\bar{\Omega}$, as claimed.

**4. Show that the $H^k(\mathbb{R}^n)$ norm and the norm*

$$\left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right)^{1/2}$$

are equivalent. [Recall that $(\partial^\alpha f)\hat{} = (\mathrm{i}\xi)^\alpha \hat{f}$ and that $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$.]

5. Show that

$$\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^k} \, \mathrm{d}\xi < \infty$$

if and only if $k > n/2$.

We have

$$\int_{|x| \le R} \frac{1}{(1 + |\xi|^2)^k} \, \mathrm{d}\xi = c \int_0^R \frac{1}{(1 + r^2)^k} \, r^{n-1} \, \mathrm{d}r$$

and since

$$c' + \frac{1}{2} \int_1^R r^{n-1-2k} \, \mathrm{d}r < \int_0^R \frac{r^{n-1}}{(1 + r^2)^k} \, \mathrm{d}r < c' + \int_1^R r^{n-1-2k}$$

the integral is finite iff $n - 1 - 2k < -1$, i.e. iff $k > n/2$.

6. Show that the unbounded function

$$f(x) = \log\log\left(1 + \frac{1}{|x|}\right)$$

is an element of $H^1(B(0,1))$*, where* $B(0,1)$ *is the unit ball in* $\mathbb{R}^2$*.*

First note that

$$\int_B \left[\log\log\left(1 + \frac{1}{|x|}\right)\right]^2 \,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^1 r\log\log(1+1/r)\,\mathrm{d}r\,\mathrm{d}\theta$$

is finite, since the integrand is bounded.

Away from $x = 0$ the derivative of f is

$$\partial_i f(x) = g_i(x) = \frac{1}{\log(1+1/|x|)}\,\frac{x_i}{|x|^2(1+|x|)},$$

whence

$$\int_B |g|^2\,\mathrm{d}x\,\mathrm{d}y = 2\pi\int_0^1 \frac{1}{\log(1+1/r^2)}\,\frac{1}{r(1+r)^2}\,\mathrm{d}r.$$

With the substitution $u = 1/r$ the integral becomes

$$\int_1^\infty \frac{1}{u+(1/u)}\,\frac{1}{\log(1+u^2)}\,\mathrm{d}u < \int_1^\infty \frac{1}{u(\log u)^2}\,\mathrm{d}u < \infty.$$

It remains to check that g really is the weak derivative of f throughout $B(0,1)$: to this end take $\varphi \in C_c^\infty(B(0,1))$, and then away from the singularity at $x = 0$ we have

$$\int_{B(0,1)\setminus B(0,\epsilon)} f\partial_i\varphi = -\int_{B(0,1)\setminus B(0,\epsilon)} g_i\varphi + \int_{|x|=\epsilon} f\varphi\, n_i\,\mathrm{d}S,$$

where n_i is the outward normal on $\partial B(0,\epsilon)$. The boundary term can be bounded according to

$$\left|\int_{|x|=\epsilon} f\varphi\, n_i\,\mathrm{d}S\right| \le \|\varphi\|_\infty \int_{|x|=\epsilon} \log\log(1+\epsilon^{-1})\,\mathrm{d}S \le C\epsilon\log\log(1+\epsilon^{-1}),$$

which tends to zero as $\epsilon \to 0$, showing that g_i is indeed the weak derivative $\partial_i f$.

**7. Show that if* $f \in H^k(\mathbb{R}^n)$ *the for any* $0 < \alpha < \min(k - n/2, 1)$ *there exists a constant* C *such that*

$$|f(x) - f(y)| \le C\|f\|_{H^k}|x-y|^\alpha.$$

8. In the proof of the Arzelà-Ascoli Theorem, it is relatively straightforward to find a subsequence such that $f_{n_j}(x_k)$ converges for each $k \in \mathbb{N}$, where the $\{x_k\}$ are points in K such that for every $x \in K$ and $\delta > 0$, there exists an N such that

$$|x - x_k| < \delta \qquad \textit{for some } k \in \{1, \ldots, N\}.$$

(Can you prove the existence of such a collection of points? And of such a subsequence?) Given this, use the equicontinuity of the $\{f_n\}$ to show that f_{n_j} converges uniformly on K.

