Motivic Integration

Victoria Hoskins

30 March 2007

Supervisor: Dr. Dmitriy Rumynin

Abstract

Motivic integration was introduced by Kontsevich to prove a conjecture by Batyrev, which stated that crepant resolutions of a complex projective Calabi-Yau variety with at worst canonical Gorenstien singularities have the same Hodge numbers. This paper gives an exposition of the arithmetic motivic measure theory developed by Denef and Loeser. We define a 'motivic measure' which takes values in a motivic ring built of finite formal sums of varieties. The paper covers the construction of the motivic integral and explains how Kontsevich used motivic integration to prove Batyrev's conjecture on Calabi-Yau varieties.

1 Introduction - Chow motives and divisors

This paper assumes familiarity with the main concepts outlined by Reid in Undergraduate Algebraic Geometry [18]. This first section will introduce important concepts and constructions which will be used throughout the paper. We will assume our field k is a commutative field which is algebraically closed, although often we will be working in the specific case of the complex numbers.

Recall for an algebraic variety X, we let $k[X]$ denote the ring of polynomial functions on X and $k(X)$ the field of rational functions on X. A function $f \in k(X)$ is *regular* at $P \in X$ if there exists an expression $f = g/h$ with $g, h \in k[X]$ such that $h(P) \neq 0$. We then define the local ring of X at P to be,

$$\mathcal{O}_{X,P} = \{f \in k(X) \; : \; f \text{ is regular at } P\}.$$

Definition 1.1 : The sheaf of holomorphic functions, $\mathcal{O}$, and the sheaf of meromorphic functions, $\mathcal{K}$.
Let $X = \mathbb{A}_k^r$ and let $\mathcal{F}(X)$ denote the sheaf of germs of functions on X with values in k. We define a subsheaf $\mathcal{O}$ of $\mathcal{F}(X)$ in the following way. For $x \in X$, let $\mathcal{O}_x$ be the subring of $k(x_1, ...x_r)$ given by the set of rational function which are regular at x. Then *the sheaf of holomorphic functions* $\mathcal{O}$ is the sheaf defined by these local rings $\mathcal{O}_x$.

Now suppose V is an irreducible algebraic subvariety of X, the operation of restriction of a homomorphism defines a canonical homomorphism $\epsilon_x : \mathcal{F}(X)_x \longrightarrow \mathcal{F}(V)_x$. Let $\mathcal{O}_{V,x} := \epsilon_x(\mathcal{O}_x)$, these stalks form a subsheaf of $\mathcal{F}(V)$ which we will denote $\mathcal{O}_V$.

Let U be a nonempty open subset of V, then $\mathcal{A}_U := \Gamma(U, \mathcal{O}_V)$ is an integral ring whose elements are regular functions on U, this follows from the irreducibility of V (see [19]). Thus we can talk about the field of quotients of $\mathcal{A}_U$ which we will denote $\mathcal{K}_U$. If $U \subseteq W$, the homomorphism $\rho_U^W : \mathcal{A}_U \longrightarrow \mathcal{A}_W$, given by restriction of a section, is injective and thus we get a well defined isomorphism $\varphi_U^W : \mathcal{K}_U \longrightarrow \mathcal{K}_W$. The system of presheaves $(\mathcal{K}_U, \varphi_U^W)$ defines a *sheaf of fields* $\mathcal{K}_V$, where the stalk at x, $\mathcal{K}_{V,x}$ is canonically isomorphic to the field of quotients of $\mathcal{O}_{V,x}$.

We call $\mathcal{O}_V$ the sheaf of holomorphic functions on V and $\mathcal{K}_V$ the sheaf of meromorphic functions on V. These sheaves can be defined in greater generality, a more detailed treatment can be found in [19] and [20].

Definition 1.2 Let X be a nonsingular algebraic variety. A *prime divisor* on X is an irreducible subvariety of codimension one. We let $\mathrm{Div}(X)$ denote the free abelian group generated by the set of prime divisors on X. A *Weil divisor* is an element of $\mathrm{Div}(X)$; that is, a finite formal sum of prime divisors $D = \sum_{i=1}^{r} n_i V_i$ where V_i are prime divisors and n_i are integers. We say a divisor is *effective* if each coefficient in the sum is nonnegative, this is written as $D \geq 0$.

Example 1.3 Let C be a nonsingular projective algebraic curve, the irreducible subvarieties of codimension one are simply the points of C. Furthermore for $f \in k(C)^*$ we can define a divisor $\mathrm{div}(f)$ by evaluating the order of vanishing of f at each point $p \in C$. Given $g \in k[C]^*$ we define $\upsilon_p(g)$ to be the order of vanishing of g at p. Then for $f = g/h \in k(X)^*$ we define $\upsilon_p(f) = \upsilon_p(g) - \upsilon_p(h)$ and since a nonzero rational function has only finitely many zeros and poles this defines a divisor

$$\mathrm{div}(f) := \sum_{p \in C} \upsilon_p(f) p$$

Such divisors are called *principle divisors*. This allows us to introduce an equivalence relation on $\mathrm{div}(X)$; we say D_1 is *linearly equivalent* to D_2,

$$D_1 \sim D_2 \;\Leftrightarrow\; D_1 - D_2 = \mathrm{div}(f) \text{ for some } f \in k(C).$$

In fact every divisor D on a smooth variety X is locally principle (see [20]), so in a sufficiently small neighbourhood U of a point it is described by $\mathrm{div}(f_U)$. We say f_U is a *local defining equation* for D on U. In fact f_U determines a divisor $\mathrm{div}(f_U)$ which is given as a sum over subvarieties of codimension one, with the coefficients determined by the order of vanishing of f_U along the subvariety. This will be defined concretely in section 1.1. If we take an open covering of $X = \cup_\alpha U_\alpha$ where on each U_α, the divisor D is principle with local defining equation f_α on U_α. The functions $f_\alpha f_\beta^{-1}$ are regular on $U_\alpha \cap U_\beta$, and thus are elements of $\mathcal{O}_X(U_\alpha \cap U_\beta)^*$.

Definition 1.4 A *Cartier divisor* is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$. More specifically, a Cartier divisor on a complex algebraic variety X is given by an open cover $\{U_i\}$ of X with corresponding local sections $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$

such that $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^*$.

A Cartier divisor on a smooth complex variety X defines a line bundle $L = \cup_i(\mathbb{A}^1 \times U_i)$ over X such that we identify the points

$$(a, x) \sim (b, x) \text{ if } x \in U_i \cap U_j \text{ and } b = f_i/f_j(x)a$$

The sheaf of sections of L coincides with $\mathcal{O}(D)$ for some divisor D on X, where $\mathcal{O}(D)$ is the sheaf defined by the sections

$$\Gamma(U, \mathcal{O}(D)) := \{f \in \mathbb{C}(U) \,:\, \operatorname{div}(f) + D \geq 0\} \quad \text{for } U \subseteq X$$

From the comments above one can see that the definition of divisors given by Weil and Cartier agree for smooth varieties over the complex numbers.

1.1 Construction of the Chow Ring

The Chow ring of an algebraic variety is a geometric analogue of the cohomology ring. Elements of the Chow ring are formed from subvarieties and a product is induced by the idea of intersecting subvarieties.

Definition 1.5 The Chow Group A *cycle* on a complex algebraic variety X is a finite formal sum of irreducible subvarieties $\alpha = \sum n_i[V_i]$ where n_i are integers and V_i are irreducible subvarieties. We say α is a *k-cocyle* if each of the V_i are k-codimensional subvarieties of X. The *Chow group* $A^k(X)$ is formed by taking the group of k-cocycles on X modulo *rational equivalence*, which we explain below.

Let W be a $(k-1)$-codimensional subvariety of X and $r \in \mathbb{C}(W)^*$ be a nonzero rational function on W, then we define a k-cocycle $\operatorname{div}(r)$ on X by

$$\operatorname{div}(r) := \sum \operatorname{ord}_V(r)\,[V]$$

where this sum is over all codimension one subvarieties V of W. The order of vanishing along V is a function $\operatorname{ord}_V \colon \mathbb{C}(W)^* \longrightarrow \mathbb{Z}$ analagous to that defined for divisors. If W is nonsingular along V then the local ring $\mathcal{O}_{W,V}$ is a discrete valuation ring so has a unique maximal ideal with single generator z. Then $r = uz^m$ for some unit u and integer m, in this case $\operatorname{ord}_V(r) = m$. More formally we define $\mathrm{ord}_V(r)$ to be the length of the $\mathcal{O}_{W,V}$-module $\mathcal{O}_{W,V}/(r)$, which will be finite as X is Noetherian. Given a sequence of $\mathcal{O}_{W,V}$-modules:

$$0 = N_0 \subseteq N_1 \subseteq ... \subseteq N_n = \mathcal{O}_{W,V}/(r)$$

where N_i are all $\mathcal{O}_{W,V}$-modules and N_{i+1}/N_i is simple for $i = 1, ..., n-1$, we say $\mathcal{O}_{W,V}/(r)$ has length n, thus $\mathrm{ord}_V(r) = n$.

