
1. Show that if $X$ is complete then $L^p(0, T; X)$ is complete.

2. Show that if $X$ is a Hilbert space then $L^2(0, T; X)$ is a Hilbert space.

3. We say that $f : (0, T) \rightarrow X$ is integrable if the function $\langle L, f(t) \rangle : (0, T) \rightarrow \mathbb{R}$ is integrable for every $L \in X^*$, and there exists a $y \in X$ such that

$$\langle L, y \rangle = \int_0^T \langle L, f(t) \rangle \, dt \quad \text{for all} \quad L \in X^*;$$

in this case we define

$$\int_0^T f(t) \, dt = y.$$

Show that if $X$ is reflexive and

$$\int_0^T \|f(t)\|_X \, dt < \infty$$

then such a $y$ exists and is well-defined.

4. Show that if $f : (0, T) \rightarrow X$ is integrable (as in the previous exercise) then

$$\left\| \int_0^T f(t) \, dt \right\|_X \leq \int_0^T \|f(t)\|_X \, dt.$$

5. Show that if $f \in C^0([0, T])$ and $\dot{f} \in L^1(0, T)$ is the weak time derivative of $f$, then for any $\varphi \in C^1([0, T])$

$$\int_0^T \dot{f}(t)\varphi(t) \, dt = -\int_0^T f(t)\dot{\varphi}(t) \, dt + f(T)\varphi(T) - f(0)\varphi(0).$$

(This is what lies behind the argument about the initial condition being attained when using weak convergence methods; cf. our Lemma 5.2.)

6. Suppose that $f \in L^2$ and that $g \in H^1_0$. Show that $f \in H^{-1}$, in the sense that

$$F(g) := (f, g)_{L^2}$$

defines an element $F \in H^{-1}$. (This is what it means that say that $F \in H^{-1}$ is also in $L^2$.)
7. Given \( f \in H^{-1} \), extend \( f \) to a linear functional \( F \in (H^1)^* \), such that 
\[
F(x) = f(x) \text{ for all } x \in H^1_0 \text{ and } \| F \|_{(H^1)^*} = \| f \|_{H^{-1}}. 
\]
[Hint: use the Riesz Representation Theorem. This allows one to extend the proof of Lemma 5.3 to the case \( f \in L^2(0,T; H^1), \ dot{f} \in L^2(0,T; H^{-1}) \), which is useful for higher regularity of solutions.]

8. Suppose that \( f \in L^2(0,T; H^1_0) \) and that \( \dot{f} \in L^2(0,T; H^{-1}) \). If \( u_0 \in H^1_0 \cap H^2 \) show that \( u \in C^1([0,T]; H^{-1}) \). [Hint: consider the equation for \( v = \dot{u} \), and use the Sobolev embedding \( H^2(0,T; H^{-1}) \subset C^1([0,T]; H^{-1}) \) (we have not explicitly proved such a result).]

9. Prove the Hahn-Banach theorem in a Hilbert space. [Show first that \( f \) extends to the closure of \( U \) if \( U \) is not closed; then consider \( \overline{U} \) and \( (\overline{U})^\perp \).]

*10. Suppose that \( x \) and \( y \) are linearly independent elements of \( X \). Show that there exists a constant \( c \) such that
\[
|\alpha\|x\| + \beta c \leq \|\alpha x + \beta y\|
\]
for every \( \alpha, \beta \in \mathbb{R} \). [If you can prove this you can prove the Hahn–Banach Theorem... more or less.]

*11. It is relatively easy to show that any element \( \phi \) of the sequence space \( \ell^\infty \) gives rise to a linear functional on \( \ell^1 \) via
\[
L_\phi(x) = \sum_{k=1}^{\infty} \phi_k x_k
\]
with \( \|L_\phi\|_{(\ell^1)^*} = \|\phi\|_{\ell^\infty} \). Show that any \( L \in (\ell^1)^* \) can be written as \( L_\phi \) for some \( \phi \in \ell^\infty \), and hence that \( (\ell^1)^* = \ell^\infty \). [Hint: let \( \{e_k\}_{k=1}^{\infty} \) be the basis for \( \ell^1 \) where \( e_k \) consists of all zeros apart from a 1 in the \( k \)th position and write \( x = \sum_{k=1}^{\infty} x_k e_k \). What is \( L_\phi \)?]

12. Show that weak convergence in \( X^* \) implies weak-* convergence; and that if \( X^* \) is reflexive then weak-* convergence implies weak convergence.
ADDITIONAL QUESTION (TO BE ASSESSED)

Let \( \Omega \subset \mathbb{R}^2 \), and consider the nonlinear reaction-diffusion equation

\[
\frac{\partial u}{\partial t} - \Delta u = \beta u - u^3 \quad u|_{\partial \Omega} = 0
\]

with \( u(x, 0) = u_0(x) \in L^2 \).

Assuming that \( u \) is smooth, show that there exists a constant \( K \) (depending on \( \beta \) and \( T \)) such that

\[
\sup_{0 \leq t \leq T} \| u(t) \|_{L^2}^2 + \int_0^T \| \nabla u(t) \|_{L^2}^2\, dt + \int_0^T \| u(t) \|_{H^1}^4\, dt \leq K \| u_0 \|_{L^2}^2. \tag{1}
\]

Deduce that

\[
\int_0^T \| \dot{u}(t) \|_{H^{-1}}^2\, dt \leq C(\| u_0 \|_{L^2}^2),
\]

i.e. that \( u_t \in L^2(0, T; H^{-1}) \).

[Hint: The only awkward term is \(-u^3\). Recall the definition of the \( H^{-1} \) norm, and the fact that \( H^1(\Omega) \subset L^p(\Omega) \) for any \( p < \infty \) if \( \Omega \subset \mathbb{R}^2 \) to show that \( \|(u^3)\|_{H^{-1}} \leq \|u\|_{L^3}^3 \); then use the \( L^p \) interpolation inequality (q9 of sheet 2) to bound the \( L^3 \) norm in terms of the \( L^2 \) and \( L^4 \) norm; then use (1).]

These estimates would form the basis of a proof based on Galerkin approximations

\[
\dot{u}_n - \Delta u_n = \beta u_n - P_n(u_n^3) \quad u_n(0) = P_n u_0.
\]

(Note that \( P_n u_n^3 \neq u_n^3 \) in general, i.e. the nonlinear term introduces ‘other modes’ in the eigenfunction expansion.) If you try to prove directly that \( \{u_n\} \) is Cauchy you will run into problems with the nonlinear term.

Which estimate from (1) – which will hold uniformly for the \( u_n \) – would you expect to enable us to show convergence of \( P_n u_n^3 \) to \( u^3 \) in some weak sense, and what is this convergence?