We give the full proof. For each $n \in \mathbb{N}$ we have

$$K \subset \bigcup_{x \in K} B(x, 2^{-n}).$$

since K is compact, a finite number of these balls still cover K, giving a finite set of points such that any $k \in K$ lies within 2^{-n} of one of these points. Taking the sequence of these points created with ever larger choices of n gives the set $\{x_k\}$ in the question.

We now perform the famous 'diagonal argument'. Since $f_n(x_1)$ is a bounded sequence of real numbers, we can take a subsequence of $\{f_n\}$, call it $f_{1,n}$, such that $f_{1,n}(x_1)$ converges. Now observe that $f_{1,n}(x_2)$ is a bounded sequence of real numbers, so we can take a subsequence of $f_{1,n}$, call it $f_{2,n}$, such that $f_{2,n}(x_2)$ converges. Note that $f_{2,n}(x_1)$ is a subsequence of $f_{1,n}(x_1)$; since $f_{1,n}(x_1)$ converges, $f_{2,n}(x_2)$ still converges. Now observe that $f_{2,n}(x_3)$ is a bounded sequence of real numbers... We continue inductively, obtaining nested subsequences $f_{k,n}$ such that

$$f_{k,n}(x_j) \text{ converges as } n \to \infty \text{ for every } j = 1, \ldots, k.$$

The 'diagonal subsequence' $f_k^* = f_{k,k}$ is such that $f_k^*(x_j)$ converges for every $j \in \mathbb{N}$.

We can now answer the question as posed. Choose $\epsilon > 0$; since the f_n are equicontinuous, there exists a $\delta > 0$ such that for every $n \in \mathbb{N}$

$$|x - y| < \delta \qquad \Rightarrow \qquad |f_n^*(x) - f_n^*(y)| < \epsilon.$$

We know that there exists an M such that for every $x \in K$ there is an x_i with $i \leq M$ such that $|x - x_i| < \delta$. Now take N large enough that

$$|f_n^*(x_i) - f_m^*(x_i)| < \epsilon/3 \qquad \text{for all} \qquad m, n \geq N \quad \text{and} \quad i = 1, \ldots, M.$$

Then for any $x \in K$ choose an x_i such that $i \leq M$ and $|x - x_i| < \delta$; it follows that

$$\begin{aligned} &|f_n^*(x) - f_m^*(x)| \\ &\quad \leq |f_n^*(x) - f_n^*(x_i)| + |f_n^*(x_i) - f_n^*(x_j)| + |f_n^*(x_j) - f_n^*(x)| \\ &\quad \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

which shows that $\{f_n^*\}$ is Cauchy in the sup norm and hence uniformly convergent on K.

9. Suppose that $f, g, h \in H^1(\mathbb{R}^3)$. Show that

$$\left| \int fg\, \nabla h \, \mathrm{d}x \right| \leq c\|f\|_{L^3}\|g\|_{H^1}\|h\|_{H^1}.$$

Note that if $f, g, h \in H^1(\mathbb{R}^3)$ then $f, g, h \in L^p(\mathbb{R}^3)$ for any $2 \leq p \leq 6$. So first we use the the result of question 8 on Problems 2 with exponents $p = 3$, $q = 6$, and $r = 2$, to give

$$\left| \int fg\, \nabla h \, \mathrm{d}x \right| \leq \|f\|_{L^3}\|g\|_{L^6}\|\nabla h\|_{L^2}.$$

Now use the fact that $\|g\|_{L^6} \leq c\|g\|_{H^1}$ and that $\|\nabla h\|_{L^2} \leq \|h\|_{H^1}$ to give the required inequality.