We say a k-cocycle α is *rationally equivalent to zero*, denoted $\alpha \sim 0$, if there exists a finite number of $(k-1)$-codimensional subvarieties W_i of X and rational functions $r_i \in \mathbb{C}(W_i)^*$ such that $\alpha = \sum_i \mathrm{div}(r_i)$. Then two k-cocycles are rationally equivalent if their difference is rationally equivalent to zero. The group of k-cocycles modulo rational equivalence is the k^{th} Chow group $A^k(X)$. We define the Chow group to be the direct sum of the k^{th} Chow groups

$$A^*(X) := \bigoplus_k A^k(X).$$

The reasoning behind taking cocycles modulo rational equivalence will become clear later when we extend the Chow group to a ring by defining an intersection pairing. Since not all varieties intersect in the way we desire we may have to perform a deformation and thus the intersection pairing is only well defined up to rational equivalence.

1.1.1 Pullbacks and pushforwards of cocycles

Let $f : X \to Y$ be a proper morphism of algebraic varieties. Since f is proper it sends subvarieties $V \subset X$ to subvarieties $W := f(V) \subset Y$. Hence f induces an embedding $\mathbb{C}(W) \hookrightarrow \mathbb{C}(V)$ and either $\dim(W) < \dim(V)$ or they have equal dimension in which case the field extension is finite. Define

$$f_*[V] = \begin{cases} [\mathbb{C}(V) : \mathbb{C}(W)][W] & \text{if } \dim(W) = \dim(V), \\ 0 & \text{if } \dim(W) < \dim(V). \end{cases}$$

This map extends linearly to a pushforward homomorphism on the group of cocycles.

Theorem 1.6 ([12]) *If $f : X \to Y$ is a proper morphism and $\alpha \sim 0$ is a cocycle on X then $f_*\alpha \sim 0$.*

Thus we get an induced homomorphism $f_* : A^*(X) \to A^*(Y)$ and A^* is a covariant functor from the category of algebraic varieties with proper morphisms to the category of groups.

Now let $f : X \to Y$ be a flat morphism of relative dimension n, geometrically this means the fibres $f^{-1}(y)$ have dimension $n = \dim(X) - \dim(Y)$. The most important examples will be projections from a cartesian product onto one of its factors and the projection map of a vector bundle onto its base space. In fact a subvariety $V \subseteq Y$ of codimension p gives a subvariety of X of codimension p since if $\dim(Y) = d$ then V has dimension $d - p$ so its preimage $f^{-1}(V)$ has dimesnion $d - p + n$ and thus codimension p in X. We define the pull back of a subvariety $V \subseteq Y$ of codimension p to be

$$f^*[V] := [f^{-1}(V)].$$

This also respects rational equivalence (see [12]) and so extends linearly to a pullback homomorphism

$$f^* : A^p(Y) \to A^p(X)$$

and thus A^* is a contravariant functor from the category of algebraic varieties with flat morphisms to the cateory of groups.

1.1.2 The intersection pairing

We give the Chow group $A^*(X)$ a ring structure by taking intersections of varieties. Ideally we want to take the intersection of two subvarieties and get another subvariety, but we can only do this if they intersect in a 'good way' (see below) as we need the intersection pairing to be a map,

$$A^p(X) \otimes A^q(X) \to A^{p+q}(X).$$

Definition 1.7 For two subvarieties $V \subseteq A^p(X)$ and $W \subseteq A^q(X)$, we say V and W have a *good intersection* if each irreducible component Z_i of their intersection has codimension $p + q$; that is,

$$\dim(Z_i) = \dim(V) + \dim(W) - \dim(X).$$

In this case, the intersection product is given by

$$V \cdot W = \sum_i m(Z_i)[Z_i]$$

where $m(Z_i)$ is the intersection multiplicity. In the case of divisors on X, we define $V \cdot W = [V \cap W]$, provided their intersection is good ([13]).

However, not all varieties intersect in this way, in which case it is necessary to perform a slight deformation in order to allow the varieties to have a good intersection.

Example 1.8 Consider $X = \mathbb{P}^2$ and both V and W are the same projective line L, then $V \cap W = L$ has dimension one, but

$$\dim(V) + \dim(W) \text{ - } \dim(X) = 1 + 1 - 2 = 0$$

Clearly this is not a good intersection. By Bézout we expect the intersection of two projective lines in the projective plane to be a point with multiplicity one. To define the intersection pairing we would need to deform either V or W so their intersection has the correct dimension, however there are many ways one can do this. So the intersection product will be a point which is defined up to rational equivalence. This explains the purpose of taking cocycles modulo rational equivalence in the construction of the Chow group.

Now consider two general varieties, whose intersection is not necessarily good, we will describe a method for calculating the intersection pairing which will be defined up to rational equivalence. Firstly one observes that calculating the intersection pairing of two varieties is equivalent to calculating the intersection of their cartesian product with the diagonal $\triangle_X$ in $X \times X$.

The diagonal embedding $\delta : X \hookrightarrow X \times X$ is a regular map of codimension n, where $n = \dim(X)$. Let V be the cartesian product of a p-cocycle and a q-cocyle we want to intersect, so that $V \subseteq X \times X$ is a $(p+q)$-cocycle. The first stage in defining the intersection product is to define the product $X \cdot V$. To do this we let $f : V \to X \times X$ denote the natural embedding and complete the fiber square below. We get $W = X \cap V = f^{-1}(\triangle_X)$ where $\triangle_X$ is the image of X under the diagonal embedding and a morphism $g : W \longrightarrow X$.

$$\begin{array}{ccc} W & \xrightarrow{i} & V \\ \downarrow g & & \downarrow f \\ X & \xrightarrow{\delta} & X \times X \end{array}$$

We take the pullback of the normal bundle $N_X(X \times X)$ under g and denote it by $N := g^* N_X(X \times X)$, this is a bundle of rank n on W and we let $\pi : N \longrightarrow W$ denote the corresponding projection. In fact, the flat pullback of this projection is an isomorphism $\pi^* : A^{p+q}(W) \longrightarrow A^{p+q}(N)$ is an isomorphism (see [12]), and therefore its inverse, $(\pi^*)^{-1} : A^{p+q}(N) \longrightarrow A^{p+q}(W)$ is well defined. The normal cone $C_W V$ to W in V is a closed subcone of N of codimension $p+q$ and so determines a $p+q$ cocycle on N. We can now define the product of X and V by

$$X \cdot V = (\pi^*)^{-1}[C_W V].$$

The intersection pairing is the composite map

$$A^p(X) \otimes A^q(X) \xrightarrow{\times} A^{p+q}(X \times X) \xrightarrow{\delta^*} A^{p+q}(X)$$

where the first map is given by taking the cartesian product of varieties and the second map is the Gysin homomorphism induced by δ. The Gysin homomorphism is given by

$$\delta^*(\sum_i n_i[V_i]) := \sum_i n_i \, X \cdot V_i.$$

We write $x \otimes y := \delta^*(x \times y)$ for two cocycles $x, y \in A^*(X)$, often the intersection pairing is also denoted by $x \cdot y$. This operation gives $A^*(X)$ a ring structure. A good explaination of cycles and rational equivalence can be found in [12].

Definition 1.9 : The Category of Chow Motives
An object in the category of Chow motives, $\mathcal{M}_{\mathbb{C}}$, is a triple (X, p, m) where X is a smooth complex projective variety of dimension d, p is a projective element of $A^d(X \times X)$ (that is, it satisfies $p^2 = p$) and m is an integer. A morphism between Chow motives (X, p, m) and (Y, p', n) is an element of $p' A^{d+n-m}(Y \times X)\, p$. The composition of morphisms is given by

$$A^*(X \times Y) \times A^*(Y \times Z) \longrightarrow A^*(X \times Z)$$

$$(\alpha, \beta) \mapsto \pi_{XZ*}(\pi_{XY}^*(\alpha)\pi_{YZ}^*(\beta))$$

where $\pi_{XY} : X \times Y \times Z \to X \times Y$ denotes the projection map and $\pi_{XY}^* : A^*(X \times Y) \to A^*(X \times Y \times Z)$ is the pullback and $\pi_{XY*} : A^*(X \times Y \times Z) \to A^*(X \times Y)$ is the pushforward.

The Grothendieck group of Chow motives, $K_0(\mathcal{M}_{\mathbb{C}})$ is the free abelian group generated by isomorphism classes of objects in $\mathcal{M}_{\mathbb{C}}$ modulo the subgroup generated by elements of the form $[(X,p,m)] - [(Y,q,n)] - [(Z,r,k)]$ whenever $(X,p,m) \cong (Y,q,n) \oplus (Z,r,k)$. The binary operation $\oplus$ does not always define a Chow motive, however in the case $n = k$ it does and we define

$$(Y,q,n) \oplus (Z,r,n) = (Y \amalg Z, (q,r), n)$$

where (q,r) is the cocycle consisting of pairs of subvarieties.

We define the product of two motives to be

$$(X,p,m) \otimes (Y,p',n) := (X \times Y, p \otimes p', m+n)$$

which gives $K_0(\mathcal{M}_{\mathbb{C}})$ a ring structure.

2 The additive motivic measure

In this section X will be a smooth complex algebraic variety.

2.1 Pseudo finite fields and ring formulae

Traditionally in order to construct a measure, we would first construct measurable sets, which we would like to be able to assign a value to. In motivic measure theory we work with formulae rather than sets so firstly we need to decide what will be measurable.

Definition 2.1 A ring formula over a field k is a syntactically correct formula built from a countable collection of variables, parentheses and the symbols

$$\forall, \quad \exists, \quad \vee, \quad \wedge, \quad \neg, \quad 0, \quad 1, \quad (+), \quad (-), \quad (*), \quad (=).$$

Definition 2.2 A field K is *pseudo-algebraically closed* if every absolutely irreducible variety over the field has a K-rational point. Furthermore a field K is a *pseudo finite field*, denoted PFF, if it is an infinite pseudo-algebraically closed field, which has exactly one field extension of degree n inside a fixed algebraic closure of K, for every integer $n > 0$.

