Character table of $SL_2(\mathbb{F}_q[z]/\langle z^2\rangle)$

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Abstract

Here we give a full description of the character table of $SL_2(\mathbb{F}_q[z]/\langle z^2\rangle)$.

0 Introduction, Facts About $\mathbb{F}_q$

For all of this work we will take $\mathbb{F}_q$ to be the finite field of order q where q is prime and $q > 2$. We will discuss a few facts about $\mathbb{F}_q$, firstly we have that the additive group of $\mathbb{F}_{q^n}$ is cyclic generated by 1 and that $(\mathbb{F}^*_{q^n}, \cdot)$ the multiplicative group is cyclic for any $n \in \mathbb{N}$, in the specific case where $n = 1$ we fix a cyclic generator of $\mathbb{F}^*_q$ and call it $\zeta \in \mathbb{F}^*_q$. We define $x \in \mathbb{F}^*_q$ to be square if $\exists y \in \mathbb{F}^*_q$ s.t. $x = y^2$ and call x square free otherwise. It is easy to see that the square elements form a subgroup of the multiplicative group of $\mathbb{F}^*_q$ and that if we multiply any two square free elements the result is a square element. If we consider $x \in \mathbb{F}^*_q$ a square free element then we have $\sqrt{x} \notin \mathbb{F}_q$ and this gives rise to a basis $1, \sqrt{x}$ of the quadratic extension of $\mathbb{F}_q$, $\mathbb{F}_{q^2}$ and so we can view $\mathbb{F}_q$ as a subfield of $\mathbb{F}_{q^2}$. It is straightforward to see that the number of square and square free elements in $\mathbb{F}^*_q$ are equal and hence there are $\frac{q-1}{2}$ of each, if we fix a square free element $\varepsilon \in \mathbb{F}^*_q$ we see that all the square elements are given by $\{\alpha^2 | 1 \leq \alpha \leq \frac{q-1}{2}\}$ and the square free elements are given by $\{\alpha^2\varepsilon | 1 \leq \alpha \leq \frac{q-1}{2}\}$. Any further results required about $\mathbb{F}_q$ will be mentioned when used later on, we are now ready to begin.

1 The Conjugacy Classes of $SL_2(\mathbb{F}_q[z]/\langle z^2\rangle)$

For this chapter let $G = SL_2(\mathbb{F}_q[z]/\langle z^2\rangle)$. In order to find the conjugacy classes of G it is first important to note that G is a semi direct product of groups $SL_2(\mathbb{F}_q)$ and an abelain group N isomorphic to $sl_2(\mathbb{F}_q)$ under addition due to the following homomorphism:

$$\phi : G \to SL_2(\mathbb{F}_q)$$
$$\phi\left(\begin{pmatrix} x_1 + a_1 z & x_2 + a_2 z \\ x_3 + a_3 z & x_4 + a_4 z \end{pmatrix}\right) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

ϕ acts as the identity on $SL_2(\mathbb{F}_q) \leq G$ and has kernel

$$Ker(\phi) = \left\{\begin{pmatrix} x_1 + a_1 z & x_2 + a_2 z \\ x_3 + a_3 z & x_4 + a_4 z \end{pmatrix} \in G \,\middle|\, \phi\left(\begin{pmatrix} x_1 + a_1 z & x_2 + a_2 z \\ x_3 + a_3 z & x_4 + a_4 z \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix} 1 + a_1 z & a_2 z \\ a_3 z & 1 + a_4 z \end{pmatrix} \in G\right\}$$

$$= \left\{\begin{pmatrix} 1 + a_1 z & a_2 z \\ a_3 z & 1 + a_4 z \end{pmatrix} \,\middle|\, \begin{vmatrix} 1 + a_1 z & a_2 z \\ a_3 z & 1 + a_4 z \end{vmatrix} = 1, a_i \in \mathbb{F}_q\right\}$$

$$= \left\{\begin{pmatrix} 1 + a_1 z & a_2 z \\ a_3 z & 1 + a_4 z \end{pmatrix} \,\middle|\, 1 + (a_1 + a_4)z + (a_1 a_4 + a_2 a_3)z^2 = 1, a_i \in \mathbb{F}_q\right\}$$

But since we are over $\mathbb{F}_q[z]/\langle z^2\rangle$ this becomes

$$= \left\{\begin{pmatrix} 1 + a_1 z & a_2 z \\ a_3 z & 1 + a_4 z \end{pmatrix} \,\middle|\, 1 + (a_1 + a_4)z = 1, a_i \in \mathbb{F}_q\right\}$$

and hence

$$= \left\{ \left(\begin{array}{cc} 1+a_1z & a_2z \\ a_3z & 1+a_4z \end{array} \right) \middle| (a_1+a_4)=0, a_i \in \mathbb{F}_q \right\}$$

$$= \left\{ Id + \left(\begin{array}{cc} a_1 & a_2 \\ a_3 & -a_1 \end{array} \right) z \middle| a_i \in \mathbb{F}_q \right\}$$

$$= \{ Id + \mathbf{x}z | \mathbf{x} \in sl_2(\mathbb{F}_q) \}$$

This homomorphism means that G is a semi direct product of $SL_2(\mathbb{F}_q)$ (which we shall call H from now) and the $Ker(\phi)$. Multiplication in this kernel behaves very nicely, due to z^2 being zero, as follows, if $Id+\mathbf{x}z, Id+\mathbf{y}z \in Ker(\phi)$ then $(Id+\mathbf{x}z)\cdot(Id+\mathbf{y}z) = Id+(\mathbf{x}+\mathbf{y})z$. This shows that the kernel is abelian and there is an obvious isomorphism to the additive group of $sl_2(\mathbb{F}_q)$. The $Ker(\phi)$ will be called N from now on, and G is thus the semi direct product of N and H. We can easily determine that $|N| = q^3$ and less obviously $|H| = q(q^2-1)$ together giving $|G| = |H|\cdot|N| = q^4(q^2-1)$. The order of H can be determined by considering $\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in H$,the first column of a matrix in H just needs to be any nonzero vector in $\mathbb{F}_q \times \mathbb{F}_q$ so q^2-1 choices, the second column of a matrix needs to ensure $ad-bc=1$ which gives q choices for the second column, thus a total of $q(q^2-1)$

In order to define the conjugacy class representatives of G we shall take the following route, first we will find the conjugacy class reps of N in G which will only split N into conjugacy classes as N is normal in G. Then we will find the remaining conjugacy classes of G by looking at the conjugacy classes of H and and considering the semidirect product.

As G is a semidirectproduct of N and H, finding the conjugacy classes of N in G comes down to finding representatives of the orbits that partion N by the conjugation action of G on elements in N. So if $n \in N$ and $g \in G$ then g can be uniquely written as $g = h \cdot \bar{n}$ for some $h \in H$ and $\bar{n} \in N$ and $gng^{-1} = h\bar{n}n\bar{n}^{-1}h^{-1}$ but since N is an abelian group this is simply hnh^{-1}. So we only need to consider the action of H on N to find its conjugacy classes. Looking more closely at this action we see if $n = Id+\mathbf{x}z \in N$ (i.e. $\mathbf{x} \in sl_2(\mathbb{F}_q)$) and $\mathbf{y} \in H$ then $\mathbf{y}n\mathbf{y}^{-1} = \mathbf{y}(Id+\mathbf{x}z)\mathbf{y}^{-1} = Id+\mathbf{y}\mathbf{x}\mathbf{y}^{-1}z$, so is simply a case of finding orbit representatives of H acting on $sl_2(\mathbb{F}_q)$ by conjugation. It is important to note that any two elements in the same orbit must have the same eigenvalues as they will be similar matrices, this means if two matrices have different eigenvalues they must represent different orbits, this leads to our first set of conjugacy class representatives:

$$\left\{ Id + \left(\begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) z \middle| \alpha \in \left\{ 1, 2, \cdots, \frac{q-1}{2} \right\} \right\}$$

Each one clearly has different eigenvalues so they give distinct conjugacy classes with stabilisers by the action of H of order $q-1$ as follows:

$$Stab_H\left(Id + \left(\begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) z \right) = \left\{ h \in H \middle| h\left(Id + \left(\begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) z \right) h^{-1} = Id + \left(\begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) z \right\}$$

$$= \left\{ h \in H \middle| h \left(\begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) h^{-1} = \left(\begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) \right\}$$

$$= \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in H \middle| \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) \left(\begin{array}{cc} d & -b \\ -c & a \end{array} \right) = \left(\begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) \right\}$$

$$= \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in H \middle| \left(\begin{array}{cc} (ad+bc)\alpha & -2ab\alpha \\ +2cd\alpha & -(ad+bc)\alpha \end{array} \right) = \left(\begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) \right\}$$

So we have a series of 4 equations that all need to be simulteanously satisfied

$$-2ab\alpha = 0 \tag{1}$$

$$2cd\alpha = 0 \tag{2}$$

$$ad - bc = 1 \tag{3}$$

$$ad + bc = 1 \tag{4}$$

and the number of solutions to these equations gives the order of the stabiliser. Notice (1)$\Rightarrow$ either $a=0$ or $b=0$, if we assume $a=0$ then (3) and (2) together imply $d=0$ but this gives $bc=1$ and $bc=-1$ so is a contradiction $\therefore$ $b=0$ by (1). Then (2) and (3) together imply $c=0$ leaving $ad=1$ in (3) and (4), for which there are $q-1$ solutions.

This means that we have $q-1$ elements in H and hence $q^3(q-1)$ elements in G that fix our representative (which we will call x for brevity) as if $h \in H$ fixes x then $\forall n \in N$ $hnx(hn)^{-1} = hnxn^{-1}h^{-1} = hxh^{-1} = x$ and $|N| = q^3$. Hence by the orbit-stabiliser theorem the size of each conjugacy class associated to one of the representatives above is:

$$\left|Orb_G\left(Id + \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} z\right)\right| = \frac{|G|}{\left|Stab_G\left(Id + \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} z\right)\right|} = \frac{q^4(q^2-1)}{q^3(q-1)} = q(q+1)$$

Another set of conjugacy class representatives comes from looking at square free elements of $\mathbb{F}_q$:

$$\left\{Id + \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} z \,|\alpha \in \mathbb{F}_q \setminus A\right\}$$

Where $A = \{x^2 | x \in \mathbb{F}_q\}$. These matrices have eigenvalues $\pm\sqrt{\alpha} \notin \mathbb{F}_q$ hence give distinct conjugacy classes to those found above, by considering $\sqrt{\alpha}$ over the quadratic extension and $|\mathbb{F}_q \setminus A| = \frac{q-1}{2}$ tells us we have $\frac{q-1}{2}$ conjugacy classes of this type. Now to determine the size of these conjugacy classes as above,

$$Stab_H\left(Id + \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} z\right) = \left\{h \in H \,\middle|\, h\left(Id + \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} z\right)h^{-1} = Id + \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} z\right\}$$

$$= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H \,\middle|\, \begin{pmatrix} db - ac\alpha & a^2\alpha - b^2 \\ d^2 - c^2\alpha & -db + ca\alpha \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}\right\}$$

Again this results in 4 equations that need to be solved simultaneously:

$$db - ac\alpha = 0 \tag{5}$$

$$d^2 - c^2\alpha = 1 \tag{6}$$

$$a^2\alpha - b^2 = \alpha \tag{7}$$

$$ad - bc = 1 \tag{8}$$

assuming $b \neq 0$ (5) gives $d = \frac{ac\alpha}{b}$ putting this in equation (8) yields $(a^2\alpha - b^2)c = b$ and using (7) we get $c = \frac{b}{\alpha}$ and thus we have $d = a$, if we assume $b = 0$ this implies $c = 0$ and we are looking at solutions of (7) so the number of solutions simply comes from the total number of solutions to (7), this question can be answered by the following theorem:

Theorem 1.1. *([1] 6.26. page 282) Let f be a nondegenerate quadratic form over $\mathbb{F}_q$,q odd, in an even number n of indeterminates. Then for $b \in \mathbb{F}_q$ the number of solutions of the equation $f(x_1, \cdots, x_n) = b$ in $\mathbb{F}_q^n$ is*

$$q^{n-1} + \upsilon(b)q^{(n-2)/2}\psi((-1)^{n/2}\Delta)$$

where ψ is the quadratic character of $\mathbb{F}_q$, $\Delta = det(f)$ and υ is the integer valued function on $\mathbb{F}_q$ s.t $\upsilon(b) = -1$ for $b \in \mathbb{F}_{q^*}$ and $\upsilon(0) = q-1$. (the quadratic character ψ is defined on $\mathbb{F}_{q^*}$ as $\psi(c) = 1$ if c is a square of an element in $\mathbb{F}_{q^*}$ and $\psi(c) = -1$ otherwise)

We can view (7) as the non degenerate quadratic form over $\mathbb{F}_q$:

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \alpha$$

Hence the number of solutions is simply $q + \upsilon(\alpha)\psi(\alpha) = q - \psi(\alpha) = q + 1$ so

$$\left|Stab_H\left(Id + \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} z\right)\right| = q + 1$$

and hence

$$\left|Stab_G\left(Id + \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} z\right)\right| = q^3(q+1)$$

and hence the size of the conjugcy class is:

$$\left|Orb_G\left(Id + \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} z\right)\right| = \frac{|G|}{\left|Stab_G\left(Id + \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} z\right)\right|} = q(q-1)$$

All the remaining conjugacy classes in H come from matrices with 0 eigenvalues, the following three group elements are the remaining conjugacy class representatives; $\mathbf{a} = Id + \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix} z$, $\mathbf{b} = Id + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z$ and $\mathbf{c} = Id + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} z$, where ε is some fixed square free element. It is easy to see that non of these are conjugate to the previous conjugacy class representatives as the conjugation action of N preserves rank, this argument also shows that **c** is not conjugate to **a** and **b**. So it is left to show that **a** is not conjugate to **b** and to find the sizes of the classes that these elements represent. First we will show **a** is not conjugate to **b**, suppose they are then $\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q)$ s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which as before leads to a set of simultaneous equations which are $c = 0$, $a = d\varepsilon$ and $ad = 1$(this comes from the determinant being 1 and c=0) combining the last two gives $a^2/\varepsilon = 1$ which implies $\varepsilon = a^2$ which is a contradiction as ε is square free, thus they are not conjugate.