**10. Reinterpret Poisson's equation*

$$-\Delta u = f \qquad u|_{\partial\Omega} = 0$$

as an abstract variational problem

$$(u, \phi)_{H_0^1} = f(\phi) \qquad \textit{for all } \phi \in H_0^1(\Omega)$$

with $f \in H^{-1}$, and show that given $f \in H^{-1}$ the equation has a unique solution $u \in H_0^1$.

Problems Four.

1. Let V be the subspace of $H^1(\Omega)$ consisting of functions with zero integral over Ω,

$$V = \{f \in H^1(\Omega) : \int_\Omega f = 0\}.$$

Arguing by contradiction, use the fact that $H^1(\Omega)$ is compactly embedded in

$L^2(\Omega)$ to show that there is a constant $C > 0$ such that

$$\|f\|_{L^2} \le C\|\nabla f\|_{L^2} \qquad \textit{for all} \quad f \in V.$$

(You may assume that if $\nabla f = 0$ almost everywhere then f is constant.)

Suppose that the result is not true; then there exists a sequence $f_n \in V$ such that

$$\|f_n\|_{L^2} \ge n\|\nabla f_n\|_{L^2}.$$

Replacing f_n by $f_n/\|f_n\|_{L^2}$ we obtain a sequence with $\|f_n\|_{L^2} = 1$ such that

$$\|\nabla f_n\|_{L^2} \le n^{-1}.$$

In particular f_n is a bounded sequence in H^1, and so has a subsequence that converges in L^2 to some $g \in H^1$ with $\|g\|_{L^2} = 1$ and $\int g = 0$ (since V is closed in the L^2 norm).

However, for any $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega g\partial_j\phi \, \mathrm{d}x = \lim_{n\to\infty} \int_\Omega f_n \partial_j \phi \, \mathrm{d}x = -\lim_{n\to\infty} \int_\Omega (\partial_j f_n)\phi \, \mathrm{d}x = 0$$

since $f_n \to g$ in L^2 and $\|\nabla f_n\|_{L^2} \le 1/n$. It follows that g is constant on Ω, contradiction $\int g = 0$ and $\|g\|_{L^2} = 1$.

2. Examine the argument used to prove Theorem 3.2 and convince yourself that the following is also true: If $f \in L^2(\Omega)$, and $u \in H^1(\Omega)$ satisfies $Lu = f$, then for every $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists a constant $c_{\Omega',\Omega''}$ such that

$$\|u\|_{H^2(\Omega')} \le c_{\Omega'}(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega'')}).$$

**3. Suppose that $u \in H_0^1(\Omega)$ is a solution of*

$$(\nabla u, \nabla v) = (f, v) \qquad \textit{for all} \quad v \in H_0^1(\Omega).$$

Show that if $f \in H^k(\Omega)$ then $u \in H^{k+2}(\Omega')$ for any $\Omega' \subset\subset \Omega$. [Take $w \in C_c^\infty(\Omega)$, which is dense in $H_0^1(\Omega)$, and consider the test function $v = (-1)^{|\alpha|}\partial^\alpha w$, where $|\alpha| = k$. You could do the same analysis with a more general $B(u, v)$ as in the notes, but it takes longer to write down.]

**4. Poisson's equation with Neumann boundary conditions is: find u such that*

$$-\Delta u = f \ \textit{in}\ \Omega \qquad \textit{and} \qquad \nabla u \cdot n|_{\partial\Omega} = 0,$$

(where n is the outward normal). By taking the inner product with a test function $\varphi \in C^\infty(\Omega)$ derive the weak form of the equation: find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \qquad \textit{for all} \quad v \in H^1(\Omega).$$

**5. Show that*

$$-\Delta u + \lambda u = f \ \textit{in}\ \Omega \qquad \textit{and} \qquad \nabla u \cdot n|_{\partial\Omega} = 0,$$

has a unique weak solution $u \in H^1(\Omega)$ (i.e. solution of the weak form of the equation) $u \in H^1(\Omega)$ for every $\lambda > 0$.