PFFs arise in model theory and there are not many nice examples, however Ax [1] characterised PFFs as infinite ultra products of finite fields.

Proposition 2.3 (Ax [1]) *Two ring formulae give the same solutions over $\mathbb{F}_p$ for a sufficiently large prime p if and only if they are the same when interpreted in K, for every pseudo finite field K containing $\mathbb{Q}$.*

PFFs are infinite fields which have every elementary property shared by finite fields, they are the infinite models of the theory of finite fields. They were introduced to allow problems involving finite fields to be solved using PFFs instead. We use PFFs to define an equivalence relation on ring formulae.

Definition 2.4 The *Grothendieck group of the theory of pseudo finite fields over k* is the free abelian group of ring formulae over k modulo the following scissor and congruence relations.
Scissor relation If $\phi_1 \vee \phi_2$ is a disjunction of ring formulae, then $[\phi_1 \vee \phi_2] = [\phi_1] + [\phi_2] - [\phi_1 \wedge \phi_2]$.
Congruence relation $[\phi] = [\phi']$ if there is a ring formula ψ such that for each pseudo-finite field K containing k, ψ defines a bijection between the tuples in K satisfying ϕ and the tuples in K satisfying ϕ'.
This congruence relation identifies formulae which are 1-sheeted coveras of each other, however we can generalise this to a broader condition.
Congruence relation (covers) If ϕ is a n-sheeted cover of ϕ', then $[\phi] = n[\phi']$.

This abelian group becomes a ring by introducing the notion of products of ring formulae. Let $\phi_1 (x_1, ..., x_n)$ be a formula in free variables $x_1, ..., x_n$ and $\phi_2 (y_1, ..., y_m)$ be a formula in free variables $y_1, ..., y_m$, distinct from the variables of ϕ_1 then we set $[\phi_1] \cdot [\phi_2] = [\phi_1 (x_1, ..., x_n) \wedge \phi_2 (y_1, ..., y_m)]$. We define $K_0(PFF_k)$ to be the ring of the theory of pseudo finite fields over k.

2.2 The Motivic Ring

Definition 2.5 Let $\mathcal{V}_{\mathbb{C}}$ denote the category of complex algebraic varieties. We form a Grothendieck group from the free abelian goup generated by the objects of $\mathcal{V}_{\mathbb{C}}$ by quotienting out by relations of the form $[X]$=$[Z]$+$[X \backslash Z]$

where Z is a closed subvariety of X and by a congruence relation, which is given in definition 2.7. This group will be denoted $K_0\left(\mathcal{V}_{\mathbb{C}}\right)$.

Definition 2.6 An *additive invariant of algebraic varieties* is a map from the category of complex algebraic varieties to a ring S,

$$\lambda : \mathcal{V}_{\mathbb{C}} \longrightarrow S$$

satisfying

I) $\lambda(X) = \lambda(Y) \quad$ if $X \cong Y$,

II) $\lambda(X) = \lambda(Y) + \lambda(X \backslash Y) \quad$ if $Y \subseteq X$ is a closed subset,

III) $\lambda(X \times Y) = \lambda(X)\lambda(Y)$.

Such a map extends to take values on constructible sets, which are disjoint unions of locally closed varieties. Let

$$W = \bigsqcup_{i=1}^{r} Z_i$$

be a constructible set, where Z_i are locally closed, then we define

$$\lambda(W) := \sum_{i=1}^{r} \lambda(Z_i).$$

There are many examples of additive invariants of algebraic varieties but probably the most common example is the Euler characteristic which takes values in $\mathbb{Z}$.

We want to introduce a congruence relation on the Grothendieck group of varieties and to do this we describe what it means for two varieties to be isomorphic as virtual Chow motives. We construct an additive invariant of algebraic varieties known as the virtual motive, mapping into the Grothendieck ring of Chow motives. In section 1 we described the construction of the Grothendieck ring of Chow motives. We let $\mathcal{M}_{\mathbb{C}}$ denote the category of Chow motives, keeping in mind we are working over the complex numbers.

Gillet and Soulé [14] illustrate a canonical morphism from $K_0(\mathcal{V}_{\mathbb{C}})$ to $K_0(\mathcal{M}_{\mathbb{C}})$ by constructing a contravariant functor $M : \mathcal{V}_{\mathbb{C}} \longrightarrow \mathcal{M}_{\mathbb{C}}$ given by $M(X) = (X, 1_X, 0)$ where 1_X denotes the identity element in the ring $A^*(X \times X)$ which is given by the class of the diagonal $\triangle_X \subset X \times X$. If d denotes the dimension of X then it is clear that 1_X is a d-cocycle in $X \times X$ and since it is the identity element it is also projective. So this does indeed define a Chow motive. Since M is an additive invariant of algebraic varieties it will factor through $K_0(\mathcal{V}_{\mathbb{C}})$ and so by taking the class of its image in $K_0(\mathcal{M}_{\mathbb{C}})$, M extends to a canonical morphism $K_0(\mathcal{V}_{\mathbb{C}}) \rightarrow K_0(\mathcal{M}_{\mathbb{C}})$.

Definition 2.7 We say two nonsingular projective varieties are *equal as virtual Chow motives* if their images under this morphism M have the same class in $K_0(\mathcal{M}_{\mathbb{C}})$.
Congruence relation: We say $[X]$=$[Y]$ if X and Y are both nonsingular projective varieties that give the same virtual Chow motive.

Let [ ]: Ob$(\mathcal{V}_{\mathbb{C}}) \longrightarrow K_0(\mathcal{V}_{\mathbb{C}})$ denote the natural quotient map sending a variety X to its class, $[X]$, in the Grothendieck group. We can give this Grothendieck group of varieties the structure of a ring using the following convention $[X]\cdot[Y]=[X \times Y]$. We call this ring the *Grothendieck ring of complex algebraic varieties* and denote it by $K_0(\mathcal{V}_{\mathbb{C}})$. We introduce symbols $1 :=$[point] and $\mathbb{L} := [\mathbb{A}^1_{\mathbb{C}}]$.

The congruence relation identifies more varieties than simply those which are isomorphic, since varieties which are not isomorphic as varieties may become isomorphic when viewed as Chow motives.

Example 2.8 Let $f : E \longrightarrow E'$ be an isogeny of elliptic curves, that is f is a surjective morphism with finite kernel. Clearly isogenous elliptic curves E and E' are not always isomporphic as varieties but they are always isomorphic when viewed as Chow motives. To prove this consider the corresponding Chow motives, $M = (E, 1_E, 0)$ and $M' = (E', 1_{E'}, 0)$. Recall a morphism from M to M' is an element of the form $1_{E'}A^1(E' \times E)1_E$ and a morphism from M' to M has the form $1_E A^1(E \times E')1_{E'}$. So to show M and M' are isomorphic we need to find $\alpha \in A^1(E' \times E), \beta \in A^1(E \times E')$ such that

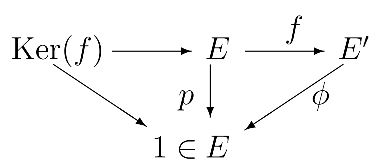
$$1_{E'} \cdot \alpha \cdot 1_E \cdot 1_E \cdot \beta \cdot 1_{E'}$$

is an isomorphism, which is equivalent to showing $\alpha \cdot \beta$ is an isomorphism. An obvious choice for a codimension one subvariety of $E \times E'$ would be the incidence curve. We want to choose the cocycles α and β to be codimension one subvarieties V and W such that one is isomorphic to E and the other is isomorphic to E'. To do this we construct a morphism from E' to E in the following way.

Recall $f : E \longrightarrow E'$ is an isogeny of elliptic curves so is surjective with finite kernel, say $\mathrm{Ker}(f) = \{e_1, \ldots, e_n\} \subseteq E$. We know $E/\mathrm{Ker}(f) \cong E'$ so using this we want to construct a map $\phi : E' \longrightarrow E$. Since the kernel is a subgroup of order n we know $e_j^n = 1$ for all $e_j \in \ker(f)$, hence we constuct a group homomorphism

$$p : E \longrightarrow E$$
$$e \longmapsto e^n.$$

Clearly $\ker(f) \subseteq \ker(p)$, however there may be a few other points in the kernel of p, in fact it will contain any torsion point of order n. We constuct ϕ as illustrated in the following picture



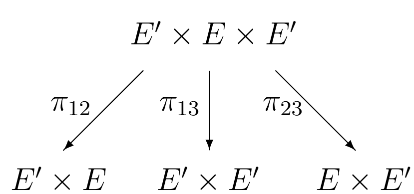
$$p(e) = \phi \circ f(e).$$

Choose the 1-cocyles to be

$$V = \{(e', e) \in E' \times E \ : e = \phi(e') \text{ for } e' \in E'\} \subseteq E' \times E,$$

$$W = \{(e, e') \in E \times E' \ : e' = f(e) \text{ for } e \in E\} \subseteq E \times E'.$$

Since V and W do not lie in the same Chow ring to define their intersection product we will pull them back to $A^1(E' \times E \times E')$ and do the multiplication in this ring, then pushforward the resulting cocycle to $A^1(E' \times E')$. Now let π_{ij} denote the projection from $E' \times E \times E'$ onto the i^{th} and j^{th} sets as below.