Now to find the size of the conjugacy classes, for **c** the size is one as it is the identity element of the group, to find the size of the class associated to **a** we will use a similar method to that above, and exactly the same method applies in order to find the conjugacy class size of **b** yielding the same result.

$$Stab_H\left(Id + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z\right) = \left\{h \in H \middle| h\left(Id + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z\right)h^{-1} = Id + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z\right\}$$
$$= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H \middle| \begin{pmatrix} -ac & a^2 \\ -c^2 & +ca \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$$

Again this is simply a set of simultaneous equations, $c = 0$, $a^2 = 1$ and $ad = 1$, so there are two choices for a each of which uniquely defines d, for each choice of a b can take any value hence:

$$\left|Stab_H\left(Id + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z\right)\right| = 2q$$

$$\therefore \left|Orb_G\left(Id + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z\right)\right| = \frac{|G|}{\left|Stab_G\left(Id + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z\right)\right|} = \frac{q^4(q^2-1)}{2q^4} = \frac{q^2-1}{2}$$

These are all the conjugacy classes in N as if we total the number of elements we have covered we have:

$$q(q+1)\frac{q-1}{2} + q(q-1)\frac{q-1}{2} + 2\frac{q^2-1}{2} + 1 = q^3 = |N|$$

To construct the remaining conjugacy classes we will need the conjugacy classes of $SL_2(\mathbb{F}_q)$, the following table is taken from [2] (page 71):

	$Representatives$	$No.elementsinClass$	$No.classes$
(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	1
(2)	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1	1
(3)	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\frac{q^2-1}{2}$	1
(4)	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\frac{q^2-1}{2}$	1
(5)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\frac{q^2-1}{2}$	1
(6)	$\begin{pmatrix} -1 & \varepsilon \\ 0 & -1 \end{pmatrix}$	$\frac{q^2-1}{2}$	1
(7)	$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, x \neq \pm 1$	$q(q+1)$	$\frac{q-3}{2}$
(8)	$\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}, x \neq \pm 1$	$q(q-1)$	$\frac{q-1}{2}$

Now it is important to notice that (7) and (8) can be expressed in the form of cylcic elements which will be of use later when discussing representations in order to construct the character table. Firstly (7) can be re-expressed as;

$$\left\{\left(\begin{array}{cc}\zeta^n & 0\\ 0 & \zeta^{-n}\end{array}\right)\middle| 1\leq n\leq (q-3)/2, \zeta \textit{ generates the cylcic group } \mathbb{F}_q^*\right\}$$

This is possible as $\left(\begin{array}{cc}x & 0\\ 0 & x^{-1}\end{array}\right)$ is conjugate to $\left(\begin{array}{cc}x^{-1} & 0\\ 0 & x\end{array}\right)$. Now looking at (8),let L be the group $\left\{\left(\begin{array}{cc}x & y\\ y\varepsilon & x\end{array}\right)\in H\right\}$ can view this as a subgroup of the multiplicative group of $\mathbb{F}_{q^2}$,the quadratic extension of $\mathbb{F}_q$. This can be done using the following homorphism $\varphi : L\rightarrow \mathbb{F}_{q^2}^*$, $\left(\begin{array}{cc}x & y\\ y\varepsilon & x\end{array}\right)\mapsto x+y\sqrt{\varepsilon}$. This homomorphism is injective as 1 and $\sqrt{\varepsilon}$ form a basis of $\mathbb{F}_{q^2}^*$ over $\mathbb{F}_q$ and hence $L\cong Im(\varphi)\leq \mathbb{F}_{q^2}^*$. Now as $\mathbb{F}_{q^2}^*$ is cyclic and we have a subgroup of a cyclic group we can conclude that L is cyclic, so is generated by some $\xi\in L$. $|L| = q+1$ as L is simply

$$\left\{\left(\begin{array}{cc}x & y\\ y\varepsilon & x\end{array}\right)\middle|\left(\begin{array}{cc}x & y\end{array}\right)\left(\begin{array}{cc}1 & 0\\ 0 & -\varepsilon\end{array}\right)\left(\begin{array}{c}x\\ y\end{array}\right)=1\right\}$$

so by Theorem 2.1 we get $|L| = q+\upsilon(1)\psi(\varepsilon) = q-\psi(\varepsilon) = q+1$. So the conjugacy representatives of (8) can be viewed as the following;

$$\left\{\xi^n \middle| 1\leq n\leq \frac{q-1}{2}\right\}$$

as $\left(\begin{array}{cc}x & y\\ \varepsilon y & x\end{array}\right)$ is conjugate to $\left(\begin{array}{cc}x & -y\\ -\varepsilon y & x\end{array}\right)=\left(\begin{array}{cc}x & y\\ \varepsilon y & x\end{array}\right)^{-1}$ and Id and -Id are already accounted for.

Also to save some time later we can change the conjugacy class representatives of (3),(4),(5) and (6). Firstly (5) and (6) will now have representatives $\left(\begin{array}{cc}-1 & 0\\ -1 & -1\end{array}\right)$ and $\left(\begin{array}{cc}-1 & 0\\ -\varepsilon & -1\end{array}\right)$ respectively, these can easily be checked to be conjugates by conjugating them by $\left(\begin{array}{cc}0 & -1\\ 1 & 0\end{array}\right)$. To change the representatives of (3) and (4) we will need a short lemma.

Lemma 1.2. *(i)If $q\equiv 1(mod4)$ and $b\in\mathbb{F}_q^*$ is a square (square free) element then $-b$ is a square (square free) element .*

(ii)If $q\equiv 3(mod4)$ and $d\in\mathbb{F}_q^*$ is a square (square free) element then $-d$ is square free (square).

Proof. (i) if $q\equiv 1(mod4)$ then $\exists a_1, a_2\in\mathbb{F}_q^*$ s.t. $a_1^2+a_2^2 = q$ and since the square elements form a multiplicative subgroup of $\mathbb{F}_q^*$ generated by ζ^2 we get that $b = xa_1^2$ where $x = y^2$, where $y\in\mathbb{F}_q^*$. So we have $b+y^2a_2^2 = y^2a_1^2+y^2a_2^2 = y^2q = 0$, hence $-b = y^2a_2^2 = (ya_2)^2$. If b is square free then $b = \varepsilon a_1^2$ for ε sqaure free and so we get $\varepsilon a_1^2+\varepsilon a_2^2 = \varepsilon q = 0$ which gives us $-b = -\varepsilon a_1^2 = a_2^2\varepsilon$ and hence $-b$ is square free.

(ii) If $q\equiv 3(mod4)$ then $q = 1+(q-1)$, $(q-1)$ is square free, as suppose not, then we have $x\in\mathbb{F}_q^*$ with $x^2 = (q-1) = -1$ thus $|x| = 4$ in the group F_q^* but this contradicts Lagrange's Theorem as $|F_q^*| = q-1\equiv 2(mod4)$. Now given any square element $b\in\mathbb{F}_q^*$ we have $b\cdot 1+b(q-1) = 0$ so $-b = b(q-1)$ and $b(q-1)$ is square free. Now if b is square free we have $-b = b(q-1)$ but since both b and $q-1$ are square free we get that $b(q-1) = -b$ is a square element. □

Now if we look at the following:

$$\left(\begin{array}{cc}0 & -\zeta^{-n}\\ \zeta^n & 0\end{array}\right)\left(\begin{array}{cc}1 & x\\ 0 & 1\end{array}\right)\left(\begin{array}{cc}0 & \zeta^{-n}\\ -\zeta^n & 0\end{array}\right)=\left(\begin{array}{cc}1 & 0\\ -x\zeta^{2n} & 1\end{array}\right)$$

So if $q\equiv 1(mod4)$ so -1 is a square element we get that $\left(\begin{array}{cc}1 & 1\\ 0 & 1\end{array}\right)$ is conjugate to $\left(\begin{array}{cc}1 & 0\\ 1 & 1\end{array}\right)$ and $\left(\begin{array}{cc}1 & \varepsilon\\ 0 & 1\end{array}\right)$ is conjugate to $\left(\begin{array}{cc}1 & 0\\ \varepsilon & 1\end{array}\right)$. Now if $q\equiv 3(mod4)$ since -1 is square free we get $\left(\begin{array}{cc}1 & 1\\ 0 & 1\end{array}\right)$ is conjugate to $\left(\begin{array}{cc}1 & 0\\ \varepsilon & 1\end{array}\right)$ and $\left(\begin{array}{cc}1 & \varepsilon\\ 0 & 1\end{array}\right)$ is conjugate to $\left(\begin{array}{cc}1 & 0\\ 1 & 1\end{array}\right)$. Thus we will use the following four conjugacy class representatives for (3),(4),(5) and (6) instead; $\pm\left(\begin{array}{cc}1 & 0\\ 1 & 1\end{array}\right)$ and $\pm\left(\begin{array}{cc}1 & 0\\ \varepsilon & 1\end{array}\right)$

The conjugation in G works as follows

$$\mathbf{C}(Id+\mathbf{c}z)\mathbf{B}(Id+\mathbf{b}z)(\mathbf{C}(Id+\mathbf{c}z))^{-1}$$

$$=\mathbf{C}(\mathbf{B}+(\mathbf{cB}-\mathbf{Bc})z+\mathbf{Bb}z)\mathbf{C}^{-1}$$

$$= \mathbf{CBC}^{-1}(Id + (\mathbf{CB}^{-1}\mathbf{cBC}^{-1} - \mathbf{CcC}^{-1} + \mathbf{CbC}^{-1})z)$$

So an element of the form $\mathbf{B}(Id+\mathbf{b}z)$ can only be conjugate to an element of the form $\mathbf{A}(Id+\mathbf{a}z)$ if $\mathbf{B}$ is conjugate to $\mathbf{A}$ in H. So given $\mathbf{A}(Id+\mathbf{a}z) \in G$ then it is conjugate to $\mathbf{B}(Id+\mathbf{a}z)$ for a unique $\mathbf{B}$ from the table above and some $\mathbf{b} \in sl_2(\mathbb{F}_q)$. So to define the rest of the conjugacy classes of G we have to find all of the $\mathbf{b}_i \in sl_2(\mathbb{F}_q)$ for each $\mathbf{B}$ in the table above such that $\mathbf{B}(Id+\mathbf{b}_i z)$ and $\mathbf{B}(Id+\mathbf{b}_j z)$ are not conjugate for $i \neq j$ in G. We will work through the above table from (1) to (8), (1) is simply the case in N which was dealt with at the start, the same procedure also works for (2), this is all due to the fact that Id,-Id commute with everything.

Lemma 1.3. *for (3)* $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *the conjugacy classes are:*

$$\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \left(Id + \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} z \right) | b \in \mathbb{F}_q \right\}$$

Proof. suppose that two distinct elements are conjugate then $\exists \mathbf{C}(Id+\mathbf{c}z) \in G$ s.t.

$$\mathbf{C}(Id+\mathbf{c}z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \left(Id + \begin{pmatrix} 1 & b_1 \\ 0 & -1 \end{pmatrix} z \right) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \left(Id + \begin{pmatrix} 1 & b_2 \\ 0 & -1 \end{pmatrix} z \right) \mathbf{C}(Id+\mathbf{c}z)$$

where $b_1 \neq b_2$. Comparing the non z coefficients means that $\mathbf{C}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{C}$, this implies $\mathbf{C}$ is of the form $\begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix}$ for any $c \in \mathbb{F}_q$. If we assume $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and letting $\gamma = \begin{pmatrix} x & y \\ w & -x \end{pmatrix}$ comparing the z coefficients in the equation above gives:

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} x & y \\ w & -x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & b_1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & b_2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} x & y \\ w & -x \end{pmatrix}$$

which gives

$$\begin{pmatrix} x+y+1 & y+b_1 \\ cx+w+cy-x+1 & cy-x+(1+c)b_1-1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+b_2c+x & b_2+y \\ 1+(b_2-1)c+(1+c)x+w & b_2-1+(1+c)y-x \end{pmatrix} \tag{9}$$

But this implies $b_1 = b_2$ due to the top right entries which is a contradiction. There is an almost identical argument if we assume $\mathbf{C} = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$ thus showing non of the elements are conjugate, to finish the claim we need to show we have covered all the possible elements $\mathbf{A}(Id+\mathbf{a}z)$ s.t. $\mathbf{A}$ is conjugate to $\mathbf{B}$. This means we need to have $q^3 \cdot \frac{q^2-1}{2}$ as the total number of elements from all the conjugacy class representatives in the claim. This will follow by looking at the order of the stabiliser of the conjugation action. The order of the stabiliser is the number of solutions to (9) where $b_1 = b_2$ plus the number of solutions to same system but where $\mathbf{C} = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$. From (9) we get 2 simultaneous equations:

$$y = bc \tag{10}$$

$$x = \frac{2c+cy-bc}{2} \tag{11}$$

Solving the same system where $\mathbf{C} = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$ leads to the following stabiliser:

$$Stab_G\left(\mathbf{B}\left(Id + \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} z\right)\right) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix} \left(Id + \begin{pmatrix} \pm\frac{c(2+bc-b)}{2} & bc \\ w & \mp\frac{c(2+bc-b)}{2} \end{pmatrix} z \right) \right\}$$