6. Show that

$$-\Delta u = f \qquad \nabla u \cdot n|_{\partial\Omega} = 0$$

has a solution $u \in H^1(\Omega)$ iff $\int f_\Omega = 0$. [Hint: use question 1]

If $(\nabla u, \nabla v) = (f, v)$ for every $v \in H^1(\Omega)$ then one can choose $v = 1$ on Ω to show that one must have $\int f = 0$. In the previous exercise we needed an extra $\lambda\|u\|^2$ with $\lambda > 0$ to show that u is coercive; in lectures we used a Poincaré inequality for the case of Dirichlet boundary conditions, and then we didn't need this additional factor - we can recover something similar here.

Note that if $\int f = 0$ then

$$(f, v) = \left(f, v - \int v\right),$$

since $\int v$ is a constant and $(f, 1) = 0$. Also

$$(\nabla u, \nabla v) = \left(\nabla u, \nabla(v - \int v)\right)$$

since $\nabla c = 0$ for any constant c. The weak form of the equation is therefore equivalent to

$$(\nabla u, \nabla v) = (f, v) \qquad \text{for all} \qquad v \in V,$$

where V is the space from question 1. If we now work in V rather than H^1 then we have a Poincaré inequality $\|u\|_{L^2} \le \|\nabla u\|_{L^2}$, and we can immediately deduce that for $u \in V$ we have

$$B(u, u) \ge \|\nabla u\|_{L^2}^2 \ge c\|u\|_{H^1}^2$$

and B is coercive. It follows that the equation has a unique solution in V, and hence at least one solution in H^1 (and in fact if u is a solution then $u - c$ is a solution for any $c \in \mathbb{R}$).

7. Show that any compact operator is bounded. (Consider the image of the closed unit ball.)

If $T : X \to Y$ is compact then $\overline{TB_X(0,1)}$ is compact. Any compact *set* is bounded, so $TB_X(0,1) \subset B_Y(0,M)$ for some M; it follows immediately that $\|T\|_{\mathscr{L}(X,Y)} \le M$.

8. Check that the Gram–Schmidt process produces an orthonormal set.

Suppose that $\tilde{e}_1, \ldots, \tilde{e}_n$ are orthonormal. Given e_{n+1}, we construct

$$\hat{e}_{n+1} = e_{n+1} - \sum_{j=1}^{n} (e_{n+1}, \tilde{e}_j)\tilde{e}_j \qquad \text{and} \qquad \tilde{e}_{n+1} = \frac{\hat{e}_{n+1}}{\|\hat{e}_{n+1}\|}.$$

Clearly $\|\tilde{e}_{n+1}\| = 1$; we only have to check that $\tilde{e}_{n+1}$ is orthogonal to $\tilde{e}_1, \ldots, \tilde{e}_n$. Fix $k \in \{1, \ldots, n\}$; then

$$\begin{aligned}(\tilde{e}_{n+1}, \tilde{e}_k) &= \frac{1}{\|\hat{e}_{n+1}\|}\left(e_{n+1} - \sum_{j=1}^{n}(e_{n+1}, \tilde{e}_j)\tilde{e}_j, \tilde{e}_k\right) \\ &= \frac{1}{\|\hat{e}_{n+1}\|}[(e_{n+1}, \tilde{e}_k) - (e_{n+1}, \tilde{e}_k)] = 0,\end{aligned}$$

since $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ are orthonormal.

9. Show that if $T \in \mathscr{L}(H, H)$ is invertible (i.e. the equation $Tx = y$ has a unique solution for every $y \in H$) then $\mathrm{Ker}(T) = \{0\}$*. Show that the reverse implication does not hold.*

If $Tx = y$ has a unique solution for any $y \in H$ but $\mathrm{Ker}(T)$ contains a non-zero element z then $T(x + \alpha z) = Tx = y$ for any $\alpha \in \mathbb{R}$. So invertibility of T implies that $\mathrm{Ker}(T) = \{0\}$. The problem with the converse is that while T can be such that $Tx = y$ has a unique solution for every $y \in T(H)$, it may be the case that $T(H) \neq H$ even if $\mathrm{Ker}(T) = \{0\}$. One example is the right-shift operator on ℓ^2:

$$\sigma(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots).$$

Clearly $\mathrm{Ker}(\sigma) = \{0\}$, but σ is not onto.