Then the intersection product of V and W is given by

$$V \cdot W = \pi_{13*}(\pi_{12}^*(V) \cdot \pi_{23}^*(W)) \in A^1(E' \times E').$$

In section 1 the pullpack of a flat morphism was defined to be the preimage of the variety under this map, hence

$$\tilde{V} := \pi_{12}^*(V) = \pi_{12}^{-1}(V) = V \times E' \in A^1(E' \times E \times E'),$$
$$\tilde{W} := \pi_{23}^*(W) = \pi_{23}^{-1}(W) = E' \times W \in A^1(E' \times E \times E').$$

These varieties have a good intersection, that is the dimension of their intersection equals the dimension of $\tilde{V}$ plus the dimension of $\tilde{W}$ minus the dimension of $E' \times E \times E'$. Thus the intersection pairing is simply given by their intersection.

$$\tilde{U} := \tilde{V} \cap \tilde{W} = \{(e', \phi(e'), f(\phi(e'))) \in E' \times E \times E'$$

But $f(\phi(e')) = e'^n$ since $f^{-1}(e') = \tilde{e}\ker(f)$, so $\phi(e') = \tilde{e}^n$ and since f is a group homomorphism $f(\phi(e')) = f(\tilde{e})^n = e'^n$. Let the pushforward of U by the projection π_{13} be denoted by $\tilde{U}$.

$$U := \pi_{13*}(\tilde{U}) = \{(e', e'^n) : e' \in E'\} \in A^1(E' \times E')$$

This is clearly isomorphic to the diagonal in $E' \times E'$ and so is an isomorphism. Hence two isogenous elliptic curves may be isomorphic as virtual Chow motives but not as varieties.

Definition 2.9 We define the *Motivic ring* to be the localization of $K_0(\mathcal{V}_{\mathbb{C}})$, $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}] := S^{-1}K_0(\mathcal{V}_{\mathbb{C}})$ where $S := \{1, \mathbb{L}, \mathbb{L}^2, ...\}$ and $\mathbb{L} = [\mathbb{A}^1]$ denotes the class of the affine line. Often this is denoted by $\mathcal{M}_{\mathbb{C}}$, which should not be confused with the category of Chow motives, also denoted by $\mathcal{M}_{\mathbb{C}}$. Sometimes we want to eliminate any torsion, in which case we define the localised motivic scissor ring to be, $\mathbb{S}_{mot} := K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}] \otimes \mathbb{Q}$.

2.3 Motivic Counting measure

Theorem 2.10 ([15]) *There exists a unique ring homomorphism $K_0(PPF_{\mathbb{C}}) \longrightarrow \mathbb{S}_{mot}$ such that whenever ϕ is a ring formula given by the conjunction of polynomial equations, then $[\phi]$ is sent to the affine variety defined by these polynomial equations.*

In this theorem we deliberately specified $k = \mathbb{C}$ since this is the most common choice of base field for motivic integration.

Definition 2.11 The *motivic counting measure* of a ring formula ϕ is defined to be the image of $[\phi]$ under the above homomorphism. We will also let $[\phi]$ denote its class in $\mathbb{S}_{mot}$.

Example 2.12 : Motivic Counting Measure of the set of nonzero 5^{th} powers

The following example is motivated by a calculation given by Hales [15] of the motivic counting measure of the nonzero cubes. As an extension to this we will consider the set of nonzero 5^{th} powers given by the ring formula:

$$\phi(x) : \exists\, y \text{ s.t} \left(y^5 = x\right) \wedge (x \neq 0)$$

Firstly we need to consider the case of repeated solutions by considering how many of the 5^{th} roots of unity lie in the field. To do this we solve the equation $z^5 - 1 = 0$ over our field and look for conditions on the field relating to the existence of roots of unity. Clearly 1 is always a root of unity in any field and so we can factor this root out:

$$0 = z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$$

This is equivalent to solving the following equation:

$$\left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) + 1 = 0$$

Introducing a change of variables $a = z + \frac{1}{z}$ and noticing that $a^2 = z^2 + 2 + \frac{1}{z^2}$ gives a quadratic, $a^2 + a - 1 = 0$ which has a solution in our field if the discriminant, 5, is a square number. Supposing that 5 is a square, our quadratic has roots $a_1 = (-1 - \sqrt{5})/2$ and $a_2 = (\sqrt{5} - 1)/2$. Then plugging these values into the change of variables formula we get quadratics

$$z^2 + \left(\frac{1 + \sqrt{5}}{2}\right) z + 1 = 0 \quad \text{and} \quad z^2 + \left(\frac{1 - \sqrt{5}}{2}\right) z + 1 = 0$$

which have solutions provided $(\sqrt{5} - 5)/2$ and $(-\sqrt{5} - 5)/2$ are squares respectively. Hence the number of 5^{th} roots of unity is related to the number of solutions to the equation $\left(\sqrt{5}+5+2t^2\right)\left(\sqrt{5}-5-2t^2\right) = t^4+5t^2+5 = 0$. If this equation has no solutions there is only one root of unity, if this equation has two solutions then there will be three roots of unity and if it has four solutions then all five roots of unity lie in the field, assuming 5 is a square.

Let $\mathbb{V}$ be the class in $\mathbb{S}_{\text{mot}}$ corresponding to the variety defined by $t^4 + 5t^2 + 5 = 0$. We can now split our ring formula into three disjoint cases:
$\phi = \phi_1 \vee \phi_2 \vee \phi_3$. Where ϕ_1 one corresponds to the field containing one root of unity, ϕ_2 corresponds to the field containing three roots of unity and ϕ_3 corresponds to all five roots of unity lying in the field.

Throughout this argument we assumed 5 was a square number, so we need to account for whether this is the case or not. In the case of ϕ_1 there are no roots of unity so it does not matter whether 5 is a square or not. In the other two cases, we need 5 to be a square as $\sqrt{5}$ appears in the expression for the roots unity. Hence we need another variety class $\mathbb{W}$ representing the equation $s^2 - 5 = 0$.

For ϕ_1 there is only one root of unity and therefore the nonzero points on the affine line correspond to solutions of the ring formula thus, ϕ_1 has measure $(\mathbb{L}-1)(1-\frac{\mathbb{V}}{2})(1-\frac{\mathbb{V}}{4})$. The purpose of the last two terms is to ensure that ϕ_1 has measure zero when the equation $t^4+5t^2+5=0$ has a solution; that is, in the case $\# \mid \mathbb{V} \mid = 2$ or 4. If we are in the case of ϕ_1 we would expect this equation to have no solutions, $\# \mid \mathbb{V} \mid = 0$, so the last two terms will be equal to one and thus do not alter the measure attributed to ϕ_1.

The case ϕ_2 corresponds to the equation $t^4 + 5t^2 + 5 = 0$ having two solutions and hence the field containing 3 roots of unity, therefore the nonzero points on the affine line give a 3-fold cover of ϕ_2. In this case $\# \mid \mathbb{V} \mid = 2$ and by using a similar technique we can make the measure attributed to ϕ_2 vanish in the case that $\# \mid \mathbb{V} \mid = 0$ or 4. As $\mathbb{W}$ can either have two solutions or no solutions we see that multiplying by $\mathbb{W}/2$ takes into account whether 5 is square or not. Thus the measure of ϕ_2 is thus given by

$$\left(\frac{\mathbb{L}-1}{3}\right)\left(\frac{\mathbb{W}}{2}\right)\left(1-\frac{\mathbb{V}}{4}\right)\mathbb{V}.$$

In a similar way, we calculate the measure of ϕ_3 to be

$$\left(\frac{\mathbb{L}-1}{5}\right)\left(\frac{\mathbb{W}}{2}\right)\left(\frac{\mathbb{V}}{2}-1\right)\left(\frac{\mathbb{V}}{4}\right).$$

The total motivic counting measure of the set of nonzero 5^{th} powers is

$$\begin{aligned}[\phi] \quad &= \quad (\mathbb{L}-1)\left(1-\frac{\mathbb{V}}{2}\right)\left(1-\frac{\mathbb{V}}{4}\right) + \left(\frac{\mathbb{L}-1}{3}\right)\left(\frac{\mathbb{W}}{2}\right)\left(1-\frac{\mathbb{V}}{4}\right)\mathbb{V} \\ &+ \left(\frac{\mathbb{L}-1}{5}\right)\left(\frac{\mathbb{W}}{2}\right)\left(\frac{\mathbb{V}}{2}-1\right)\left(\frac{\mathbb{V}}{4}\right).\end{aligned}$$

2.4 The outer measure of a DVR formula

In the previous section we constructed a counting measure which can be used with finite fields. Now we want to constuct a universal measure which can be used on any locally compact field.

2.4.1 Valued rings

Definition 2.13 Let R be a ring accompanied by functions val and ac on R such that, val:$R \longrightarrow \mathbb{Z} \cup \infty$ and ac: $R \longrightarrow k$ where k is a field. The function val is a valuation if it satisfies

I) $\text{val}(x) = \infty$ iff $x = 0$.

II) $\text{val}(xy) = \text{val}(x)\text{val}(y)$ for $x, y \in R^*$.

III) $\text{val}(x+y) \geq min\{\text{val}(x), \text{val}(y)\}$ for $x, y \in R$.

In this case R is called a *valued ring* and the target of val and ac are called the *value group* and *residue field* respectively.