Where c and w are any elements of $\mathbb{F}_q$. Hence

$$\left| Stab_G\left(\mathbf{B}\left(Id + \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} z\right)\right)\right| = 2q^2$$

and so:

$$\left|Orb_G\left(\mathbf{B}\left(Id+\begin{pmatrix}1 & b\\ 0 & -1\end{pmatrix}z\right)\right)\right| = \frac{q^4(q^2-1)}{2q^2} = \frac{q^2(q^2-1)}{2}$$

And since we have q of such conjugacy classes we have covered the required number of elements by the disjoint conjugacy classes ☐

Lemma 1.4. *For (4),(5) and (6) from the table there are identical results.*

Proof. Simply follow the previous method. ☐

Lemma 1.5. *For (7) we have* $\mathbf{B} = \begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}, \beta \neq \pm 1$ *the conjugacy class representatives are:*

$$\left\{\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\left(Id+\begin{pmatrix}b & 0\\ 0 & -b\end{pmatrix}z\right)|b\in\mathbb{F}_q\right\}$$

Proof. As of the proof for previous conjugacy classes, first show no two of these representatives are conjugate and then that they cover the $\frac{q^4(q+1)}{2}$ elements of the form $\mathbf{A}(Id+\mathbf{a}z)$ for any $\mathbf{a}\in sl_2(\mathbb{F}_q)$ and $\mathbf{A}$ conjugate to $\mathbf{B}$. Now suppose $\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\left(Id+\begin{pmatrix}b_1 & 0\\ 0 & -b_1\end{pmatrix}z\right)$ is conjugate to $\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\left(Id+\begin{pmatrix}b_2 & 0\\ 0 & -b_2\end{pmatrix}z\right)$ by $\mathbf{C}(Id+\mathbf{c}z)$ for $b_1\neq b_2$ comparing non z coefficeints in the equation given by the fact these elements are conjugate implies $\mathbf{C}$ satisfies $\mathbf{CB}=\mathbf{BC}$ which implies $\mathbf{C}=\begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}$ for some $c\in\mathbb{F}_q^*$. Comparing the z coefficients gives the following equation

$$\begin{aligned}\mathbf{C}&\left(\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\begin{pmatrix}b_1 & 0\\ 0 & -b_1\end{pmatrix}+\begin{pmatrix}x & y\\ w & -x\end{pmatrix}\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\right)\\ &=\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\begin{pmatrix}b_1 & 0\\ 0 & -b_1\end{pmatrix}\mathbf{C}+\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\mathbf{C}\begin{pmatrix}x & y\\ w & -x\end{pmatrix}\end{aligned}$$

Where $\mathbf{c}=\begin{pmatrix}x & y\\ w & -x\end{pmatrix}$ If $\mathbf{C}=\begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}$ we get:

$$\begin{pmatrix}c\beta b_1+cx\beta & cy\beta^{-1}\\ c^{-1}w\beta & -b_1\beta^{-1}c^{-1}-xc^{-1}\beta^{-1}\end{pmatrix}=\begin{pmatrix}\beta b_2c+xc\beta & cy\beta\\ wc^{-1}\beta^{-1} & -b_2c^{-1}\beta^{-1}-xc^{-1}\beta^{-1}\end{pmatrix} \tag{12}$$

But this implies by the top left entry that $b_1=b_2$ so contradiction, hence the elements cannot be conjugate. Now to look at the size of the conjugacy classes by finding the size of the stabiliser, but this is simply $q-1$ times the number of solutions to (12) where $b=b_1=b_2$ (the $q-1$ comes from the options for $\mathbf{C}$). The system leads to four simultaneous equations:

$$c\beta b+cx\beta=\beta bc+xc\beta \tag{13}$$

$$cy\beta^{-1}=cy\beta \tag{14}$$

$$c^{-1}w\beta=wc^{-1}\beta^{-1} \tag{15}$$

$$-b\beta^{-1}c^{-1}-xc^{-1}\beta^{-1}=-bc^{-1}\beta^{-1}-xc^{-1}\beta^{-1} \tag{16}$$

These equtions have very simple solutions (14) and (15) imply $y=w=0$ as $\beta\neq\pm 1 or 0$ and $c\neq 0$, x can be anything in $\mathbb{F}_q$ hence the size of the stabiliser is $(q-1)q$. So the size of an orbit of one of our conjugacy class representatives is $\frac{q^3(q+1)}{2}$ and for each fixed $\mathbf{B}$ we have q of such conjugacy classes so covering $\frac{q^4(q+1)}{2}$ as required. ☐

Lemma 1.6. *For (8) we have* $\mathbf{B}=\begin{pmatrix}x & y\\ \varepsilon y & x\end{pmatrix}, x\neq\pm 1$ *and* ε *is square free the conjugacy class representatives are:*

$$\left\{\begin{pmatrix}x & y\\ \varepsilon y & x\end{pmatrix}\left(Id+\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}z\right)|b\in\mathbb{F}_q\right\}$$

where (x,y) *and* $(x,-y)$ *are the same conjugacy class representative.*

Proof. Following a similar route to the previous claims first we show that the conjugacy class representatives are not conjugate to each other, we already know that if they have different leading terms that they cannot be conjugate due to the conjugacy classes of $SL_2(\mathbb{F}_q)$, first we will look at the form of elements we can achieve by conjugation in N, so we are looking at elements of the form

$$(Id + \mathbf{a}z)\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right)(Id - \mathbf{a}z) \tag{17}$$

where $\mathbf{a} = \begin{pmatrix} \theta & \varphi \\ \tau & -\theta \end{pmatrix}$ putting this into (17) gives us:

$$\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} (x(\varphi\varepsilon - \tau) + 2\theta\varepsilon y)y & 2\theta yx + y^2(\varphi\varepsilon - \tau) + b \\ -\varepsilon(2\theta yx + y^2(\varphi\varepsilon - \tau)) & -(x(\varphi\varepsilon - \tau) + 2\theta\varepsilon y)y \end{pmatrix} z\right)$$

If we let $\tilde{c} = y(\varphi\varepsilon - \tau)$ and $\tilde{d} = 2\theta y$ and noting $(x^2 - y^2\varepsilon) = 1$ as $\begin{pmatrix} x & y \\ \alpha y & x \end{pmatrix} \in H$ then we have

$$\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} x\tilde{c} + \varepsilon y\tilde{d} & x\tilde{d} + y\tilde{c} + b \\ -\varepsilon(x\tilde{d} + y\tilde{c}) & -x\tilde{c} - \varepsilon y\tilde{d} \end{pmatrix} z\right) \tag{18}$$

Now if we suppose that $\exists g = \mathbf{A}(Id + \mathbf{a}z) \in G$ s.t.

$$g\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} z\right)g^{-1} = \begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} 0 & \hat{b} \\ 0 & 0 \end{pmatrix} z\right) \tag{19}$$

for some $b \neq \hat{b}$ then $\mathbf{A}\mathbf{B}\mathbf{A}^{-1} = \mathbf{B}$ implies $\mathbf{A} \in H$ is of the form $\begin{pmatrix} a & \frac{c}{\varepsilon} \\ c & a \end{pmatrix}$ (of which there are $q+1$ of by theorem 1.1). Then

$$g\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} z\right)g^{-1}$$

$$= \mathbf{A}\,(Id + \mathbf{a}z)\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} z\right)(Id - \mathbf{a}z)\,\mathbf{A}^{-1}$$

Letting $\mathbf{a} = \begin{pmatrix} \theta & \varphi \\ \zeta & -\theta \end{pmatrix}$ and using the substitutions as in equation (18) we have:

$$= \mathbf{A}\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} x\tilde{c} + \varepsilon y\tilde{d} & x\tilde{d} + y\tilde{c} + b \\ -\varepsilon(x\tilde{d} + y\tilde{c}) & -x\tilde{c} - \varepsilon y\tilde{d} \end{pmatrix} z\right)\mathbf{A}^{-1}$$

which

$$= \begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \mathbf{A}\begin{pmatrix} x\tilde{c} + \varepsilon y\tilde{d} & x\tilde{d} + y\tilde{c} + b \\ -\varepsilon(x\tilde{d} + y\tilde{c}) & -x\tilde{c} - \varepsilon y\tilde{d} \end{pmatrix}\mathbf{A}^{-1} z\right)$$

as $\mathbf{A}\mathbf{B}\mathbf{A}^{-1} = \mathbf{B}$. Substituting $\mathbf{A} = \begin{pmatrix} a & \frac{c}{\varepsilon} \\ c & a \end{pmatrix}$, $r = x\tilde{c} + \varepsilon y\tilde{d}$ and $p = x\tilde{d} + y\tilde{c}$ we get:

$$= \begin{pmatrix} x & y \\ \alpha y & x \end{pmatrix}\left(Id + \begin{pmatrix} a^2r - 2cap - acb + \frac{rc^2}{\varepsilon} & \frac{c^2p}{\varepsilon} + a^2p - 2\frac{car}{\varepsilon} + a^2b \\ 2cra - a^2\varepsilon p - c^2p - c^2b & -a^2r + 2cap + acb - \frac{rc^2}{\varepsilon} \end{pmatrix} z\right)$$

And two final substitutions $f = a^2r - 2cap - acb + \frac{rc^2}{\varepsilon}$ and $g = 2cra - a^2\varepsilon p - c^2p - c^2b$ so,

$$= \begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} f & -\frac{g}{\varepsilon} + a^2b - \frac{c^2b}{\varepsilon} \\ g & -f \end{pmatrix} z\right)$$

but since $\mathbf{A} \in H$ implies $a^2 - \frac{c^2}{\varepsilon} = 1$ finally giving

$$= \begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}\left(Id + \begin{pmatrix} f & -\frac{g}{\varepsilon} + b \\ g & -f \end{pmatrix} z\right)$$

but this contradicts equation (19) as if $g, f = 0$ implies $b = \hat{b}$ hence non of the representatives are conjugate. Now it is left to show we have covered $q^4(q-1)$ elements for each $\mathbf{B}$. This is simply a case of looking at the size of the stabiliser of each $\mathbf{B}(Id + \mathbf{b}z)$, where $\mathbf{b} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, which we will do by solving the following equation:

$$\mathbf{B}(Id + \mathbf{b}z) = \mathbf{A}(Id + \mathbf{a}z)\mathbf{B}(Id + \mathbf{b}z)(Id - \mathbf{a}z)\mathbf{A}^{-1}$$

we know that $\mathbf{B}$ and $\mathbf{A}$ must commute, so

$$= \mathbf{B}(Id + \mathbf{A}(\mathbf{B}^{-1}\mathbf{a}\mathbf{B} + \mathbf{b} - \mathbf{a})\mathbf{A}^{-1}z)$$

Lemma 1.7. *If* $\mathbf{A} = \begin{pmatrix} a & \frac{c}{\varepsilon} \\ c & a \end{pmatrix} \in H$ *(iff* $\mathbf{B}$ *and* $\mathbf{A}$ *commute) then* $\mathbf{A}\mathbf{d}\mathbf{A}^{-1} = \mathbf{b}$ *implies* $\mathbf{d} = \begin{pmatrix} bac & ba^2 \\ -bc^2 & -bac \end{pmatrix}$, *where* $\mathbf{d} \in sl_2(\mathbb{F}_q)$

Proof. Suppose $\mathbf{A}\mathbf{d}\mathbf{A}^{-1} = \mathbf{b}$, $\mathbf{d} = \begin{pmatrix} x & y \\ w & -x \end{pmatrix} \in sl_2(\mathbb{F}_q)$ and $\mathbf{A} = \begin{pmatrix} a & \frac{c}{\varepsilon} \\ c & a \end{pmatrix}$. Then $\mathbf{A}\mathbf{d}\mathbf{A}^{-1} = \mathbf{b}$ is equivalent to $\mathbf{A}\mathbf{d} = \mathbf{b}\mathbf{A}$ which gives the following four simultaneous equations:

$$ax + \frac{cw}{\varepsilon} = bc \tag{20}$$

$$ay - \frac{cx}{\varepsilon} = ba \tag{21}$$

$$cx + aw = 0 \tag{22}$$

$$cy - ax = 0 \tag{23}$$

as $\mathbf{A} \in SL_2(\mathbb{F}_q)$ we have either a *or* $c \neq 0$. If we assume $a \neq 0$ (22) implies $w = \frac{-cx}{a}$ and (23) implies $x = \frac{cy}{a}$ so $w = -\frac{c^2 y}{a^2}$. Putting $x = \frac{cy}{a}$ into equation (21) gives:

$$ay - \frac{c^2 y}{a\varepsilon} = ba$$

which is iff

$$(a^2 - \frac{c^2}{\varepsilon})y = ba^2$$

and since $a^2 - \frac{c^2}{\varepsilon} = 1$ we have $y = ba^2$ substituting this solution into (20) varifies this is a solution and hence this is the unique solution. If $c \neq 0$ the result follows from an almost identical argument. Using this solution gives the matrix stated in the lemma. ☐