10. Suppose that $f \in L^2(\Omega)$ is such that $(f, v) = 0$ for every $v \in H^1_0(\Omega)$. Show that $f = 0$.

We know that $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$; and that $C_c^\infty(\Omega) \subset H_0^1(\Omega)$. So given any $v \in L^2(\Omega)$, write $v = \lim_{n\to\infty} v_n$, where $v_n \in H_0^1(\Omega)$ and the series converges in $L^2(\Omega)$. Then $(f, v) = 0$ for every $v \in L^2(\Omega)$, in particular for $v = f$¡ which shows that $\|f\|_{L^2}^2 = 0$ and hence $f = 0$ (in L^2, i.e. almost everywhere).

11. Show that two eigenfunctions u and v of our elliptic problem with distinct eigenvalues are orthogonal in $H_0^1(\Omega)$ (and not just in $L^2(\Omega)$).

This is in the notes. ??? JUST LAPLACIAN

12. Take $f \in H^{-1}$ and consider the equation

$$B(u, v) = f(v) \qquad \textit{for every} \quad v \in H_0^1,$$

where $B : H_0^1 \times H_0^1 \to \mathbb{R}$ is bilinear and bounded ($|B(u, v)| \leq \alpha \|u\|_{H^1} \|v\|_{H^1}$). Define a linear operator $A : H_0^1 \to H^{-1}$ such that this equation can be written as

$$Au = f,$$

an equation in H^{-1}. How does this relate to the argument we used in the proof of the Lax–Milgram Lemma?

Note that for each fixed $u \in H_0^1$ the map $g : H_0^1 \to \mathbb{R}$

$$g(v) = B(u, v)$$

is linear and bounded ($|g(v)| \leq [\alpha \|u\|_{H^1}] \|v\|_{H^1}$) it follows that $g \in H^{-1}(\Omega)$. We define $Au = g$; then A is linear and since

$$\|Au\|_{H^{-1}} = \|g\|_{H^{-1}} \leq \alpha \|u\|_{H^1}$$

it follows that A is bounded with $\|A\|_{\mathscr{L}(H_0^1, H^{-1})} \leq \alpha$. We can therefore write the equation as $Au = f$.

In the proof of the Lax–Milgram Lemma we instead viewed the equation in H_0^1 using the Riesz Representation Theorem; every element of H^{-1} can be derived from an element of H_0^1 via an inner product, so any inequality in H^* can be viewed as a corresponding equality in H (one switches to and fro via the 'Riesz map' $u \mapsto (\cdot, u)$ and its inverse).

Problems Five.

**1. We say that $f : (0, T) \to X$ is integrable if the function $\langle L, f(t) \rangle$:*

$(0,T) \to \mathbb{R}$ is integrable for every $L \in X^*$, and there exists a $y \in X$ such that

$$\langle L, y\rangle = \int_0^T \langle L, f(t)\rangle \,\mathrm{d}t \qquad \textit{for all} \qquad L \in X^*;$$

in this case we define

$$\int_0^T f(t)\,\mathrm{d}t = y.$$

Show that if X is reflexive and

$$\int_0^T \|f(t)\|_X \,\mathrm{d}t < \infty$$

then such a y exists and is well-defined.