Example 2.14 Let $\mathbb{C}[[t]]$ be the ring of formal power series with complex coefficients. For $x \neq 0 \in \mathbb{C}[[t]]$, let k be the smallest integer such that the coefficient a_k of t^k is nonzero, so we can write $x = \sum_{i=k}^{\infty} a_i t^i$. Then we can define functions val and ac on $\mathbb{C}[[t]]$ as follows, let val(x)=k and ac(x)=a_k.

We set val(0)=∞ and ac(0)=0. In this case $\mathbb{C}[[t]]$ is a valued ring with $\mathbb{Z}_{\geq 0} \cup \infty$ and $\mathbb{C}$ being the value group and residue field respectively.

Definition 2.15 A syntactically well-formed expression involving variables, parentheses, quantifiers, the function symbols val and ac, along with the usual ring operations of the valued ring and residue fields and the group operations and inequalities on the value group is called a *DVR formula*. The above example of $\mathbb{C}[[t]]$ gives a structure for the DVR language.

Theorem 2.16 (Hensel's Lemma) *Let K be a field, complete with respect to a non-archimedean absolute value $|\ |$ and let R denote its ring of integers. Let $f_j \in R[x_1, ..., x_n]$ for $1 \leq j \leq n$ be a system of polynomial equations. Suppose $(a_1, ..., a_n) \in R^n$ is an approximate solution satisfying*

$$| f_j(a_1, ..., a_n) | < | \mathrm{Det}(J(a_1, ..., a_n)) |^2 \quad \text{for } 1 \leq j \leq n$$

where $J(a_1, ..., a_n)$ is the Jacobian matrix of the system of polynomial equations with respect to $(a_1, ..., a_n)$. Then $(a_1, ..., a_n)$ can be lifted to a solution $(b_1, ..., b_n) \in R^n$ such that

$$f_j(b_1, ..., b_n) = 0 \quad \text{and } | b_j - a_j | < | \mathrm{Det}(J(a_1, ..., a_n)) | \quad \text{for } 1 \leq j \leq n$$

Remark 2.17 Hensel's Lemma is well known in the case of the p-adic numbers but this more general version gives conditions on a polynomial to ensure it has roots in a given neighbourhood. Fields satisfying Hensel's Lemma are called Henselian. We need criteria for the existence of roots of polynomials to be able to eliminate quantifiers from a DVR formula. Hensel's lemma can in fact can be simplified in the case of a valued field K. It then becomes if $f \in K[x]$ is a polynomial whose coefficients have nonnegative valuation and $x \in K$ satisfies $\mathrm{val}(f(x)) > 0$ and $\mathrm{val}(f'(x)) = 0$, then there exists a $y \in K$ such that $f(y) = 0$ and $\mathrm{val}(y - x) > 0$.

Theorem 2.18 ([15]) *Let K be a valued field of characteristic zero, with val: $K \longrightarrow \mathbb{Z} \cup \infty$ and ac: $K \longrightarrow k$, where the residue field, k, also has characteristic zero. Let ϕ be a DVR formula then if K is Henselian there is another formula ϕ' without quantifiers such that*

$$\forall (x, \eta, m) \in K^n \times k^p \times (\mathbb{Z} \cup \infty)^r \qquad \phi^K(x, \eta, m) = \phi'^K(x, \eta, m).$$

2.4.2 Tiles, outer measure and the motivic measure

In traditional measure theory one would now tile the space with cubes of a fixed volume and then approximate the volume using a counting measure. We then take the limit as the volume of the cubes tends to zero to get the outer measure of the space. In motivic measure theory we can do exactly the same thing.

Definition 2.19 Let K be a valued field, for $z = (z_1, ..., z_n) \in K^n$ we define the cube of width m, centred at z, to be the set

$$\{(x_1, ..., x_n) \in K^n : \text{val}(x_i - z_i) \geq m, \text{ for } i = 1, ..., n\}.$$

Example 2.20 If $K = \mathbb{C}[[t]]$ is the ring of formal power series with complex coefficients and $z = \sum_{i=k}^{\infty} a_i t^i$ is a formal power series then a cube of width m around z is given by the set of power series whose first m coefficients agree with those of z.

$$\{x \in \mathbb{C}[[t]] : \text{val}(x - z) \geq m\} = \{x \in \mathbb{C}[[t]] : x = \sum_{i=k}^{m-1} a_i t^i + \sum_{i=m}^{\infty} b_i t^i : b_i \in \mathbb{C}\}$$

This can be seen as holding the power series up to a fixed level then shaking the tail to produce a cube. More precisely let $\rho_m : \mathbb{C}[[t]] \longrightarrow \mathbb{C}[[t]]/(t^m) \cong \mathbb{C}^m$ denote the truncation map

$$\sum_{i=0}^{\infty} a_i t^i \overset{\rho_m}{\longmapsto} \sum_{i=0}^{m-1} a_i t^i \longmapsto (a_0, ...a_{m-1})$$

Then the cube of width m centred at z is given by $\rho_m^{-1}(z)$.

Definition 2.21 Let ϕ be a DVR formula in free variables $(x_1, ..., x_n)$. An *outer ring formula approximation* to ϕ at level m is a ring formula, ϕ_m, in nm free variables u_{ij} such that over every field k we have

$$\{u \in k^{nm} : \ \phi_m(u)\} =$$

$$\{u \in k^{nm} : \ \exists\, a_1, ..., a_n \quad \phi(a_1, ..., a_n) \wedge \text{val}(a_i - \sum_{j=0}^{m-1} u_{ij} t^j) \geq m\}.$$

A point is a solution of the approximation ϕ_m if it is sufficiently close to a true solution of ϕ. Thus $[\phi_m]$ counts the number of centres of cubes of width m in which there is a solution.

For the example of $\mathbb{C}[[t]]$, if ϕ was a formula in free variables $(x_1, ...x_n) \in \mathbb{C}[[t]]^n$, the outer ring formula ϕ_m at level m is

$\{u \in \mathbb{C}^{nm} \, : \, \phi_m(u)\} =$

$$\{u \in \ \mathbb{C}^{nm} \, : \, \exists \, a_1, ...a_n \in \mathbb{C}[[t]] \quad \phi(a_1, ..., a_n) \wedge \mathrm{val}(a_i - \sum_{j=0}^{m-1} u_{ij} t^j) \geq m\}.$$

Points satisfying ϕ_m are n-tuples of power series truncated to level m which are sufficiently close (determined by val and m) to a real solution of ϕ.

Definition 2.22 Let ϕ be a DVR formula in free variables $(x_1, ...x_n)$. We define the *outer measure* of ϕ at level m to be

$$\frac{[\phi_m]}{\mathbb{L}^{nm}} \in \mathbb{S}_{\mathrm{mot}} \, .$$

The numerator counts the number of cube centres that contains a solution to the DVR formula and the denominator is a scaling term to take into account the volume of the cubes.

Definition 2.23 Let ϕ be a DVR formula in free variables $(x_1, ...x_n)$. We define the *motivic measure* of ϕ to be

$$\lim_{m \to \infty} [\phi_m] \mathbb{L}^{-nm}.$$

3 Motivic Integration

Throughout this section let $\mathcal{M}_{\mathbb{C}} := K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ be the motivic ring.

3.1 The Space of formal arcs of a complex manifold

Definition 3.1 Let Y be a complex manifold of dimension n, and $y \in Y$ be a point. A *k-jet* over y is a morphism

$$\gamma_y : \text{Spec } \mathbb{C}[t]/(t^{k+1}) \longrightarrow Y \quad \text{with} \quad \gamma_y\,(\text{Spec}\,\mathbb{C}) = y.$$

If we take local coordinates at y then we can view the space of k-jets over y as the space of n-tuples of polynomials of degree k whose constant terms are zero (this ensures a 0-jet will map to y as required). We can see a k-jet over y as a germ of a curve at y which has been truncated so it is defined by only the directions of the i^{th} order tangent vectors for $i = 0, .., k$. Let $J_k\,(Y)$ denote the bundle over Y whose fibre over $y \in Y$ is the space of k-jets over y. A *formal arc* over y is a morphism

$$\gamma_y : \text{Spec } \mathbb{C}[[t]] \longrightarrow Y \quad \text{with} \quad \gamma_y\,(\text{Spec}\,\mathbb{C}) = y.$$

Similarly if we take local coordinates at y then we can view the space of k-jets over y as the space of n-tuples of power series whose constant terms are zero. The *arc space* over Y is the bundle $\pi_0 : J_\infty\,(Y) \longrightarrow Y$ whose fibre over $y \in Y$ is the space of formal arcs over y. For each $k \in \mathbb{Z}_{\geq 0}$ the inclusion $\mathbb{C}[t]/(t^{k+1}) \longrightarrow \mathbb{C}[[t]]$ induces a surjection $\pi_k : J_\infty\,(Y) \longrightarrow J_k\,(Y)$.

Definition 3.2 A subest $C \subseteq J_\infty(Y)$ is a *cylinder set* if $C = \pi_k^{-1}(B_k)$ for some $k \in \mathbb{Z}_{\geq 0}$ and $B_k \subseteq J_k(Y)$ a *constructible* subset. Recall that a subset of a variety is constructible it if is a finite, disjoint union of (Zariski) locally closed subvarieties.