Now using this lemma on

$$\mathbf{B}(Id + \mathbf{b}z) = \mathbf{B}(Id + \mathbf{A}(\mathbf{B}^{-1}\mathbf{a}\mathbf{B} + \mathbf{b} - \mathbf{a})\mathbf{A}^{-1}z)$$

means $\mathbf{B}^{-1}\mathbf{a}\mathbf{B} + \mathbf{b} - \mathbf{a} = \begin{pmatrix} bac & ba^2 \\ -bc^2 & -bac \end{pmatrix}$ so as $\mathbf{B} = \begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}$ and letting $\mathbf{a} = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}$ results in the following four simultaneous equations:

$$ex^2 + xf\varepsilon y + e\varepsilon y^2 - gyx - e = bac \tag{24}$$

$$fx^2 + eyx - gy^2 + eyx - f + b = ba^2 \tag{25}$$

$$-\varepsilon eyx - f\varepsilon^2 y^2 + gx^2 - \varepsilon eyx - g = -bc^2 \tag{26}$$

$$-ex^2 - xf\varepsilon y - e\varepsilon y^2 + gyx + e = -bac \tag{27}$$

First notice that $equation(24) = -equation(27)$ so they are linearly dependent, and rearranging (25) gives:

$$f(x^2 - 1) + 2eyx - gy^2 = b(a^2 - 1)$$

which using the following two facts, $x^2 - \varepsilon y^2 = 1$ and $a^2 - \frac{c^2}{\varepsilon} = 1$ gives

$$f\varepsilon y^2 + 2eyx - gy^2 = \frac{bc^2}{\varepsilon} \tag{28}$$

and using the same ideas equation(26) becomes

$$-\varepsilon(f\varepsilon y^2 + 2eyx - gy^2) = -bc^2 \tag{29}$$

So $equation(29) = -\varepsilon equation(28)$ which implies equation(26) is a scalar multiple of (25), so we have reduced the system to just equations (24) and (25), which is

$$\begin{pmatrix} \varepsilon y^2 + x^2 - 1 & x\varepsilon y & -yx \\ -2\varepsilon yx & -\varepsilon^2 y^2 & \varepsilon y^2 \end{pmatrix} \begin{pmatrix} e \\ f \\ g \end{pmatrix} = \begin{pmatrix} bac \\ -bc^2 \end{pmatrix}$$

If we can show the existence of one solution to this system then we have a linear subspace of solutions over $\mathbb{F}_q$ i.e. q distinct solutions. First notice that $y \neq 0$ as $\mathbf{B} = \begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}$ and $x \neq \pm 1$ then we can construct a solution, let $f = 0$ then we have the following two simultaneous equations:

$$(\varepsilon y^2 + x^2 - 1)e - yxg = bac \tag{30}$$

$$-2\varepsilon yxe + \varepsilon y^2 g = -bc^2 \tag{31}$$

using $x^2 - \varepsilon y^2 = 1$ (30) becomes

$$2\varepsilon y^2 e - yxg = bac$$

and since $y \neq 0$ we have $e = \frac{bac+yxg}{2\varepsilon y^2}$ substituting this into (31) we get:

$$\frac{-2\varepsilon yxbac}{2\varepsilon y^2} - \frac{2\varepsilon y^2 x^2 g}{2\varepsilon y^2} + \varepsilon y^2 g = -bc^2$$

$$(\varepsilon y^2 - x^2)g = -bc^2 + \frac{-2\varepsilon yxbac}{2\varepsilon y^2}$$

which gives us a solution for g since $\varepsilon y^2 - x^2 = -1$. Therefore for each $\mathbf{A} = \begin{pmatrix} a & \frac{c}{\varepsilon} \\ c & a \end{pmatrix}$ we have q elements of the form $\mathbf{A} + (Id + \mathbf{a}z)$ that stabilise $\mathbf{B} + (Id + \mathbf{b}z)$ and since there are q+1 choices for $\mathbf{A}$ we have:

$$|Stab_G(\mathbf{B} + (Id + \mathbf{b}z))| = q(q+1)$$

and hence,

$$|Orb_G(\mathbf{B} + (Id + \mathbf{b}z))| = \frac{q^4(q^2-1)}{q(q+1)} = q^3(q-1)$$

so for each $\mathbf{B}$ we have covered a total of $q \cdot q^3(q-1)$ where the q at the front comes from the q options for $\mathbf{b}$ □

This all gives us:

Theorem 1.8. *The following* $\mathbf{B}(Id + z\mathbf{b})$ *are conjugacy class representatives for* $SL_2(\mathbb{F}_q[z]/\langle z^2 \rangle)$

	$\mathbf{B}$	$\mathbf{b}$	$b \in \mathbb{F}_q$	$No.elementsinClass$
(1)	$\pm Id$	$\mathbf{0}$	n/a	1
(2)	$\pm Id$	$\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$	$1 \leq b \leq \frac{q-1}{2}$	$q(q+1)$
(3)	$\pm Id$	$\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$	$\sqrt{b} \notin \mathbb{F}_q$	$q(q-1)$
(4)	$\pm Id$	$\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$	n/a	$\frac{q^2-1}{2}$
(5)	$\pm Id$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	n/a	$\frac{q^2-1}{2}$
(6)	$\pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$	any	$\frac{q^2(q^2-1)}{2}$
(7)	$\pm \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$	any	$\frac{q^2(q^2-1)}{2}$
(8)	$\begin{pmatrix} x & 0 \\ 0 & x^-1 \end{pmatrix}, x \neq \pm 1$	$\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$	any	$q^3(q+1)$
(9)	$\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}, x \neq \pm 1$	$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$	any	$q^3(q-1)$

where in (8) $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ *and* $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$ *are the same, which in future will be represented by* $\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}, 1 \leq n \leq \frac{q-3}{2}$*, and in (9)* $(x.y)$ *and* $(x, -y)$ *are the same. Where* ε *is some fixed square free element in* $\mathbb{F}_q$*.*

It will be important later to view the representatives of (11) in the following form $\xi^n \left(Id + z \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right)$ where ξ is a fixed cyclic generator of $H_{[0,\varepsilon,1]}$ (see Lemma 2.2).

A couple of quick checks can be done at this point, firstly it can easily be shown that the total number of elements in all the conjugacy classes is equal to $|G|$ and secondly the number of conjugacy classes we have is equal to the total number quoted in [3].

2 Serre's Induced Irreducible Characters of $SL_2(\mathbb{F}_q[z]/\langle z^2 \rangle)$

Now to find the representation of $SL_2(\mathbb{F}_q[z]/\langle z^2 \rangle)$ we will follow the following method from [4] (8.2 page 62):

Let N and H be two subgroups of the group G, with N normal. Make the following hypothesis:

(i) N is abelian

(ii) G is the semidirect product of H by N

Since N is abelian its irreducible characters are of degree 1 and form a group $X = Hom(N, \mathbb{C}^*)$. The group G acts on X by

$$(s\chi)(a) = \chi(s^{-1}as) \quad for\, s \in G,\ \chi \in X,\ a \in N$$

Let $(\chi_i)_{i \in X/H}$ be a system of representatives for the orbits of H in X. For each $i \in X/H$, let H_i be the subgroup of H consisting of h such that $h\chi_i = \chi_i$, and let $G_i = N \cdot H_i$ be the corresponding subgroup of G. Extend the function χ_i to G_i by setting

$$\chi_i(ah) = \chi(a) \quad for\, a \in N,\ h \in H_i.$$

Using the fact that $h\chi_i = \chi_i$ for all $h \in H_i$, we see that χ_i is a character of degree 1 of G_i. Now let ρ be an irreducible representation of H_i; by composing ρ with the cannonical projection $G_i \to H_i$ we obtain an irreducible representation $\tilde{\rho}$ of G_i. Finally, by taking the tensor product of χ_i and $\tilde{\rho}$ we obtain an irreducible representation $\chi_i \otimes \tilde{\rho}$ of G_i; let $\theta_{i,p}$ be the corresponding induced representation of G.

Theorem 2.1. *[4]*

(a) $\theta_{i,p}$ *is irreducible*

(b) If $\theta_{i,p}$ *and* $\theta_{i',p'}$ *are isomorphic, then* $i = i'$ *and* ρ *is isomorphic to* ρ'*.*

(c) Every irreducible representation of G is isomorphic to one of the $\theta_{i,p}$*.*

So using the notation from the previous chapter we have $G = SL_2(\mathbb{F}_q[z]/\langle z^2 \rangle)$ is the semi direct product of N and H where N is an abelian group thus allowing us to apply the above process, first we will look at the irreducible reps/characters of N. Since N is abelian this is a very easy process we will set up the following notation to identify the irreducible characters:

$$\chi_{[I,J,K]} \left(Id + \begin{pmatrix} x & y \\ v & -x \end{pmatrix} z \right) = \omega^{xI}\omega^{yJ}\omega^{vK} = \omega^{xI+yJ+vK}$$

Where $[I, J, K] \in \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$ and $\omega = exp\left(\frac{2\pi i}{q}\right)$. These are all the irreducible characters of N and the action $(s\chi_{[I,J,K]})(a) = \chi_{[I,J,K]}(s^{-1}as) \quad for\ s \in G,\ \chi \in X,\ a \in N$ if we let $s = \begin{pmatrix} e & f \\ g & h \end{pmatrix}(Id + \mathbf{b}z)$ and $a = Id + \begin{pmatrix} x & y \\ v & -x \end{pmatrix} z$ then,

$$(s\chi_{[I,J,K]})(a) = \chi_{[I,J,K]}(s^{-1}as) = \omega^{Ihex-Ifev+Ihgy+Ifgx+2Jhfx-Jf^2v+Jh^2y-2Kgex+Ke^2v-Kg^2y}$$

$$= \chi_{[I(he+fg)+2Jhf-2Kge,Ihg+Jh^2-Kg^2,-Ife-Jf^2+Ke^2]}(a)$$

Now we will look for orbit representatives of this action and find the stabilisers for these representatives.

We start by looking at $\chi_{[I,0,0]}$ $(I \neq 0)$ then if $s = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ we get $s\chi_{[I,0,0]} = \chi_{[I(eh+fg),Ihg,-Ife]}$, there are two points of interest here, when is $\chi_{[I',0,0]}$ in the orbit of $\chi_{[I,0,0]}$ and what is the stabiliser, $H_{[I,0,0]}$, in H. The first question is simple, if $s\chi_{[I,0,0]} = \chi_{[I(eh+fg),Ihg,-Ife]} = \chi_{[I',0,0]}$ then we have the following system of equations:

$$(1)\, I(eh + fg) = I' \quad (2)\, Ihg = 0 \quad (3)\, -Ife = 0 \quad (4)\, eh - fg = 1$$

where (4) is because $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in H$. The combination of (2),(3) and (4) imply either $e = h = 0$ & $fg = -1$ or $f = g = 0$ & $eh = 1$ this implies $I' = \pm I$ so $\chi_{[I,0,0]}$ lie in distinct orbits for $1 \leq I \leq \frac{q-1}{2}$. Now to look at the stabiliser in H is given by the $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ s.t. $s\chi_{[I,0,0]} = \chi_{[I,0,0]}$ but from the above work this is

$$\left\{ \begin{pmatrix} e & f \\ g & h \end{pmatrix} | f = g = 0, \ \& \ eh = 1 \right\}$$

Hence we have $|H_{[I,0,0]}| = |Stab_H(\chi_{[I,0,0]})| = (q-1)$ and hence $|Orb_H(\chi_{[I,0,0]})| = q(q+1)$. And now we will find the irreducible representations of $H_{[I,0,0]}$. Noticing that $H_{[I,0,0]} \cong \mathbb{F}_q^*$ we can express each $\begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$ as $\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}$ where ζ generate $\mathbb{F}_q^*$ thus giving the irreducible representations as

$$\rho_{[I,0,0],l}\left(\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\right) = \exp \frac{2\pi l n}{q-1}$$

where $1 \leq l \leq q-1$.

Next we look at $\chi_{[0,1,0]}$, firstly we look at which $\chi_{[0,J,0]}$ we get from the action. We get $s\chi_{[0,1,0]} = \chi_{[2hf,h^2,-f^2]}$, this means $f = 0$ so we can get the $\frac{q-1}{2}$ $\chi_{[0,J,0]}$ where $J \in \{x | x = h^2, h \in \mathbb{F}_q^*\}$ by taking $s = \begin{pmatrix} h^{-1} & 0 \\ 0 & h \end{pmatrix}$. Now the $Stab_H(\chi_{[0,1,0]})$ just requires $f = 0$ and $h^2 = 1$ so,

$$Stab_H(\chi_{[0,1,0]}) = \left\{ \begin{pmatrix} h^{-1} & 0 \\ g & h \end{pmatrix} | h^2 = 1 \,\&\, g \in \mathbb{F}_q \right\}$$

So we have $|Stab_H(\chi_{[0,1,0]})| = 2q$ and so $|Orb_H(\chi_{[0,1,0]})| = \frac{q^2-1}{2}$. Now to look at the irreducible representations of $H_{[0,1,0]} = Stab_H(\chi_{[0,1,0]})$. Its easy to see that every element can be uniquely expressed as $(-Id)^n \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}$, where $1 \leq n \leq 2$ and $g \in \mathbb{F}_q$, and that $H_{[0,1,0]}$ is abelian so the representations are simply,

$$\rho_{[0,1,0],l_1,l_2}\left((-Id)^n \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}\right) = (-1)^{l_1 n} \exp\left(\frac{2\pi i g l_2}{q}\right) \tag{32}$$

where $l_1 \in \{0, 1\}$ and $0 \leq l_2 \leq q-1$.

Now if we look at $\chi_{[0,\varepsilon,0]}$ where ε is some fixed square free element we get identical results in the same way as those above and it is clear that they are not in the same orbit.

If we look at $\chi_{[0,0,0]}$ we get that $Stab_H(\chi_{[0,0,0]}) = H_{[0,0,0]} = H$ and hence we get $|Orb_H(\chi_{[0,0,0]})| = 1$ and the irreducible representations are simply the irreducible representations of H which can be found in [2].