2. Show that if $f : (0,T) \to X$ is integrable (as in the previous exercise) then

$$\left\| \int_0^T f(t)\,\mathrm{d}t \right\|_X \leq \int_0^T \|f(t)\|_X \,\mathrm{d}t.$$

Lemma 6.2 shows that given any $y \in X$ there exists an $L \in X^*$ such that $\|L\|_{X^*} = 1$ and $\langle L, y\rangle = \|y\|_X$. Using the definition of the integral from the previous exercise it follows that

$$\|y\|_X = \langle L, y\rangle = \int_0^T \langle L, f(t)\rangle \,\mathrm{d}t \leq \int_0^T \|f(t)\|_X \,\mathrm{d}t$$

since $\|L\|_{X^*} = 1$.

3. Show that if X is complete and reflexive then $L^2(0,T;X)$ is complete. [Hint: take a sequence $\{f_n\} \in L^2(0,T;X)$, and for each $L \in X^*$ show that $\langle L, f_n(t)\rangle \in L^2(0,T)$ is Cauchy in L^2. A Cauchy sequence has a subsequence that converges almost everywhere; follow an argument similar to that of question 1, i.e. working in X^{**}, to obtain a candidate limit $f \in L^2(0,T;X)$; finally show that $f_n \to f$.]

First, observe that $\langle L, f_n(t)\rangle \in L^2(0,T)$, since

$$\int_0^T |\langle L, f_n(t)\rangle|^2 \,\mathrm{d}t \leq \int_0^T \|L\|_{X^*}^2 \|f_n(t)\|_X^2 \,\mathrm{d}t = \|L\|_{X^*}^2 \int_0^T \|f_n(t)\|_X^2 \,\mathrm{d}t.$$

The same argument shows that $\langle L, f_n(t)\rangle$ is a Cauchy sequence in $L^2(0,T)$.

By completeness of $L^2(0,T)$, it follows that there is a function $f_L(\cdot) \in L^2(0,T)$ such that

$$\langle L, f_n \rangle \to f_L$$

in $L^2(0,T)$. It follows (Corollary 2.15) that there exists a subsequence f_{n_j} such that $\left\langle L, f_{n_j}(t) \right\rangle$ converges to $f_L(t)$ for almost every t. At these t define a linear map $I(t) : X^* \to \mathbb{R}$ by

$$\langle I(t), L \rangle = f_L(t) = \lim_{j\to\infty} \left\langle L, f_{n_j}(t) \right\rangle .$$

Then $I \in L^2(0,T;X^{**})$, since for almost every t we have

$$|\,\langle I(t), L \rangle\,| = \lim_{j\to\infty} |\left\langle L, f_{n_j}(t) \right\rangle| \le \|L\|_* \|f_{n_j}(t)\|_X,$$

i.e.

$$\|I(t)\|_{X^{**}} \le \lim_{j\to\infty} \|f_{n_j}(t)\|_X,$$

and so

$$\int_0^T \|I(t)\|_{X^{**}}^2 \,\mathrm{d}t \le \int_0^T \lim_{j\to\infty} \|f_{n_j}(t)\|_X^2 \,\mathrm{d}t \le M$$

since $\{f_n\}$ is Cauchy in $L^2(0,T;X)$. Since X is reflexive, we can find an $f(t) \in X$ such that $\langle I(t), L \rangle = \langle L, f(t) \rangle$. Since $\|f(t)\|_X = \|I(t)\|_{X^{**}}$, it follows that $f \in L^2(0,T;X)$. Since $\langle I(t), L \rangle = f_L(t)$ by definition, it follows that

$$\langle L, f_n \rangle \to \langle L, f \rangle$$

in $L^2(0,T)$, i.e. $\langle L, f_n - f \rangle \to 0$ in $L^2(0,T)$.

I can't finish the proof at the moment...

4. Show that if X is a Hilbert space then $L^2(0,T;X)$ is a Hilbert space.

We have completeness from question 3. The norm on $L^2(0,T;X)$ can be derived from the inner product

$$(f,g) = \int_0^T (f(t), g(t))_X \,\mathrm{d}t,$$

which shows that $L^2(0,T;X)$ is a Hilbert space. (It is easy to check that this is an inner product.)