$J_k(Y)$ is a bundle of rank nk over Y, the fibre over $y \in Y$ is given by

$$\begin{aligned} J_k(Y)_y &= \{\text{ space of } k\text{-jets over } y\} \\ &= \{(\textstyle\sum_{i=1}^k a_{i1}x_1^i, \sum_{i=1}^k a_{i2}x_2^i, ..., \sum_{i=1}^k a_{in}x_n^i) \;:\; a_{ij} \in \mathbb{C}\} \quad \cong \mathbb{A}_{\mathbb{C}}^{nk}. \end{aligned}$$

where $(x_1, ..., x_n)$ are local coordinates. This is easy to check since in the case $k = 1$, $J_k(Y)$ is the tangent bundle which has rank n. Therefore we can see $J_k(Y)$ as an affine space of dimension $n(k+1)$, so a subvariety of $J_k(Y)$ is given by algebraic conditions on the i^{th} order tangent directions ($i = 0, ..., k$). We can then give $J_k(Y)$ the Zariski topology where the closed sets are those defined by polynomial conditions on the $(a_{ij})_{0\leq i\leq k\,,\,1\leq j\leq n}$. Hence we can view a cylinder set $C = \pi_k^{-1}(B_k)$ locally as a direct product of an infinite dimensional affine space with some variety in a finite dimensional affine space given by algebraic conditions on the directions of the i^{th} order tangent vectors for $i = 0, ..., k$.

Definition 3.3 We define a finitely additive motivic measure on the algebra of cylinder sets by

$$\tilde{\mu} \,:\, \{\text{cylinder sets in } J_\infty(Y)\} \longrightarrow \mathcal{M}_{\mathbb{C}}$$

$$\pi_k^{-1}(B_k) \longmapsto [B_k]\mathbb{L}^{-n(k+1)}.$$

Since the map [ ] sending a variety to its class in $\mathcal{M}_{\mathbb{C}}$ is additive on disjoint unions of constructible sets it follows that

$$\tilde{\mu}\left(\bigsqcup_{i=1}^{l} C_i\right) = \sum_{i=1}^{l} \tilde{\mu}(C_i) \quad \text{for disjoint cylinder sets } C_1, ..., C_l.$$

The motivic integral is defined using level sets of functions defined on $J_\infty(Y)$, we will soon see that it will be useful to exend $\tilde{\mu}$ to a measure μ defined on the collection of countable disjoint unions of cylinder sets. Kontsevich [16] completed the ring $\mathcal{M}_{\mathbb{C}}$, with respect to a filtration so that $\lim_{i\to\infty}[\mathbb{L}^{-i}] = 0$.

Definition 3.4 Let $\widehat{\mathcal{M}_{\mathbb{C}}}$ denote the completion of the ring $\mathcal{M}_{\mathbb{C}}$ with respect to the filtration

$$... \supseteq F^{-1}\mathcal{M}_{\mathbb{C}} \supseteq F^{0}\mathcal{M}_{\mathbb{C}} \supseteq F^{1}\mathcal{M}_{\mathbb{C}} \supseteq ...$$

where for each $m \in \mathbb{Z}$, $F^m\mathcal{M}_{\mathbb{C}} \lhd \mathcal{M}_{\mathbb{C}}$ is generated by elements of the form $[V] \cdot \mathbb{L}^{-i}$ for $\dim(V) \leq i - m$. Notice that $\mathbb{L}^{-m} \in F^m\mathcal{M}_{\mathbb{C}}$ and $F^n\mathcal{M}_{\mathbb{C}}F^m\mathcal{M}_{\mathbb{C}} \subseteq F^{n+m}\mathcal{M}_{\mathbb{C}}$. Let $\phi : \mathcal{M}_{\mathbb{C}} \longrightarrow \widehat{\mathcal{M}_{\mathbb{C}}}$ denote the natural completion map.

We define $\mu' := \phi \circ \tilde{\mu}$ to be an extension taking values in the ring $\widehat{\mathcal{M}_{\mathbb{C}}}$. Since this ring is complete, talking about a sequence of cylinder sets converging to zero with respect to μ' makes sense and so we can define the measure of a countable disjoint union of cylinder sets.

Definition 3.5 We extend μ' to a measure $\mu : \mathcal{C} \longrightarrow \widehat{\mathcal{M}_{\mathbb{C}}}$ where

$$\mathcal{C} = \{A, A^c : A = \bigsqcup_{i \in \mathbb{N}} C_i \quad \text{for cylinder sets } C_i \text{ s.t } \mu'(C_i) \to 0 \ \text{ as } \ i \to \infty\}$$

by

$$\mu(\bigsqcup_{i \in \mathbb{N}} C_i) := \sum_{i \in \mathbb{N}} \mu'(C_i).$$

3.2 Effective divisors and their integrals

Definition 3.6 Let Y be a complex manifold, D an effective divisor on Y and g a local defining equation for D on a neighbourhood $U \subseteq Y$. Recall we have a correspondence between Weil and Cartier divisors over smooth complex varieties, thus given a divisor we can find such a local defining equation. Let γ_u be a formal arc over $u \in U$, we define the *intersection number* $\gamma_u \cdot D$ to be the order of vanishing of the formal power series $g(\gamma_u(t))$ at $t = 0$. Then we associate to a divisor D a function $F_D : J_\infty(Y) \longrightarrow \mathbb{Z}_{\geq 0} \cup \infty$ given by $F_D(\gamma_u) = \gamma_u \cdot D$. If we decompose D into a linear combination of prime divisors, $D = \sum_{i=1}^r a_i D_i$ then $g = \prod_{i=1}^r {g_i}^{a_i}$ where g_i is the defining equation for D_i, hence $F_D = \sum_{i=1}^r a_i F_{D_i}$.

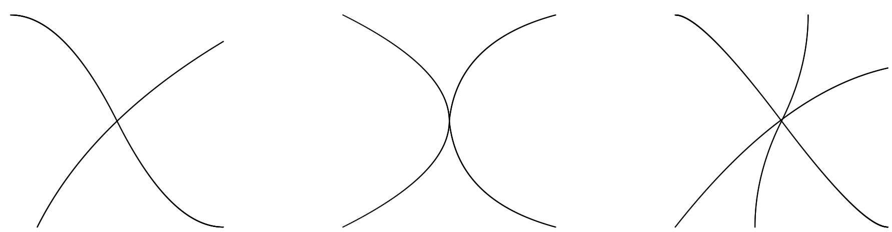
Example 3.7 Consider $Y = \mathbb{A}^2_{\mathbb{C}}$, condiser the two coordinate axes, $D_j = \{(z_1, z_2) \in \mathbb{C}^2 \ : \ z_j = 0\}$ for $j = 1, 2$, and the divisor $D = aD_1 + bD_2$. Let us consider F_{D_1}, since $D_1 : (z_1 = 0)$, on any open neighbourhood U a local defining equation for D_1 would be $g(z_1, z_2) = z_1$. A formal arc at $u \in U$ is given by a pair of power series, for example $\gamma_u(t) = (\sum_{i=k}^\infty c_i t^i, \sum_{i=l}^\infty d_i t^i)$ then $g(\gamma_u(t)) = \sum_{i=k}^\infty c_i t^i$, so $F_{D_1}(\gamma_u) = k$. Similarly $F_{D_2}(\gamma_u) = l$ and so we have $F_D(\gamma_u) = ak + bl$.

We want to integrate F_D over $J_\infty(Y)$ by considering the measure of the level sets $F_D^{-1}(s) \subseteq J_\infty(Y)$ for $s \in \mathbb{Z}_{\geq 0} \cup \infty$. However if we restrict to the case where D has simple normal crossings, it can be shown $F_D^{-1}(s)$ is a cylinder set for $s \in \mathbb{Z}_{\geq 0}$ and $F_D^{-1}(\infty)$ is a countable union of cylinder sets which has measure zero. Therefore the level sets $F_D^{-1}(s)$ are measureable. For proofs of the above statements see Craw [5].

Definition 3.8 Let $D = \sum_{i=1}^{r} a_i D_i$ be a divisor we say D has *simple normal crossings* at a point $y \in Y$ if there is a neighbourhood U of y, with local coordinates $z_1, ... z_n$ such that a local defining equation for D on U is of the form $g = {z_1}^{a_1} \cdots {z_{j_y}}^{a_{j_y}}$ for some $j_y \leq n$. Geometrically we see this condition as the divisor looking locally like the intersection of up to n hyperplanes.

Example 3.9 Divisors on $\mathbb{R}^2$ are finite formal sums of curves, looking at where these curves intersect will determine whether a divisor has simple normal crossings. Consider the following pictures.

Picture A Picture B Picture C



Picture A is an example of a divisor on $\mathbb{R}^2$ with simple normal crossings, at the intersection this divisor will locally look like $xy = 0$. Picture B shows two curves meeting tangentially, which means at the intersection the corresponding divisor will locally look like two copies of a hyperplane given by $x^2 = 0$. Thus picture B also has simple normal crossings. The final picture shows three curves meeting in one comment point, locally this will look like $xy(x - y) = 0$ so the corresponding divisor does not have simple normal crossings.