Finally if we look at $\chi_{[0,x,1]}$ where x is some square free element first look at $s\chi_{[0,x,1]} = \chi_{[2(xhf-ge),xh^2-g^2,-xf^2+e^2]} = s\chi_{[0,x',1]}$ for some square free $x' \neq x$, this gives us four equations;

$$xhf - ge = 0 \tag{33}$$

$$xh^2 - g^2 = x' \tag{34}$$

$$e^2 - xf^2 = 1 \tag{35}$$

$$eh - gf = 1 \tag{36}$$

It is straightforward to check there are no solutions if any of e, f, g *or* $h = 0$ so can assume they are all non zero so (33) gives $f = \frac{ge}{xh}$ substituting this into (36) gives $e(h - \frac{g^2}{xh}) = 1$ but using (34) in this gives $e = \frac{xh}{x'}$ and so $f = \frac{g}{x'}$, substituting these into (35) gives $xh^2 - g^2 = \frac{x'^2}{x}$ which using (34) implies $x = x'$ which is a contradiction hence there are no solutions so $\chi_{[0,x,1]}$ define distinct orbits. Now to look at the stabiliser of $\chi_{[0,x,1]}$ this is the set of solutions to the above equations where $x = x'$, so (34) is now $xh^2 - g^2 = x$. Notice (34) implies either g *or* $h \neq 0$, assuming $h \neq 0$ this gives similar to before $f = \frac{g}{x}$ and $e = h$ now if we assume that $g \neq 0$ we get exactly the same result, hence:

$$H_{[0,x,1]} = Stab_H(\chi_{[0,x,1]}) = \left\{ \begin{pmatrix} h & \frac{g}{x} \\ g & h \end{pmatrix} \middle| h^2 - \frac{g^2}{x} = 1 \right\}$$

and by theorem 2.1 we have $|Stab_H(\chi_{[0,x,1]})| = q + 1$ and hence $|Orb_H(\chi_{[0,x,1]})| = q(q-1)$. Now to find the irreducible representations of $H_{[0,x,1]}$.

Lemma 2.2. *$H_{[0,x,1]}$ is cyclic.*

Proof. Can view $H_{[0,x,1]}$ as acting on the quadratic extension of $\mathbb{F}_q$ as multiplication as follows. Let $e_1 = 1$ and $e_2 = \frac{1}{\sqrt{x}}$ then e_1 and e_2 form a basis of the quadratic extension over $\mathbb{F}_q$ then we can define the action of $H_{[0,x,1]}$ on the quadratic extension by $\begin{pmatrix} h & \frac{g}{x} \\ g & h \end{pmatrix}(ae_1 + be_2) = (he_1 + ge_2)(ae_1 + be_2)$. So we can view $H_{[0,x,1]}$ as a subgroup of the multiplicative group of the quadratic extension, as each element of $H_{[0,x,1]}$ acts as a unique element of the multiplicative group. Hence $H_{[0,x,1]}$ is the subgroup of the multiplicative group of the quadratic extension which is cyclic and hence $H_{[0,x,1]}$ is also cyclic. $\square$

Hence we have $H_{[0,x,1]} =< \eta >$ for some $\eta \in H_{[0,x,1]}$ s.t. $|\eta| = q+1$ and so the irreducible representations of $H_{[0,x,1]}$ are simply $\rho_{[0,x,1],l}(\eta^n) = \exp(\frac{2\pi i l n}{q+1})$ where $0 \leq l \leq q+1$.

We can now carry out a couple of quick checks , we can check that we have in fact found all of the orbit representatives of X under the action of H by simply totalling the sum of the sizes of each orbit which will yield $q^3 = |X|$ as required. We have not ignored the case $G_{[0,0,0]}$ the required characters will be given a little later as we will quote them from [2] and induce them at the same time.

3 The Irreducible Characters of $SL_2(\mathbb{F}_q[z]/\langle z^2\rangle)$

We are now in a position to induce the characters we have from the previous bit of work, throughout this section we will use e^x to denote $exp(x)$. We start by inducing our characters of $G_{[I,0,0]} = \left\{ \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix} \left(Id + z \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \right) \right\}$ for $1 \leq I \leq \frac{q-1}{2}$,

$$\chi_{[I,0,0]} \otimes \tilde{\rho}_{[I,0,0],l} \left(\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix} \left(Id + z \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \right) \right) = e^{\frac{2\pi i l n}{q-1}} e^{\frac{2\pi i \alpha I}{q}}$$

We will induce this character (and all the other characters later) using the following [5](proposition 21.23) if ψ is a character of a subgroup H of G, and suppose that $x \in G$.

(1) if no element of x^G lies in H, then $(\psi \uparrow G)(x) = 0$

(2) if some element of x^G lies in H, then

$$(\psi \uparrow G)(x) = |C_G(x)| \left(\frac{\psi(x_1)}{|C_H(x_1)|} + \cdots + \frac{\psi(x_m)}{|C_H(x_m)|} \right)$$

Where $H \cap x^G$ splits into m conjugacy classes in H $x_1, \cdots, x_m$ are the m conjugacy class representatives of these classes and $\psi \uparrow G$ is the induced character.

So in order to use this we need to determine the conjugacy classes of $G_{[I,0,0]}$ which will be our next step.

Lemma 3.1. *The conjugacy classes of $G_{[I,0,0]}$ are the following*

Rep	*var*	*size*	$\lvert C_{G_{[I,0,0]}}\rvert$
$\pm\left(Id + z\begin{pmatrix} \alpha & 0 \\ 1 & -\alpha \end{pmatrix}\right)$	$\alpha \in \mathbb{F}_q$	$\frac{q-1}{2}$	$2q^3$
$\pm\left(Id + z\begin{pmatrix} \alpha & 0 \\ \varepsilon & -\alpha \end{pmatrix}\right)$	$\alpha \in \mathbb{F}_q$	$\frac{q-1}{2}$	$2q^3$
$\pm\left(Id + z\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}\right)$	$\alpha \in \mathbb{F}_q$	1	$(q-1)q^3$
$\pm\left(Id + z\begin{pmatrix} \alpha & 1 \\ c & -\alpha \end{pmatrix}\right)$	$c, \alpha \in \mathbb{F}_q$	$\frac{q-1}{2}$	$2q^3$
$\pm\left(Id + z\begin{pmatrix} \alpha & \varepsilon \\ c & -\alpha \end{pmatrix}\right)$	$c, \alpha \in \mathbb{F}_q$	$\frac{q-1}{2}$	$2q^3$
$\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id + z\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}\right)$	$\zeta^n \neq \pm 1, \alpha \in \mathbb{F}_q$	q^2	$(q-1)q$

Proof. for the first five rows of the table we will only need to consider the case where $\pm$ is $+$, the negative case follows due to $-Id$ commuting with everything. On the first five rows conjugation acts as follows, if $\mathbf{A}(Id+\mathbf{a}z), Id+z\mathbf{b} \in G_{[I,0,0]}$ then $\mathbf{A}(Id+\mathbf{a}z)(Id+z\mathbf{b})(Id-\mathbf{a}z)\mathbf{A}^{-1} = (Id+z\mathbf{A}\mathbf{b}\mathbf{A}^{-1})$ so we only need to consider the conjugation action of $\left\{\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\right\}$ on $sl_2(\mathbb{F}_q)$. If $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in sl_2(\mathbb{F}_q)$ then we get $\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\begin{pmatrix} \zeta^{-n} & 0 \\ 0 & \zeta^{n} \end{pmatrix} = \begin{pmatrix} \alpha & \beta\zeta^{-2n} \\ \gamma\zeta^{2n} & -\alpha \end{pmatrix}$ thus it is (quite) obvious that the first five rows of representatives in the table are in fact not conjugate to each other using the fact that as ε is square free then so is $a^2\varepsilon\ \forall a \in \mathbb{F}_q^*$. It is also clear from this action that non of the first five rows can be conjugate to the last row. To show the sizes is a simple case of looking at the stabilisers of each representative, for row three it is clear that everything in $G_{[I,0,0]}$ stabilises the rep, and for the other top five rows we get the same stabiliser as the example below

$$\left| Stab_{G_{[I,0,0]}}\left(Id + z\begin{pmatrix} \alpha & 1 \\ c & -\alpha \end{pmatrix}\right)\right| = \left|\left\{\pm\left(Id + z\mathbf{a}\right)|\mathbf{a} \in sl_2(\mathbb{F}_q)\right\}\right| = 2q^3$$

thus completing the first five rows. Now to look at the final row, these have a more complicated conjugation action as follows, let $\mathbf{A}(Id+\mathbf{a}z), Id+z\mathbf{b} \in G_{[I,0,0]}$ where $\mathbf{A}(Id+\mathbf{a}z) = \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id + z\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}\right)$ and $(Id+z\mathbf{b}) = \left(Id + z\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right)$ then

$$(Id+z\mathbf{b})\mathbf{A}(Id+\mathbf{a}z)(Id-z\mathbf{b}) = \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id + z\begin{pmatrix} \alpha & (\zeta^{-2n}-1)b \\ (\zeta^{2n}-1)c & -\alpha \end{pmatrix}\right)$$

and since $\zeta^n \neq \pm 1$ we have $\zeta^{\pm 2n} - 1 \neq 0$ Hence we can produce any $\mathbf{A}(Id+\mathbf{c}z)$ where $\mathbf{c} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ for any $\beta, \gamma \in \mathbb{F}_q$ and α determined by $\mathbf{a}$. Then noticing that by conjugateing the above result further by $\mathbf{B} = \begin{pmatrix} \zeta^m & 0 \\ 0 & \zeta^{-m} \end{pmatrix}$ yields no further conjugates as will simply commute with $\mathbf{A}$ and will not effect the value of α in $\mathbf{c}$ (as was seen when looking at the top five rows in the table) hence all of our representatives are not conjugate to one and another and the size of each conjugacy class is simply the size of set of all the elements we can obtain by conjugation i.e.

$$\left|\left\{\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id + z\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\right)\middle| \beta, \gamma \in \mathbb{F}_q\right\}\right| = q^2$$

Thus completing the conjugacy classes of $G_{[I,0,0]}$ □

Next we need to see how the conjugacy classes of G split across $G_{[I,0,0]}$ in order to induce the characters, we start by observing that where $\pm$ appears in the table of theorem 1.8 it will only be necesary to consider how the $+$ classes split as the $-$ classes will split in an identical manner.

Lemma 3.2.

x	$x^G \cap G_{[I,0,0]}$	b
Id	$Id^{G_{[I,0,0]}}$	n/a
$\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)$	$\begin{pmatrix} \pm b & 0 \\ \varepsilon & \mp b \end{pmatrix}^{G_{[I,0,0]}} \cup \begin{pmatrix} \pm b & 0 \\ 0 & \mp b \end{pmatrix}^{G_{[I,0,0]}} \cup \left(\cup_{a\in\mathbb{F}_q}\begin{pmatrix} a & 1 \\ f(a) & -a \end{pmatrix}^{G_{[I,0,0]}}\right) \cup \left(\cup_{a\in\mathbb{F}_q}\begin{pmatrix} a & \varepsilon \\ g(a) & -a \end{pmatrix}^{G_{[I,0,0]}}\right)$	$1 \leq b \leq \frac{q-1}{2}$
$\left(Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\right)$	$\left(\cup_{c\in\mathbb{F}_q}\begin{pmatrix} c\varepsilon & \varepsilon \\ -c^2\varepsilon & -c\varepsilon \end{pmatrix}^{G_{[I,0,0]}}\right) \cup \begin{pmatrix} 0 & 0 \\ h(q) & 0 \end{pmatrix}^{G_{[I,0,0]}}$	n/a
$\left(Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$	$\left(\cup_{c\in\mathbb{F}_q}\begin{pmatrix} c & 1 \\ -c^2 & -c\varepsilon \end{pmatrix}^{G_{[I,0,0]}}\right) \cup \begin{pmatrix} 0 & 0 \\ i(q) & 0 \end{pmatrix}^{G_{[I,0,0]}}$	n/a
$\left(Id + z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right)$	$\left(\cup_{c\in\mathbb{F}_q}\begin{pmatrix} c & 1 \\ j(c) & -c \end{pmatrix}^{G_{[I,0,0]}}\right) \cup \left(\cup_{c\in\mathbb{F}_q}\begin{pmatrix} c & \varepsilon \\ k(c) & -c \end{pmatrix}^{G_{[I,0,0]}}\right)$	$\sqrt{b} \notin \mathbb{F}_q$
$\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)$	$\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^{G_{[I,0,0]}} \cup \begin{pmatrix} \zeta^{-n} & 0 \\ 0 & \zeta^n \end{pmatrix}\left(Id + z\begin{pmatrix} -b & 0 \\ 0 & b \end{pmatrix}\right)^{G_{[I,0,0]}}$	$1 \leq n < \frac{q-1}{2}$

Where f to k are functions from $\mathbb{F}_q$ to $\mathbb{F}_q$ which will be defined in the proof, $1 \leq I \leq \frac{q-1}{2}$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ in the rows (2),(3),(4) and (5) means $Id + z\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$

Proof. The result about the identity is obvious, so we will start on the second row of the table, first observe that

$$\left(Id+z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G = \{Id+z\mathbf{b}|\, eigenvalues = \pm b\}$$

This can be easily seen by noticing that the conjugacy class is generated by conjugating by elements in $SL_2(\mathbb{F}_q)$ and there are no other conjugacy class representatives in G of the form $Id+z\mathbf{b}$ with eigenvalues $\pm b$. Also recall this conjugacy class in G has size $q(q+1)$. So we need to find all the conjugacy classes in $G_{[I,0,0]}$ where the representative is of the form $Id+z\mathbf{b}$ with eigenvalues $\pm b$. These are simply all the representatives in the second row of the table where $f(a)=b^2-a^2$ and $g(a)=\frac{b^2-a^2}{\varepsilon}$ to ensure we have the required eigenvalues. It can be seen that this is all of them by observing that the total of all the sizes of the conjugacy classes that these represent in $G_{[I,0,0]}$ gives us $q(q+1)$ as required.