5. Show that if $f \in C^0([0,T])$ and $\dot{f} \in L^1(0,T)$ is the weak time derivative *of f, then for any $\varphi \in C^1([0,T]) \cap C^\infty(0,T)$*

$$\int_0^T \dot{f}(t)\varphi(t)\,\mathrm{d}t = -\int_0^T f(t)\dot{\varphi}(t)\,\mathrm{d}t + f(T)\varphi(T) - f(0)\varphi(0).$$

(This is what lies behind the argument about the initial condition being attained when using weak convergence methods; cf. our Lemma 5.2.)

First note that if f has weak derivative $\dot{f}$ and $g \in C^1([0,T])$ then fg has weak derivative $f\dot{g} + \dot{f}g$, since for any $\varphi \in C_c^\infty(0,T)$

$$\begin{aligned}\int (f\dot{g} + \dot{f}g)\varphi &= \int (f\dot{g})\varphi + \dot{f}(g\varphi) \\ &= \int (f\dot{g})\varphi - f(\dot{g}\varphi) - fg\dot{\varphi} = -\int fg\dot{\varphi},\end{aligned}$$

where we have used the fact that $g\varphi \in C_c^\infty(0,T)$ (one could remove the requirement that $\varphi \in C^\infty$ by approximation). We can therefore apply the argument of Lemma 5.2 with $X = \mathbb{R}$ (and using L^1 rather than L^2) to fg, since now $fg \in L^1(0,T)$ and the weak derivative of fg is also in $L^1(0,T)$, to deduce that the above equality holds as stated.

6. Suppose that $f \in L^2$ and that $g \in H_0^1$. Show that $f \in H^{-1}$, in the sense that

$$F(g) := (f,g)_{L^2}$$

defines an element $F \in H^{-1}$. (This is what it means that say that $F \in H^{-1}$ is also in L^2.)

We have

$$|F(g)| = |(f,g)_{L^2}| \le \|f\|_{L^2}\|g\|_{L^2} \le \|f\|_{L^2}\|g\|_{H^1},$$

so F is a bounded linear functional on H_0^1.

7. Given $f \in H^{-1}$, extend f to a linear functional $F \in (H^1)^*$, such that $F(x) = f(x)$ for all $x \in H_0^1$ and $\|F\|_{(H^1)^*} = \|f\|_{H^{-1}}$. [Hint: use the Riesz Representation Theorem. This allows one to extend the proof of Lemma 5.3 to the case $f \in L^2(0,T;H^1)$, $\dot{f} \in L^2(0,T;H^{-1})$, which is useful for higher regularity of solutions.]

Given $f \in H^{-1}$, the Riesz Representation Theorem guarantees that there exists a $u \in H_0^1$ such that

$$f(v) = (u,v)_{H^1} \qquad \text{for all} \qquad v \in H_0^1 \qquad \text{and} \qquad \|f\|_{H^{-1}} = \|u\|_{H^1}.$$

If we define a linear map $F : H^1 \to \mathbb{R}$ by

$$F(v) = (u, v)_{H^1}$$

with the u above the clearly this agrees with f for $v \in H^1_0$; it is linear, and it is bounded on H^1 since

$$|F(v)| \leq \|u\|_{H^1}\|v\|_{H^1}.$$

It follows that $\|F\|_{(H^1)^*} \leq \|f\|_{H^{-1}}$, and since $H^1_0 \subset H^1$, we have equality here.

8. Suppose that $f \in L^2(0, T; H^1_0)$ and that $\dot{f} \in L^2(0, T; H^{-1})$. If $u_0 \in H^1_0 \cap H^2$ show that $u \in C^1([0, T]; H^{-1})$. [Hint: consider the equation for $v = \dot{u}$, and use the Sobolev embedding $H^2(0, T; H^{-1}) \subset C^1([0, T]; H^{-1})$ (we have not explicitly proved such a result).