Definition 3.10 Let Y be a complex manifold of dimension n and $D = \sum_{i=1}^{r} a_i D_i$ be an effective divisor with simple normal crossings. We define the motivic integral of the pair (Y, D) to be

$$\int_{J_\infty(Y)} F_D \, d\mu \, := \sum_{s \in \mathbb{Z}_{\geq 0} \cup \infty} \mu(F_D^{-1}(s)) \cdot \mathbb{L}^{-s}$$

This definition is not very useful for computations however Craw [5] used a partition of Y to find a slightly more user friendly formula. We partition our manifold into subvarieties, defined in terms of the divisor D, in a way in which the preimages of this partition under $\pi_0 : J_\infty(Y) \longrightarrow Y$ give a partition of the arc space into cylinder sets. For a divisor $D = \sum_{i=1}^r a_i D_i$ and $J \subseteq \{1, ..., r\}$ any subset we define

$$D_J \ := \begin{cases} \cap_{j\in J} D_j & \text{if } J \neq \emptyset \\ Y & \text{if } J = \emptyset \end{cases}$$

$$\text{and} \quad D_J^o \ := \ D_J \backslash \bigcup\nolimits_{i\in\{1,\ldots,r\}\backslash J} D_i \ .$$

Theorem 3.11 ([5]) *Let Y be a complex manifold of dimension n and $D = \sum_{i=1}^r a_i D_i$ be an effective divisor on Y with only simple normal crossings. The motivic integral of the pair (Y, D) is*

$$\int_{J_\infty(Y)} F_D \, d\mu \ = \sum_{J\subseteq\{1,\ldots,r\}} [D_J^o] \cdot \left(\prod_{j\in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1} \right) \cdot \mathbb{L}^{-n}$$

where the sum is over all subsets $J \subseteq \{1, ..., r\}$ including $J = \emptyset$.

Example 3.12 Let $Y = \mathbb{P}^1_{\mathbb{C}}$ be the Riemann Sphere, since $\dim_{\mathbb{C}} Y = 1$ a divisor on Y is a finite sum of points, $D = \sum_{i=1}^r n_i P_i$. Then for each $J \subseteq \{1, ..., r\}$ we calculate D_J^o:

$J = \emptyset$	$D_J = \mathbb{P}^1_{\mathbb{C}}$	$D_J^o = \mathbb{P}^1_{\mathbb{C}} \backslash \{P_1, ..., P_r\}$
$J = \{i\}$	$D_J = P_i$	$D_J^o = \{P_i\}$
$J = \{i, j\}$	$D_J = P_i \cap P_j = \emptyset$	$D_J^o = \emptyset$
$J = \{i_1, .., i_n\}$ $(n > 1)$	$D_J = \cap_{j=1}^n P_{i_j} = \emptyset$	$D_J^o = \emptyset$

Using the above theorem we calculate the integral of $(\mathbb{P}^1_{\mathbb{C}}, D)$ to be

$$\int_{J_\infty(Y)} F_D \, d\mu \ = \ [\mathbb{P}^1_{\mathbb{C}} \backslash \{P_1, ..., P_r\}] \cdot \mathbb{L}^{-1} + \sum_{i=1}^r [P_i] \cdot \left(\frac{\mathbb{L} - 1}{\mathbb{L}^{n_i+1} - 1} \right) \cdot \mathbb{L}^{-1}.$$

We can rewrite the following term

$$[\mathbb{P}^1_{\mathbb{C}} \backslash \{P_1, ..., P_r\}] = [\mathbb{P}^1_{\mathbb{C}}] - [\{P_1, ..., P_r\}] = [\mathbb{P}^1_{\mathbb{C}}] - ([P_1] + ... + [P_r])$$

using the fact that [Point] $= 1$ and $[\mathbb{A}^1_{\mathbb{C}}] = \mathbb{L}$ and by writing $\mathbb{P}^1_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}} \cup \infty$. This gives

$$[\mathbb{P}^1_{\mathbb{C}} \backslash \{P_1, ..., P_r\}] = \mathbb{L} + 1 - r.$$

Hence,

$$\int_{J_\infty(Y)} F_D \, d\mu \ = \ 1 + (1-r) \cdot \mathbb{L}^{-1} + \sum_{i=1}^{r} \left(\frac{\mathbb{L} - 1}{\mathbb{L}^{n_i+1} - 1} \right) \cdot \mathbb{L}^{-1}$$

3.3 Birational Transformation rule

Currently the motivic integral is defined for divisors on complex manifolds, but we would like to extend this definition to complex algebraic varieties where possible.

Definition 3.13 Let X be a complex algebraic variety of dimension n, the canonical bundle on X is the bundle of holomorphic n-forms on X, $\Omega^n(X) := \wedge^n(T^*X)$, where T^*X denotes the cotangent bundle of holomorphic 1-forms. There is a Cartier divisor K_X on X giving rise to the canonical bundle, it is known as the *canonical divisor*.

Definition 3.14 For a proper birational morphism of smooth complex varieties $\alpha : Y' \longrightarrow Y$ we define the *discrepancy divisor* by $W := K_{Y'} - \alpha^* K_Y$, it is the divisor of the Jacobian determinant of α. Here α^* denotes the pull back map $\mathrm{Div}(Y) \to \mathrm{Div}(Y')$ and K_Y is the canonical divisor on Y.

Theorem 3.15 (Kontsevich) *Let $\alpha : Y' \longrightarrow Y$ be a proper birational morphism between smooth varieties and let $W := K_{Y'} - \alpha^* K_Y$ be the discrepancy divisor. Then*

$$\int_{J_\infty(Y)} F_D \, d\mu \ = \ \int_{J_\infty(Y')} F_{\alpha^* D + W} \, d\mu \, .$$

Definition 3.16 Let X be a variety with a normal isolated singularity of dimension n at $x \in X$, we say x is a *Gorenstein singularity* if there exists a nowhere vanishing n-form on a punctured neighbourhood of the singularity x. For example, isolated singularities of complete intersections are Gorenstein. Recall a codimension n subvariety of a projective space is a complete intersection if it is defined by exactly n homogeneous polynomials.

Every complex algebraic variety X with at worst Gorenstein singularities can be resolved of its singularities via a proper birational morphism $\varphi : Y \longrightarrow X$ where Y is a complex manifold. We say a resolution of singularities is *crepant* if the discrepancy divisor is rationally equivalent to zero.

Definition 3.17 Let X be a complex algebraic variety with at worst Gorenstein singularities. The motivic integral of X is defined to be the motivic integral of the pair (Y, W), where $\varphi : Y \longrightarrow X$ is any resolution of singularities for which the discrepancy divisor $W := K_{Y'} - \alpha^* K_Y$ has simple normal crossings. This definition is in fact independent of the choice of resolution (see [5]).

4 The Stringy E-function

4.1 Dolbeault Cohomology

Let X be a complex variety with coordinates $(z_1, ..., z_n)$, then we can split the tangent space up into two subspaces $T(X) = T^{(1,0)}(X) \oplus T^{(0,1)}(X)$. These two spaces are spanned respectively by

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

where $z_j = x_j + iy_j$. This gives a decomposition of its dual space, the cotangent space, into holomorphic and antiholomorphic forms

$$T^*(X) = T^{*(1,0)}(X) \oplus T^{*(0,1)}(X).$$

We then define the space of smooth differential forms of type (p, q) to be

$$T^{*(p,q)}(X) = (T^{*(1,0)}(X))^{\wedge p} \otimes (T^{*(0,1)}(X))^{\wedge q}.$$

A typical element will have the form

$$\omega = \sum_{\substack{1 \leq i_1 < .. < i_p \leq n \\ 1 \leq j_1 < .. < j_q \leq n}} f_{i_1,..,i_p,j_1,..,j_q} dz_{i_1} \wedge ... \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_q}$$

for a smooth $\mathbb{C}$-valued function $f_{i_1,..,i_p,j_1,..,j_q}$. Here $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$ are the duals to $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ respectively. The exterior derivative is well known in the case of holomorphic p-forms, that is differential forms of type $(p, 0)$, and is defined in exactly the same way for forms of type $(0, q)$. We get two boundary operators

$$\partial : T^{*(p,q)}(X) \longrightarrow T^{*(p+1,q)}(X) \quad \text{and} \quad \bar{\partial} : T^{*(p,q)}(X) \longrightarrow T^{*(p,q+1)}(X)$$

where $\bar{\partial}$ is defined by

$$\bar{\partial}\omega := \sum_{\substack{1\leq i_1<...<i_{p+1}\leq n \\ 1\leq j_1<..<j_q\leq n}} \bar{d}f_{i_1,..,i_p,j_1,..,j_q} \wedge dz_{i_1} \wedge ... \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_q}$$

where

$$\bar{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

These boundary operators satisfy $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. The cohomology from the complex

$$... \xrightarrow{\bar{\partial}} T^{*(p,q-1)}(X) \xrightarrow{\bar{\partial}} T^{*(p,q)}(X) \xrightarrow{\bar{\partial}} T^{*(p,q+1)}(X) \xrightarrow{\bar{\partial}} ...$$

defines the Dolbeault cohomology of X, the complex analogue of De Rham cohomology.

$$H^{p,q}_{\bar{\partial}}(X) := \frac{\{\omega \in T^{*(p,q)}(X) \ : \ \bar{\partial}\omega = 0\}}{\bar{\partial}(T^{*(p,q-1)}(X))}$$

We define

$$\Omega^p(X) := H^{p,0}_{\bar{\partial}}(X) = \{\omega \in T^{*(p,0)}(X) \ : \ \bar{\partial}\omega = 0\}$$

to be the holomorphic p-forms on X. Note that the holomorphic 0-forms are just holomorphic functions so $\Omega^0(X) = \mathcal{O}_X$.