For the thrid row of the table we need to cover $\frac{q^2-1}{2}$ elements all with zero eigen values, to do this we look at which elements in G we can achieve by conjugation, as before we only need to consider the action of $SL_2(\mathbb{F}_q)$, thus we get,

$$Id+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G = \left\{ Id+z\begin{pmatrix} -ac\varepsilon & a^2\varepsilon \\ -c^2\varepsilon & ca\varepsilon \end{pmatrix} \middle| (a,c)\neq(0,0) \right\}$$

where a and c come from conjugating by
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q)$. $\cup_{c\in\mathbb{F}_q}\left(Id+z\begin{pmatrix} c\varepsilon & \varepsilon \\ -c^2\varepsilon & -c\varepsilon \end{pmatrix}\right)$ are all distinct conjugacy class reps in $G_{[I,0,0]}$ which are in $Id+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G$, these classes account for a total of $\frac{q(q-1)}{2}$. The remaining $\frac{q-1}{2}$ elements come from looking at the case where $a=0$ then if $q\equiv 1(mod4)$ we have $-c^2\varepsilon$ is square free and hence by choosing the right value for c we have $Id+z\begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}^{G_{[I,0,0]}} \subseteq Id+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G$ similarly if $q\equiv 3(mod4)$ we have $-c^2\varepsilon$ is square and hence we get $Id+z\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{G_{[I,0,0]}} \subseteq Id+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G$ thus by defining
$h(q)=\begin{cases} \varepsilon & if\; q\equiv 1(mod4) \\ 1 & if\; q\equiv 3(mod4) \end{cases}$ the third row is done.

An identical argument works for the fourth row of the table where
$i(q)=\begin{cases} 1 & if\; q\equiv 1(mod4) \\ \varepsilon & if\; q\equiv 3(mod4) \end{cases}$.

For the sixth row we follow a similar route to that of the second, first observe that

$$\left(Id+z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right)^G = \{Id+z\mathbf{b}|eigen-values\; of\; \mathbf{b}\; are\; \pm\sqrt{b}\}$$

and then by seeing that if we let $j(c)=b-c^2$ and $k(c)=\frac{b-c^2}{\varepsilon}$ we have distinct conjugacy classes of $G_{[I,0,0]}$ in $\left(Id+z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right)^G$ which cover the required $q(q-1)$ elements.

Finally to look at the last row of the table, it is clear from the conjugacy classes of $G_{[I,0,0]}$ that all of the classes in the table are distinct first we show that $\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id+z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right) \in \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id+z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G$ and $\begin{pmatrix} \zeta^{-n} & 0 \\ 0 & \zeta^n \end{pmatrix}\left(Id+z\begin{pmatrix} -b & 0 \\ 0 & b \end{pmatrix}\right) \in \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id+z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G$ but the top one of these is clearly true and the bottom comes from conjugating in G by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so in total we have covered $2q^2$ elements. To show these cover the required number of elements in $G_{[I,0,0]}$, we need to see what $\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id+z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G \cap G_{[I,0,0]}$ is, firstly need to observe the only terms we can achieve in $G_{[I,0,0]}$ must have $\begin{pmatrix} \zeta^{\pm n} & 0 \\ 0 & \zeta^{\mp n} \end{pmatrix}$ as the leading term in the semidirect product, this fact comes straight from the conjugacy classes of $SL_2(\mathbb{F}_q)$, the only elements in $SL_2(\mathbb{F}_q)$ which we can conjugate by which keep us in $G_{[I,0,0]}$ are of the form $\begin{pmatrix} \zeta^m & 0 \\ 0 & \zeta^{-m} \end{pmatrix}$ and

$\begin{pmatrix} 0 & -\zeta^m \\ \zeta^{-m} & 0 \end{pmatrix}$. Hence the only elements we can conjugate by are of the form $\begin{pmatrix} \zeta^m & 0 \\ 0 & \zeta^{-m} \end{pmatrix}\left(Id + z\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\right)$ and $\begin{pmatrix} 0 & -\zeta^m \\ \zeta^{-m} & 0 \end{pmatrix}\left(Id + z\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\right)$, the first of the two yields the following:

$$\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id + z\begin{pmatrix} b & \zeta^{2m}(\zeta^{-2n}-1)\beta \\ \zeta^{-2m}(\zeta^{2n}-1)\gamma & -b \end{pmatrix}\right)$$

Since $\zeta^n \neq \pm 1$ we have $\zeta^{-2n} - 1 \neq 0 \neq \zeta^{2n} - 1$ and hence we can obtain a total of q^2 elements by choosing β and γ in $\mathbb{F}_q$. If we conjugate by the second type of element we get

$$\begin{pmatrix} \zeta^{-n} & 0 \\ 0 & \zeta^{n} \end{pmatrix}\left(Id + z\begin{pmatrix} -b & -\zeta^{2m}(\zeta^{2n}-1)\gamma \\ -\zeta^{-2m}(\zeta^{-2n}-1)\beta & b \end{pmatrix}\right)$$

giving us a further q^2 elements as before, so in total we have $2q^2$ elements in $x^G \cap G_{[I,0,0]}$ and so we are done. □

We are now ready to give the first portion of the charcater table of $SL_2(\mathbb{F}_q[z]/\langle z^2\rangle)$ by inducing the following chracters of $G_{[I,0,0]}$, $\tilde{\rho}_{[I,0,0],l} \otimes \chi_{[I,0,0]}$ using the formula for induction and the above proposition we have the following

Theorem 3.3. *The following characters are the induced characters of $\tilde{\rho}_{[I,0,0],l} \otimes \chi_{[I,0,0]}$ on $G_{[I,0,0]}$ to G*

Conjugacy class representative	*restrictions*	$\tilde{\rho}_{[I,0,0],l} \otimes \chi_{[I,0,0]} \uparrow G$
Id	n/a	$q(q+1)$
$-Id$	n/a	$q(q+1)(-1)^l$
$Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$	$1 \leq b \leq \frac{q-1}{2}$	$q(e^{\frac{2\pi i I b}{q}} + e^{\frac{-2\pi i I b}{q}})$
$-\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)$	$1 \leq b \leq \frac{q-1}{2}$	$(-1)^l q(e^{\frac{2\pi i I b}{q}} + e^{\frac{-2\pi i I b}{q}})$
$Id + z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$	$\sqrt{b} \notin \mathbb{F}_q$	0
$-\left(Id + z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right)$	$\sqrt{b} \notin \mathbb{F}_q$	0
$Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$	n/a	q
$-\left(Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\right)$	n/a	$(-1)^l q$
$Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	n/a	q
$-\left(Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$	n/a	$(-1)^l q$
$\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)$	$1 \leq n < \frac{q-1}{2}$	$e^{(\frac{2\pi i I b}{q} + \frac{2\pi i l n}{q-1})} + e^{(\frac{-2\pi i I b}{q} + \frac{-2\pi i l n}{q-1})}$

and is zero on all other conjugacy classes.

It is readily varifiable that all of these characters are irreducible by means of taking any ones character inner-product with itself yielding 1 as would be expected due to Theorem 2.1.

Next we shall induce the characters from
$G_{[0,1,0]} = \left\{ (-1)^n \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}\left(Id + z\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\right) \middle| n \in \{0,1\}, \alpha, \beta, \gamma, g \in \mathbb{F}_q \right\}$

Lemma 3.4. *The conjugacy classes of $G_{[0,1,0]}$ are the following:*

Rep	*var*	*size*	$\lvert C_{G_{[0,1,0]}}\rvert$
$\pm Id$	n/a	1	$2q^4$
$\pm\left(Id+z\begin{pmatrix} b & 0 \\ 0 & -b\end{pmatrix}\right)$	$b\in\mathbb{F}_q^*$	q	$2q^3$
$\pm\left(Id+z\begin{pmatrix} 0 & \beta b \\ -\beta^{-1}b & 0\end{pmatrix}\right)$	$1\leq b\leq \frac{q-1}{2}, \beta\in\mathbb{F}_q^*$	q	$2q^3$
$\pm\left(Id+z\begin{pmatrix} 0 & \beta \\ -\beta^{-1}b & 0\end{pmatrix}\right)$	$\beta\in\mathbb{F}_q^*, \sqrt{b}\notin\mathbb{F}_q$	q	$2q^3$
$\pm\left(Id+z\begin{pmatrix} 0 & \beta \\ 0 & 0\end{pmatrix}\right)$	$\beta\in\mathbb{F}_q^*$	q	$2q^3$
$\pm\left(Id+z\begin{pmatrix} 0 & 0 \\ \beta & 0\end{pmatrix}\right)$	$\beta\in\mathbb{F}_q^*$	1	$2q^4$
$\pm\begin{pmatrix} 1 & 0 \\ g & 1\end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1\end{pmatrix}\right)$	$g\in\mathbb{F}_q^*, b\in\mathbb{F}_q$	q^2	$2q^2$

Proof. The first row is obvious, the second row it is clear that b and $\tilde{b}$ are not conjugate where $\tilde{b}\neq -b$ as they give different eigenvalues, if $\tilde{b}=-b$ could only be conjugate by an element of the form $\begin{pmatrix} 1 & 0 \\ g & 1\end{pmatrix}$ but this implies $b=-b$ (see by assuming they are conjugate) but this is a contradiction as $b\in\mathbb{F}_q^*$. It is easy to see that $C_{G_{[0,1,0]}}\left(Id+z\begin{pmatrix} b & 0 \\ 0 & -b\end{pmatrix}\right)=\{\pm(Id+z\mathbf{b})\,|\,\mathbf{b}\in sl_2(\mathbb{F}_q)\}$ which has size $2q^3$ and so the conjugacy class is of size $\frac{|G_{[0,1,0]}|}{2q^3}=q$.

To see the elements in the second row are not conjugate to those in the third row if we simply look at what conjugation by $\pm\begin{pmatrix} 1 & 0 \\ g & 1\end{pmatrix}$ does to a general element of the form $Id+z\mathbf{b}$ where $\mathbf{b}=\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha\end{pmatrix}\in sl_2(\mathbb{F}_q)$,

$$\begin{pmatrix} 1 & 0 \\ g & 1\end{pmatrix}\left(Id+z\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha\end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ -g & 1\end{pmatrix}=Id+z\begin{pmatrix} \alpha-g\beta & \beta \\ 2g\alpha+\gamma-g^2\beta & g\beta-\alpha\end{pmatrix}$$

noticing that the top right entry is fixed under conjugation this shows that the representatives in rows two and three cannot be conjugate. This also shows that none of the entries in row three are conjugate to each other as if they have the same value of b then different values β fix the top right entry and the value of b determines the eigen values of the matrix. Again we get that $C_{G_{[0,1,0]}}\left(Id+z\begin{pmatrix} 0 & \beta b \\ -\beta^{-1}b & 0\end{pmatrix}\right)=\{\pm(Id+z\mathbf{b})\,|\,\mathbf{b}\in sl_2(\mathbb{F}_q)\}$ which has size $2q^3$ and so the conjugacy class is of size $\frac{|G_{[0,1,0]}|}{2q^3}=q$.

The forth row is not conjugate to any of the previous rows due to the eigen-values and no two elements in the fourth row are conjugate to each other due to the fact that the top right entry is fixed under conjugationa and the eigen-values of the matrices.

The fifth and the sixth rows of the table all have zero as there eigenvalues, to see none of these elements are conjugate to one another simply looking at the conjugation action above it is clear that they cannot be conjugate and the stabilisers are very straightforward calculations.