Lemma 5.3 implies that $f \in C^0([0, T]; L^2)$. Consider the equation for $v = \dot{u}$, which is

$$\dot{v} - \Delta v = \dot{f}(t) \qquad v(0) = \Delta u_0 + f(0).$$

This is an equation for v in our standard parabolic form, since our assumptions mean that $\dot{f} \in L^2(0, T; H^{-1})$ and $v(0) \in L^2$. It follows that the equation has a unique solution $v \in L^2(0, T; H^1)$ and $\dot{v} \in L^2(0, T; H^{-1})$, i.e. $v \in H^1(0, T; H^{-1})$. Since the solutions are unique, we must have $v = \dot{u}$, and hence $\dot{u} \in L^2(0, T; H^1) \cap H^1(0, T; H^{-1})$. It follows that u itself is an element of $H^2(0, T; H^{-1})$, and hence of $C^1([0, T]; H^{-1})$.

9. Prove the Hahn-Banach theorem in a Hilbert space. [Show first that f extends to the closure of U if U is not closed; then consider $\overline{U}$ and $(\overline{U})^\perp$.]

If f is a bounded linear functional on U and $x \in \bar{U}$, one can write $x = \lim_{n\to\infty} x_n$, where $x_n \in U$. One can define

$$f(x) = \lim_{n\to\infty} f(x_n).$$

The limit exists since f is bounded and linear, and is well-defined. Clearly

$$|f(x)| \leq \lim_{n\to\infty} \|f\|_*\|x_n\| \leq \|f\|_*\|x\|$$

so the norm of f is not increased. So assume that U is closed. In this case one can decompose any $x \in H$ as $x = u + v$, where $v \in U^\perp$. For any $x \in H$ define

$$F(x) = f(u).$$

Then F is an extension of f, and $|F(x)| = |f(u)| \leq \|f\|_*\|u\| \leq \|f\|_*\|x\|$ since $\|u\| \leq \|x\|$.

**10. Suppose that x and y are linearly independent elements of X. Show that there exists a constant c such that*

$$|\alpha\|x\| + \beta c| \leq \|\alpha x + \beta y\|$$

for every $\alpha, \beta \in \mathbb{R}$. [If you can prove this you can prove the Hahn–Banach Theorem... more or less.]

**11. It is relatively easy to show that any element $\underline{\phi}$ of the sequence space ℓ^∞ gives rise to a linear functional on ℓ^1 via*

$$L_{\underline{\phi}}(x) = \sum_{k=1}^{\infty} \phi_k x_k$$

with $\|L_{\underline{\phi}}\|_{(\ell^1)^*} = \|\underline{\phi}\|_{\ell^\infty}$. Show that any $L \in (\ell^1)^*$ can be written as $L_{\underline{\phi}}$ for some $\underline{\phi} \in \ell^\infty$, and hence that $(\ell^1)^* = \ell^\infty$. [Hint: let $\{e_k\}_{k=1}^\infty$ be the basis for ℓ^1 where e_k consists of all zeros apart from a 1 in the k^{th} position and write $\underline{x} = \sum_{k=1}^\infty x_k e_k$. What is $L\underline{x}$?]

12. Show that weak convergence in X^* implies weak- convergence; and that if X^* is reflexive then weak-* convergence implies weak convergence.*

In general we know that any element of X gives rise to an element of X^{**} via

$$G_x(f) = f(x) \qquad \text{for all} \qquad f \in X^*.$$

So if $f_n \rightharpoonup f$ in X^*, i.e. $G(f_n) \to G(f)$ for every $G \in (X^*)^* = X^{**}$, then in particular for any $x \in X$this holds for $G = G_x$, so that

$$G_x(f_n) = f_n(x) \to G_x(f) = f(x) \qquad \text{for all} \qquad x \in X,$$

i.e. $f_n \overset{*}{\rightharpoonup} f$ in X^*.

If X is reflexive then any element $G \in X^{**}$ can be written as G_x for some $x \in X$; so given $G \in X^{**}$ we have

$$G(f_n) = G_x(f_n) = f_n(x) \to f(x) = G_x(f) = G(f),$$

which is weak convergence in X^*.