4.1.1 Relation to Čech Cohomology

Let X be a complex variety, $\mathcal{U} = \{U_\alpha\}$ a cover of X and $\mathcal{F}$ a sheaf over X. We define a p-cochain to be a choice of element $f_{(\alpha_o,...,\alpha_p)} \in \mathcal{F}(U_{\alpha_o} \cap ... \cap U_{\alpha_p})$ for each possible choice of $(\alpha_o, ..., \alpha_o)$. We denote the group of p-cochains by $C^p(\mathcal{U}, \mathcal{F})$. We define a boundary homomorphism

$$\delta_p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$$

by

$$(\delta_p f)_{(\alpha_o,\ldots,\alpha_{p+1})} := \sum_{i=0}^{p+1}(-1)^i f_{(\alpha_o,..,\widehat{\alpha_i},..,\alpha_{p+1})}.$$

Since $\delta_p \circ \delta_{p+1} = 0$ this gives a complex,

$$C^0(\mathcal{U},\mathcal{F}) \stackrel{\delta_0}{\to} C^1(\mathcal{U},\mathcal{F}) \stackrel{\delta_1}{\to} C^2(\mathcal{U},\mathcal{F}) \stackrel{\delta_2}{\to} \ldots \stackrel{\delta_{p-1}}{\to} C^p(\mathcal{U},\mathcal{F}) \stackrel{\delta_p}{\to} C^{p+1}(\mathcal{U},\mathcal{F}) \stackrel{\delta_{p+1}}{\to} \ldots$$

which defines the Čech cohomology groups

$$H^p(\mathcal{U},\mathcal{F}) := \frac{\ker\delta_p}{\mathrm{Im}\delta_{p-1}}.$$

We then take an inductive limit over refinements of coverings of X to get

$$H^p(X,\mathcal{F}) := \lim_{\leftarrow} H^p(\mathcal{U},\mathcal{F}).$$

Theorem 4.1 (Dolbeault) *Let X be a complex manifold then*

$$H^q(X,\Omega^p) \cong H^{p,q}_{\bar{\partial}}(X).$$

Definition 4.2 The dimensions of the $(p,q)^{\text{th}}$-cohomology groups are called the *Hodge Numbers*

$$h^{p,q}(X) := \dim H^{p,q}_{\bar{\partial}}(X) = \dim H^q(X,\Omega^p)$$

where the last equality is due to Dolbeault's theorem.

4.2 Hodge-Deligne numbers and the stringy E-function

Definition 4.3 Let V be a vector space, a *Hodge Structure of weight $k \in \mathbb{Z}$* is a direct sum decomposition of the complexification of V, $V\otimes\mathbb{C}$ into graded pieces $V^{p,q}$ where $p+q=k$

$$V\otimes\mathbb{C} = \bigoplus_{p+q=k} V^{p,q} \quad \text{satisftying} \quad \overline{V^{p,q}} = V^{q,p}$$

Moreover if X is a smooth complex algebraic variety then we can view X as a topological space with the topology induced by that of the complex numbers. The following decomposition of the topological cohomology groups is called the Hodge Structure associated to X,

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X) = H^q(X,\Omega^p)$.

Deligne ([7], [8]) generalised the notion of classical Hodge structures to mixed Hodge structures and showed that the cohomology groups of a complex algebraic variety carry a mixed Hodge structure. This gives rise to the *Hodge-Deligne* numbers, which are a generalisation of Hodge numbers to any complex algebraic variety, not necessarily smooth or projective. For a formal definition of the Hodge-Deligne numbers see [5].

Definition 4.4 Let X be a complex algebraic variety of dimension n. We define $E(X) \in \mathbb{Z}[u,v]$, the the E-polynomial of X to be

$$E(X) \;:=\; \sum_{0\le p,q\le n} \sum_{0\le k\le 2n} (-1)^k h^{p,q}(H^k(X,\mathbb{C}))u^p v^q$$

where $h^{p,q}(H^k(X,\mathbb{C}))$ are the Hodge-Deligne numbers, which in the case of smooth projective varieties correspond to the Hodge numbers $h^{p,q}(X)$.

The E-polynomial defines a map $E : \mathcal{V}_{\mathcal{C}} \longrightarrow \mathbb{Z}[u,v]$ which is an additive invariant of algebraic varieties (for a proof see [6]) and thus factors through the Grothendieck ring of algebraic varieties, $K_o(\mathcal{V}_{\mathcal{C}})$. Hence by defining $E(\mathbb{L}^{-1}) \;:=\; (uv)^{-1}$ we can extend it to a map

$$K_o(\mathcal{V}_{\mathcal{C}})[\mathbb{L}^{-1}] \longrightarrow \mathbb{Z}[u,v,(uv)^{-1}].$$

Definition 4.5 Let X be a complex algebraic variety of dimension n with at worst Gorenstein singularities and $\pi : Y \longrightarrow X$ a resolution of singularities for which the discrepancy divisor W has only simple normal crossings. Then we define the *stringy E-function* of X to be

$$E_{\text{st}}(X) \;:=\; E\left(\int_{J_\infty(Y)} F_W d\mu \cdot \mathbb{L}^n\right).$$

4.3 Batyrev's Conjecture and proof by Kontsevich

Definition 4.6 Let M be a complex manifold, we say M is a *Hermitian manifold* if it has a smoothly varying Hermitian inner product defined on each holomorphic tangent space. If $(z_1, ..., z_n)$ are local coordinates then the Hermitian form will locally look like

$$\omega = \sum_{i,j} w_{ij}\, dz_i \wedge d\bar{z}_j.$$

If the Hermitian form is closed, that is $d\omega = 0$, then w is called a Kähler form and M is called a *Kähler manifold*. We say a Kähler manifold M is called a *Calabi-Yau manifold* if its canonical bundle is trivial, that is $K_M \sim 0$.

Example 4.7 The only type of Calabi-Yau manifold with complex dimension one is the torus corresponding to a complex elliptic curve. In two complex dimensions Calabi-Yau manifolds are categorised into $K3$ surfaces and abelian surfaces.

Batyrev conjectured that two birationally equivalent Calabi-Yau manifolds have the same Hodge numbers. Equivalently, given a complex projective Calabi-Yau variety X with at worst canonical Gorenstein singularities and two crepant resolutions of singularities of X, say X_1 and X_2, Batyrev's conjecture states both X_1 and X_2 have the same Hodge numbers.

$$\begin{array}{ccc} X_1 & & X_2 \\ & {}_{\pi_1}\searrow \quad \swarrow_{\pi_2} & \\ & X & \end{array}$$

This conjecture was motivated from his work in string theory, which predicts that as a manifold the universe is locally a product of $\mathbb{R}^4$ (space-time) with a 3-dimensional Calabi-Yau manifold.

Motivic integration was originally developed and used by Kontsevich to prove the result conjectured by Batyrev about Calabi-Yau varieties. In the case of a crepant resolution the discrepancy divisor is zero, thus the motivic integral of X is given by the motivic integral of the pair $(X_i, 0)$. Hence the stringy E-function associated to X is given by the E-polynomial associated to X_i, that is $E_{\text{st}}(X) = E(X_1) = E(X_2)$. Since X_1 and X_2 share the same E-polynomial, it is clear that the coefficients are equal. Both X_1 and X_2 are smooth and projective so these coefficients are their Hodge numbers. This shows any crepant resolutions of X will have the same Hodge numbers.

References

[1] J. Ax. The elementary theory of finite fields. *Ann. Math.*, 88(2):239–271, 1968.

[2] V. Batyrev. Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 1–32. World Sci. Publishing, River Edge, NJ, 1998.

[3] V. Batyrev. Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs. *Journal of European Math. Soc.*, 1:5–33, 1999.

[4] W. Chow. On equivalence classes of cycles in an algebraic variety. *Ann. Math.*, 64(3):450–479, 1956.

[5] A. Craw. An introduction to motivic integration. *Amer. Math. Soc*, 3:203–225, 2004.

[6] V. Danilov and A. Khovanskii. Newton polyhedra and an algorithm for computing Hodge-Deligne numbers. *Math. USSR Izvestiya*, 29:279–298, 1987.

[7] P. Deligne. La théorie de Hodge I. In *Actes du congrès international des Mathématiciens*, pages 425–430, Nice, 1970.

[8] P. Deligne. La théorie de Hodge II. *Publ. Math. I.H.E.S*, 40:5–57, 1971.

[9] J. Denef and F. Loeser. Germs of arcs on singular algebraic varieties and motivic integration. *Invent. Math.*, 135:201–232, 1999.

[10] J. Denef and F. Loeser. Motivic integration, quotient singularities and the McKay correspondence. *Compos. Math.*, 131:267–290, 2002.

[11] M. Fried and M. Jarden. *Field Arithmetic*. Springer-Verlag, 1986.

[12] W. Fulton. *Intersection Theory*. Springer, second edition, 1998.

[13] W. Fulton and R. MacPherson. Defining algebraic intersections. In *Algebraic Geometry*, volume 687 of *Lecture Notes in Mathematics*, pages 1–30, Tromsø, Norway, 1977. Springer-Verlag.

[14] H. Gillet and C. Soulé. Descent, motives and K-theory. *J. Reine Angew. Math.*, 478:127–176, 1996.

[15] T. Hales. What is motivic measure? *Bull. Amer. Math. Soc*, 42(2):119–135, 2005.

[16] M. Kontsevich. Lecture at Orsay, 1995.

[17] F. Loeser. Lecture at École Normale Supérieure, 2002.

[18] M. Reid. *Undergraduate Algebraic Geometry.* London Mathematical Society. Cambridge University Press, 1988.

[19] J.P. Serre. Faisceaux algébriques cohérents. *Ann. Math.*, 61(2):197–278, 1955.

[20] I. Shafarevich. *Algebraic Geometry II*, volume 35 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, 1996.