Finally looking towards the last row we look at the effect of conjugating by $\begin{pmatrix} 1 & 0 \\ \tilde{g} & 1\end{pmatrix}\left(Id+z\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha\end{pmatrix}\right)$ this gives us

$$\begin{pmatrix} 1 & 0 \\ g & 1\end{pmatrix}\left(Id+z\begin{pmatrix} 1+g\beta-\tilde{g}b & b \\ \tilde{g}(2+2g\beta-\tilde{g}b)-g(2\alpha+g\beta) & \tilde{g}b-1-g\beta\end{pmatrix}\right)$$

So it is clear that the leading matrix is fixed under conjugation, as would be expected as they form a cyclic group and the second matrix in the expression the top right entry is fixed, thus all the conjugacy class representatives are distinct. Now notice we can obtain any element of the form $\begin{pmatrix} 1 & 0 \\ g & 1\end{pmatrix}\left(Id+z\begin{pmatrix} \tilde{\alpha} & b \\ \tilde{\gamma} & -\tilde{\alpha}\end{pmatrix}\right)$ by choosing $\tilde{g}=0$ and picking β s.t. the top left entry is $\tilde{\alpha}$, this is always possible as $g\neq 0$, and then choosing α so that the bottom left entry is $\tilde{\gamma}$ so we have the conjugacy class is of size q^2 completing the table. □

Lemma 3.5.

x	$x^G \cap G_{[0,1,0]}$	b
Id	$Id^{G_{[0,1,0]}}$	n/a
$\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)$	$\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}^{G_{[0,1,0]}} \cup \begin{pmatrix} -b & 0 \\ 0 & b \end{pmatrix}^{G_{[0,1,0]}} \cup \left(\cup_{\beta \in \mathbb{F}_q^*} \begin{pmatrix} 0 & b\beta \\ -b\beta^{-1} & 0 \end{pmatrix}^{G_{[0,1,0]}}\right)$	$1 \leq b \leq \frac{q-1}{2}$
$\left(Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\right)$	$\left(\cup_{\sqrt{c} \notin \mathbb{F}_q^*} \begin{pmatrix} 0 & 0 \\ cf(q) & 0 \end{pmatrix}^{G_{[0,1,0]}}\right) \cup \left(\cup_{\sqrt{a} \notin \mathbb{F}_q^*} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^{G_{[0,1,0]}}\right)$	n/a
$\left(Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$	$\left(\cup_{\sqrt{c} \in \mathbb{F}_q^*} \begin{pmatrix} 0 & 0 \\ cf(q) & 0 \end{pmatrix}^{G_{[0,1,0]}}\right) \cup \left(\cup_{\sqrt{a} \in \mathbb{F}_q^*} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^{G_{[0,1,0]}}\right)$	n/a
$\left(Id + z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right)$	$\cup_{\beta \in \mathbb{F}_q^*} \begin{pmatrix} 0 & \beta \\ -\beta^{-1}b & 0 \end{pmatrix}^{G_{[0,1,0]}}$	$\sqrt{b} \notin \mathbb{F}_q$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\left(Id + z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$\bigcup_{b \in \mathbb{F}_q, v \neq 0 \text{ square}} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}\left(Id + z\begin{pmatrix} 1 & v^{-1}b \\ 0 & -1 \end{pmatrix}\right)^{G_{[0,1,0]}}$	$b \in \mathbb{F}_q$
$\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}\left(Id + z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$\bigcup_{b \in \mathbb{F}_q, v \neq 0 \text{ square}} \begin{pmatrix} 1 & 0 \\ v\varepsilon & 1 \end{pmatrix}\left(Id + z\begin{pmatrix} 1 & v^{-1}b \\ 0 & -1 \end{pmatrix}\right)^{G_{[0,1,0]}}$	$b \in \mathbb{F}_q$

Where in rows (2),(3),(4) and (5) we use $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ *denotes* $Id + z\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$.

Proof. From the equivalent result for $G_{[I,0,0]}$ rows two and five follow simply by considering eigenvalues and checking the total numbers of elements involved. The first row is an obvious result. The third row follows by looking at $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -ac\varepsilon & a^2\varepsilon \\ -c^2\varepsilon & ac\varepsilon \end{pmatrix}$, where $ad-bc=1$ so we can see clearly that all the following conjugacy classes $\cup_{\sqrt{v}\notin\mathbb{F}_q}\left(Id+z\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\right)^{G_{[0,1,0]}} \subseteq Id+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G$ by altering the value of a (to give the $\frac{q-1}{2}$ possible values of a^2 and letting $b=c=0$. This accounts for $q\frac{q-1}{2}$ of the $\frac{q^2-1}{2}$ elements required. The remaining elements come from looking at the conjugation where $a=d=0$ and $b=-c^{-1}$ which returns $Id+z\begin{pmatrix} 0 & 0 \\ -c^2\varepsilon & 0 \end{pmatrix}$ which shows that $\cup_{\sqrt{v}\notin\mathbb{F}_q} Id+z\begin{pmatrix} 0 & 0 \\ vf(q) & 0 \end{pmatrix}^{G_{[0,1,0]}} \subseteq d+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G$ where $f(q) = \begin{cases} 1 & if\ q\equiv 1(mod4) \\ \varepsilon & if\ q\equiv 3(mod4) \end{cases}$. This gives a further further $\frac{q-1}{2}$ elements, giving us the required total and thus completing the third row. The fourth row follows an identical argument.

The final two rows follow very similar arguments to each other so we will only deal with the last row. First look at the effect of conjugating in G by $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$ where $x\in\mathbb{F}_q^*$ this gives us $\begin{pmatrix} 1 & 0 \\ x^2\varepsilon & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & x^{-2}b \\ 0 & -1 \end{pmatrix}\right)$ and so we have

$\bigcup_{b\in\mathbb{F}_q, v\neq 0\ square}\begin{pmatrix} 1 & 0 \\ v\varepsilon & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & v^{-1}b \\ 0 & -1 \end{pmatrix}\right)^{G_{[0,1,0]}} \subseteq \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)^G$ This gives us a total of $\frac{q^3(q-1)}{2}$ elements in total, adding this to the identical amount from the row above and the previous rows (taking into account we can put ± 1 at the front of any of the class representatives to give the negative conjugacy classes) gives us a total of $2\cdot 2\cdot\frac{q^3(q-1)}{2}+2q^3 = |G_{[0,1,0]}|$ □

We now have all the information in order to induce the character and so we have the following theorem:

Theorem 3.6. *The following characters are the induced characters of $\tilde{\rho}_{[0,1,0],l_1,l_2}\otimes\chi_{[0,1,0]}$ on $G_{[0,1,0]}$ to G*

Conjugacy class representative	*restrictions*	$\tilde{\rho}_{[0,1,0],l_1,l_2}\otimes\chi_{[0,1,0]}\uparrow G$
Id	n/a	$\frac{q^2-1}{2}$
$-Id$	n/a	$\frac{q^2-1}{2}(-1)^{l_1}$
$Id+z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$	$1\le b\le\frac{q-1}{2}$	$\frac{q-1}{2}$
$-\left(Id+z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)$	$1\le b\le\frac{q-1}{2}$	$\frac{q-1}{2}(-1)^{l_1}$
$Id+z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$	$\sqrt{b}\notin\mathbb{F}_q$	$-\frac{q+1}{2}$
$-\left(Id+z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right)$	$\sqrt{b}\notin\mathbb{F}_q$	$\frac{q+1}{2}(-1)^{l_1+1}$
$Id+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$	n/a	$\frac{q-1}{2}+q\sum_{\sqrt{\beta}\notin\mathbb{F}_q} e^{\frac{2\pi i\beta}{q}}$
$-\left(Id+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\right)$	n/a	$(\frac{q-1}{2}+q\sum_{\sqrt{\beta}\notin\mathbb{F}_q} e^{\frac{2\pi i\beta}{q}})(-1)^{l_1}$
$Id+z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	n/a	$\frac{q-1}{2}+q\sum_{\sqrt{\beta}\notin\mathbb{F}_q} e^{\frac{2\pi i\varepsilon\beta}{q}}$
$-\left(Id+z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$	n/a	$(\frac{q-1}{2}+q\sum_{\sqrt{\beta}\notin\mathbb{F}_q} e^{\frac{2\pi i\varepsilon\beta}{q}})(-1)^{l_1}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$b\in\mathbb{F}_q$	$\sum_{\sqrt{\beta}\in\mathbb{F}_q^*} e^{\frac{2\pi i\beta l_2}{q}} e^{\frac{2\pi i\beta^{-1}b}{q}}$
$-\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$b\in\mathbb{F}_q$	$\sum_{\sqrt{\beta}\in\mathbb{F}_q^*} e^{\frac{2\pi i\beta l_2}{q}} e^{\frac{2\pi i\beta^{-1}b}{q}}(-1)^{l_1}$
$\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$b\in\mathbb{F}_q$	$\sum_{\sqrt{\beta}\in\mathbb{F}_q^*} e^{\frac{2\pi i\beta\varepsilon l_2}{q}} e^{\frac{2\pi i\beta^{-1}b}{q}}$
$-\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$b\in\mathbb{F}_q$	$\sum_{\sqrt{\beta}\in\mathbb{F}_q^*} e^{\frac{2\pi i\beta\varepsilon l_2}{q}} e^{\frac{2\pi i\beta^{-1}b}{q}}(-1)^{l_1}$

The induced characters take the value zero on any other conjugacy classes

Proof. Simply apply the induced character formula to the previous two propositions. □

If we take a closer look at this induced character, we know by theorem 2.1 that the characters inner product with itself is necesarily 1 as it is irreducible, if we carry out this lengthy calculation we come up with the following result of possible interest:

$\sum_{\sqrt{u}\notin\mathbb{F}_q}\sum_{\sqrt{v}\notin\mathbb{F}_q}(e^{\frac{2\pi i(u-v)}{q}}+e^{\frac{2\pi i(u-v)\varepsilon}{q}})=\frac{q+1}{2}$

By copying the above steps exactly we get another set of induced characters given below from $G_{[0,\varepsilon,0]}$,

Theorem 3.7. *The following characters are the induced characters of $\tilde{\rho}_{[0,\varepsilon,0],l_1,l_2}\otimes\chi_{[0,\varepsilon,0]}$ on $G_{[0,\varepsilon,0]}$ to G*

Conjugacy class representative	*restrictions*	$\tilde{\rho}_{[0,\varepsilon,0],l_1,l_2}\otimes\chi_{[0,\varepsilon,0]}\uparrow G$
Id	n/a	$\frac{q^2-1}{2}$
$-Id$	n/a	$\frac{q^2-1}{2}(-1)^{l_1}$
$Id+z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$	$1\leq b\leq\frac{q-1}{2}$	$\frac{q-1}{2}$
$-\left(Id+z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)$	$1\leq b\leq\frac{q-1}{2}$	$\frac{q-1}{2}(-1)^{l_1}$
$Id+z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$	$\sqrt{b}\notin\mathbb{F}_q$	$-\frac{q+1}{2}$
$-\left(Id+z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right)$	$\sqrt{b}\notin\mathbb{F}_q$	$\frac{q+1}{2}(-1)^{l_1+1}$
$Id+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$	n/a	$\frac{q-1}{2}+q\sum_{\sqrt{\beta}\notin\mathbb{F}_q}e^{\frac{2\pi i\beta\varepsilon}{q}}$
$-\left(Id+z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\right)$	n/a	$(\frac{q-1}{2}+q\sum_{\sqrt{\beta}\notin\mathbb{F}_q}e^{\frac{2\pi i\beta\varepsilon}{q}})(-1)^{l_1}$
$Id+z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	n/a	$\frac{q-1}{2}+q\sum_{\sqrt{\beta}\notin\mathbb{F}_q}e^{\frac{2\pi i\beta}{q}}$
$-\left(Id+z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$	n/a	$(\frac{q-1}{2}+q\sum_{\sqrt{\beta}\notin\mathbb{F}_q}e^{\frac{2\pi i\beta}{q}})(-1)^{l_1}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$b\in\mathbb{F}_q$	$\sum_{\sqrt{\beta}\in\mathbb{F}_q^*}e^{\frac{2\pi i\beta l_2}{q}}e^{\frac{2\pi i\beta^{-1}b\varepsilon}{q}}$
$-\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$b\in\mathbb{F}_q$	$\sum_{\sqrt{\beta}\in\mathbb{F}_q^*}e^{\frac{2\pi i\beta l_2}{q}}e^{\frac{2\pi i\beta^{-1}b\varepsilon}{q}}(-1)^{l_1}$
$\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$b\in\mathbb{F}_q$	$\sum_{\sqrt{\beta}\in\mathbb{F}_q^*}e^{\frac{2\pi i\beta\varepsilon l_2}{q}}e^{\frac{2\pi i\beta^{-1}b\varepsilon}{q}}$
$-\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}\left(Id+z\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\right)$	$b\in\mathbb{F}_q$	$\sum_{\sqrt{\beta}\in\mathbb{F}_q^*}e^{\frac{2\pi i\beta\varepsilon l_2}{q}}e^{\frac{2\pi i\beta^{-1}b\varepsilon}{q}}(-1)^{l_1}$

Where again the induced characters take the value zero on other conjugacy classes.

Next we will induce the characters we have from $G_{[0,0,0]}$ From the previous chapter we have $H_{[0,0,0]} = SL_2(\mathbb{F}_q)$ and so $G_{[0,0,0]} = G$ so if we let $\rho_{[0,0,0]}$ be an irreducible character of $SL_2(\mathbb{F}_q)$ then the induced character of $\tilde{\rho} \otimes \chi_{[0,0,0]} \uparrow G$ will simply be $\tilde{\rho} \otimes \chi_{[0,0,0]}$ so we simply need the irreducible characters of $SL_2(\mathbb{F}_q)$ and then to get the induced characters we will simply have $\tilde{\rho} \otimes \chi_{[0,0,0]} \uparrow G(\mathbf{A}(Id + z\mathbf{a})) = \rho(\mathbf{A})$ where $\mathbf{A} \in SL_2(\mathbb{F}_q)$ and $\mathbf{a} \in sl_2(\mathbb{F}_q)$. So from [2] we have the following $q+4$ irreducible character of $SL_2(\mathbb{F}_q)$

Rep	$1_{H_{[0,0,0]}}$	$W_l\ 1 \le l \le \frac{q-3}{2}$	$X_l\ 1 \le l \le \frac{q-1}{2}$	V	W'	W''	X'	X''
Id	1	$q+1$	$q-1$	q	$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\frac{q-1}{2}$	$\frac{q-1}{2}$
$-Id$	1	$(q+1)(-1)^l$	$(q-1)(-1)^l$	q	$\frac{(q+1)\omega}{2}$	$\frac{(q+1)\omega}{2}$	$\frac{-(q-1)\omega}{2}$	$\frac{-(q-1)\omega}{2}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	1	1	-1	0	s	t	u	v
$-\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	1	$(-1)^l$	$-(-1)^l$	0	s'	t'	v	u
$\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$	1	1	-1	0	t	s	u'	v'
$-\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$	1	$(-1)^l$	$-(-1)^l$	0	t'	s'	v'	u'
$\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix}$	1	$exp^{\frac{2\pi inl}{q-1}} + exp^{-\frac{2\pi inl}{q-1}}$	0	1	$\frac{(-1)^n+(-1)^{-n}}{2}$	$\frac{(1)^n+(-1)^{-n}}{2}$	0	0
$\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix} = \xi^n$	1	0	$-(exp^{\frac{2\pi inl}{q+1}} + exp^{\frac{2\pi inl}{q+1}})$	-1	0	0	$\frac{-((-1)^n+(-1)^{-n})}{2}$	$\frac{-((-1)^n+(-1)^{-n})}{2}$

Where $s, t = \frac{1}{2} \pm \frac{\sqrt{\omega q}}{2}$, $s' = \begin{cases} s & if\ q \equiv 1(mod4) \\ -t & if\ q \equiv 3(mod4) \end{cases}$, $t' = \begin{cases} t & if\ q \equiv 1(mod4) \\ -s & if\ q \equiv 3(mod4) \end{cases}$, $u, v = -\frac{1}{2} \pm \frac{\sqrt{\omega(q)q}}{2}$, $u' = \begin{cases} u & if\ q \equiv 1(mod4) \\ -v & if\ q \equiv 3(mod4) \end{cases}$, $t' = \begin{cases} v & if\ q \equiv 1(mod4) \\ -u & if\ q \equiv 3(mod4) \end{cases}$
and $\omega = \begin{cases} 1 & if\ q \equiv 1(mod4) \\ -1 & if\ q \equiv 3(mod4) \end{cases}$

In order to induce the final set of characters, those arising from $G_{[0,x,1]}$ where $\sqrt{x} \notin \mathbb{F}_q$ we will have to consider two cases seperately, where $q \equiv 1(mod4)$ and $q \equiv 3(mod4)$ we will deal firstly with $q \equiv 1(mod4)$.

