
We say that a sequence \( \{ \phi_n \} \in C^\infty_c(\mathbb{R}^n) \) converges to \( \phi \in C^\infty_c(\mathbb{R}^n) \) if there exists a compact set \( K \) such that \( \text{supp}(\phi_n) \subset K \) for all \( n \), and \( \partial^\alpha \phi_n \to \partial^\alpha \phi \) uniformly on \( K \) for all \( \phi \). A distribution is a bounded linear function \( f \) on \( C^\infty_c(\mathbb{R}^n) \) such that whenever \( \phi_n \to \phi \),

\[
f(\phi_n) \to f(\phi).
\]

We write \( \mathcal{D}'(\mathbb{R}^n) \) for the set of all distributions on \( \mathbb{R}^n \).

1. Show that if \( f \in \mathcal{D}'(\mathbb{R}^n) \) then \( \partial^\alpha f \) defined by

\[
[\partial^\alpha f](\phi) = (-1)^{|\alpha|} f(\partial^\alpha \phi)
\]

is also a distribution.

2. Any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) gives rise to a distribution via the definition

\[
L_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x) \, dx.
\]

Show that if \( g \) is the weak derivative \( \partial_j f \), then \( \partial_j L_f = L_g \), i.e. ‘the distribution derivative agrees with the weak derivative when it exists’.

3. Show that the approximating sequence used in Theorem 2.32 also converges uniformly on \( \bar{\Omega} \) if \( f \in H^k(\Omega) \cap C^0(\bar{\Omega}) \) (use Proposition 2.9).

4. Show that the \( H^k(\mathbb{R}^n) \) norm and the norm

\[
\left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}
\]

are equivalent. [Recall that \( (\partial^\alpha f) = (i\xi)^\alpha \hat{f} \) and that \( \| \hat{f} \|_{L^2} = \| f \|_{L^2} \).]

5. Show that

\[
\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^k} \, d\xi < \infty
\]

if and only if \( k > n/2 \).

6. Show that the unbounded function

\[
\log \log \left( 1 + \frac{1}{|x|} \right)
\]

is an element of \( H^1(B(0,1)) \), where \( B(0,1) \) is the unit ball in \( \mathbb{R}^2 \).
7. Prove Theorem 2.35; start by writing $f$ as the inverse of its Fourier transform $\hat{f}$, and use the inequality

$$|e^{i\xi \cdot x} - e^{i\xi \cdot y}| \leq C |x - y|^\alpha |\xi|^\alpha.$$

8. In the proof of the Arzelà-Ascoli Theorem, it is relatively straightforward to find a subsequence such that $f_{n_j}(x_k)$ converges for each $k \in \mathbb{N}$, where the $\{x_k\}$ are points in $K$ such that for every $x \in K$ and $\delta > 0$, there exists an $N$ such that

$$|x - x_k| < \delta \quad \text{for some } k \in \{1, \ldots, N\}.$$

(Can you prove the existence of such a collection of points? And of such a subsequence?) Given this, use the equicontinuity of the $\{f_n\}$ to show that $f_{n_j}$ converges uniformly on $K$.

9. Suppose that $f, g \in H^1(\mathbb{R}^3)$. Show that

$$\left| \int fg \nabla h \, dx \right| \leq \|f\|_{L^3} \|g\|_{H^1} \|h\|_{H^1}.$$

10. Reinterpret Poisson’s equation

$$-\Delta u = f \quad u|_{\partial \Omega} = 0$$

as an abstract variational problem

$$(u, \phi)_{H_0^1} = f(\phi) \quad \text{for all } \phi \in H_0^1(\Omega)$$

with $f \in H^{-1}$, and show that given $f \in H^{-1}$ the equation has a unique solution $u \in H_0^1$. 

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