First we need to find the conjugacy classes of $G_{[0,x,1]} = \left\{ \left(\begin{array}{cc} a & b \\ bx & a \end{array} \right)(Id + z\mathbf{b}) \middle| a^2 - b^2x = 1, \mathbf{b} \in sl_2(\mathbb{F}_q) \right\}$ We will start by finding the conjugacy classes of the form $Id + z\mathbf{b}$ so in total we will need to cover q^3 elements. As $G_{[0,x,1]}$ is a semi direct product to find the conjugacy classes of the form $Id + z\mathbf{b}$ we only need to consider the action of $\left(\begin{array}{cc} a & b \\ bx & a \end{array} \right)$, where $a^2 - b^2x = 1$. It is interesting to see that the conjugation action on our three dimensional space, $V = \{Id + z\mathbf{b} | \mathbf{b} \in sl_2(\mathbb{F}_q)\}$ is a representation over the field $\mathbb{F}_q$ of $H_{[0,x,1]} = \left\{ \left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) \middle| a^2 - b^2x = 1, \right\}$ i.e. is a modular representation. Unfortunatley though since our field is not algebraically closed it may not be possible to fully decompose our representation into three one dimensional (as the group we are representing is abelian) subrepresentations, which would be possible over an algebraically closed field by Mashke's theorem. It is straightforward to decompose into a one dimensional and a two dimenaional representation by considering the following basis of our vector space. Let $e_1 = \left(\begin{array}{cc} 0 & 1 \\ x & 0 \end{array} \right)$, $e_2 = \left(\begin{array}{cc} 0 & 1 \\ -x & 0 \end{array} \right)$ and $e_3 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$ then we get the following $\left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) e_1 \left(\begin{array}{cc} a & b \\ bx & a \end{array} \right)^{-1} = e_1$, $\left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) e_2 \left(\begin{array}{cc} a & b \\ bx & a \end{array} \right)^{-1} = -2abxe_3 + (b^2x + a^2)e_2$ and $\left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) e_3 \left(\begin{array}{cc} a & b \\ bx & a \end{array} \right)^{-1} = (a^2 + b^2x)e_3 - 2abe_2$ Hence we have split our representation into the following $V = \langle e_1 \rangle \oplus \langle e_2, e_3 \rangle$. We can now use this decomposition to find the conjugacy classes of elements in V. First notice that αe_1 is conjugate only to itself so all class representatives we are looking for will be of the form $\alpha e_1 + v$ where $v \in \langle e_2, e_3 \rangle$ for every $\alpha \in \mathbb{F}_q$. If $v = 0$ then we have q conjugacy class representatives of the form αe_1 each of size 1. Now if we suppose that $v \neq 0$ we will show that βe_2 is not conjugate to $\tilde{\beta} e_2$ where $\tilde{\beta} \neq \beta$, if we assume they are then we get

$$\left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) \beta e_2 \left(\begin{array}{cc} a & -b \\ -bx & a \end{array} \right) = \beta(-2abxe_3 + (b^2x + a^2)e_2) = \tilde{\beta} e_2$$

for some $a^2 - b^2x = 1$, but this means $ab = 0$ which implies $b = 0$ as $q \equiv 1(mod4)$ and hence $a^2 = 1$ and hence we get a contradiction as $\tilde{\beta} = \beta$, hence they are not conjugate in $G_{[0,x,1]}$. Now if we look at the stabiliser of βe_2 for $\beta \neq 0$ we get

$$|Stab_{G_{[0,x,1]}}(\beta e_2)| = q^3 \left| \left\{ \left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) \in SL_2(\mathbb{F}_q) \middle| \left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) \left(\begin{array}{cc} 0 & 1 \\ -x & 0 \end{array} \right) = \left(\begin{array}{cc} 0 & 1 \\ -x & 0 \end{array} \right) \left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) \right\} \right|$$

where the q^3 comes from conjugating by $\mathbf{A}(Id + z\mathbf{a}) \in G_{[0,x,1]}$ and the fact that all the $Id + z\mathbf{a}$ will commute. It is easy to see that from the above we get that $b = -b$ and so $b = 0$ thus $a = \pm 1$ and hence, $|Stab_{G_{[0,x,1]}}(\beta e_2)| = 2q^3$. And so we have $q(q-1)$ $(\beta \neq 0, \alpha, \beta \in \mathbb{F}_q)$ conjugacy class representatives of the form $Id + z(\alpha e_1 + \beta e_2)$ with classes of size $\frac{q+1}{2}$ to go with our q classes of size 1 of the form $Id + z\alpha e_1$.

Now we will show that βe_2 is not conjugate to γe_3 for $\gamma, \beta \in \mathbb{F}_q^*$, this comes from supposing that they are, then we have for some $\left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) \in SL_2(\mathbb{F}_q)$

$$\left(\begin{array}{cc} a & b \\ bx & a \end{array} \right) \beta e_2 \left(\begin{array}{cc} a & -b \\ -bx & a \end{array} \right) = \beta(-2abxe_3 + (b^2x + a^2)e_2) = \gamma e_3$$

but this implies that $(b^2x + a^2) = 0$ but this is impossible as $q \equiv 1(mod4)$ hence they are not conjugate. Similar to above it is straightforward to show that $Id + z\gamma e_3$ for $\gamma \in \mathbb{F}_q$ are all distinct conjugacy classes of size $\frac{q+1}{2}$. So far we have the following conjugacy class representatives, $Id + z\alpha e_1$ of size 1, $Id + z(\alpha e_1 + \beta e_2)$ of size $\frac{q+1}{2}$ and $Id + z(\alpha e_1 + \gamma e_3)$ of size $\frac{q+1}{2}$ for $\alpha \in \mathbb{F}_q, \beta, \gamma \in \mathbb{F}_q^*$, hence we have covered a total of q^3 elements of the form $Id + z\mathbf{a}$ which is all of them. We get an identical result when looking at elements of the form $-(Id + z\mathbf{a})$

Now we will look at the terms of the form $\left(\begin{array}{cc} a & b \\ bx & a \end{array} \right)(Id + z\mathbf{a})$ where $a \neq \pm 1$, then we get the following conjugacy class representatives $\left(\begin{array}{cc} a & b \\ bx & a \end{array} \right)\left(Id + z\left(\begin{array}{cc} 0 & c \\ 0 & 0 \end{array} \right)\right)$, $c \in \mathbb{F}_q$ non of these are conjugate by Lemma 1.6 and the fact that the leading term is invariant under conjugation in $G_{[0,x,1]}$ so (a, b) is not conjugate to $(a, -b)$. The stabiliser of our class representatives is found in the same way as the proof of Lemma 1.6 and has the same size as will be contained by $G_{[0,x,1]}$ so we have

$$\left|Stab_{G_{[0,x,1]}}\left(\left(\begin{array}{cc} a & b \\ bx & a \end{array}\right)\left(Id+z\left(\begin{array}{cc} 0 & c \\ 0 & 0 \end{array}\right)\right)\right)\right| = q(q+1)$$

and hence the following

Lemma 3.8. *The conjugacy classes of $G_{[0,x,1]}$ are the following*

Rep	*var*	*size*	$\lvert C_{G_{[0,x,1]}}\rvert$
$\pm\left(Id+z\left(\begin{array}{cc} 0 & \alpha \\ \alpha x & 0 \end{array}\right)\right)$	$\alpha\in\mathbb{F}_q$	1	$q^3(q+1)$
$\pm\left(Id+z\left(\begin{array}{cc} 0 & \alpha+\beta \\ (\alpha-\beta)x & 0 \end{array}\right)\right)$	$\beta,\alpha\in\mathbb{F}_q,\beta\neq 0$	$\frac{q+1}{2}$	$2q^3$
$\pm\left(Id+z\left(\begin{array}{cc} \gamma & \alpha \\ \alpha x & -\gamma \end{array}\right)\right)$	$\gamma,\alpha\in\mathbb{F}_q,\gamma\neq 0$	$\frac{q+1}{2}$	$2q^3$
$\left(\begin{array}{cc} a & b \\ bx & a \end{array}\right)\left(Id+z\left(\begin{array}{cc} 0 & c \\ 0 & 0 \end{array}\right)\right)$	$a,b,c\in\mathbb{F}_q, a^2-b^2x=1, a\neq\pm 1$	q^2	$q(q+1)$

Theorem 3.9. *The induced character of $\tilde{\rho}_{[0,\mu^2\varepsilon,1],l}\otimes\chi_{[0,\mu^2\varepsilon,1]}$ is*

Conjugacy class representative	*restrictions*	$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l}\otimes\chi_{[0,\mu^2\varepsilon,1]}\uparrow G$
Id	n/a	$q(q-1)$
$-Id$	n/a	$q(q-1)(-1)^l$
$Id+z\left(\begin{array}{cc} b & 0 \\ 0 & -b \end{array}\right)$	$1\leq b\leq\frac{q-1}{2}$	0
$-\left(Id+z\left(\begin{array}{cc} b & 0 \\ 0 & -b \end{array}\right)\right)$	$1\leq b\leq\frac{q-1}{2}$	0
$Id+z\left(\begin{array}{cc} 0 & b \\ 1 & 0 \end{array}\right)$	$\sqrt{b}\notin\mathbb{F}_q$	$-q\left(e^{\frac{4\pi i\kappa_b}{q}}+e^{\frac{-4\pi i\kappa_b}{q}}\right)$
$-\left(Id+z\left(\begin{array}{cc} 0 & b \\ 1 & 0 \end{array}\right)\right)$	$\sqrt{b}\notin\mathbb{F}_q$	$-q\left(e^{\frac{4\pi i\kappa_b}{q}}+e^{\frac{-4\pi i\kappa_b}{q}}\right)(-1)^l$
$Id+z\left(\begin{array}{cc} 0 & \varepsilon \\ 0 & 0 \end{array}\right)$	n/a	$-q$
$-\left(Id+z\left(\begin{array}{cc} 0 & \varepsilon \\ 0 & 0 \end{array}\right)\right)$	n/a	$-q(-1)^l$
$Id+z\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right)$	n/a	$-q$
$-\left(Id+z\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right)\right)$	n/a	$-q(-1)^l$
$\xi^n\left(Id+z\left(\begin{array}{cc} 0 & b \\ 0 & 0 \end{array}\right)\right)$	$1\leq n\leq\frac{q-1}{2}, b\in\mathbb{F}_q$	$e^{\frac{2\pi inl}{q+1}}e^{\frac{2\pi i\varepsilon\mu b}{q}}+e^{\frac{-2\pi inl}{q+1}}e^{\frac{-2\pi i\varepsilon\mu b}{q}}$

where $\kappa_b=\frac{\sqrt{b}}{\sqrt{x}}\in\mathbb{F}_q$ and $x=\mu^2\varepsilon$ so $1\leq\mu\leq\frac{q-1}{2}$. The character takes the value 0 on any other conjugacy classes of G.

Proof. The proof will be structured as follows, only rows 1,3,5,7,9 and 11 will be proved as the others follow as are just multiples of $-Id$. These rows will be proved in a similar method to the previous induced characters but the conjugacy classes of G will split differently over $G_{[0,x,1]}$ depending on the value of μ so how the conjugacy classes split and the induced characters will be dealt with at the same time one row at a time.

Row (1) follows simply from the formula for inducing characters.

Row (2) we know that

$$\left(Id+z\left(\begin{array}{cc} b & 0 \\ 0 & -b \end{array}\right)\right)^G = \{Id+z\mathbf{b}|\, eigenvalues=\pm b\}$$

so we simply need to find all the conjugacy class representatives of the form $Id+z\mathbf{b}$ with eigenvalues $\pm b$ so we will look at the eigenvalues of our representatives. We will refer to the reps of the form $Id+z\mathbf{b}$ simply as $\mathbf{b}$ so our reps will simply be written as αe_1, $\alpha e_1+\beta e_2$ and $\alpha e_1+\gamma e_3$ which give rise to the following eigenvalues respectively $\pm\sqrt{\alpha^2 x}$, $\pm\sqrt{(\alpha^2-\beta^2)x}$ and $\pm\sqrt{\gamma^2+\alpha^2 x}$. From this we see that we don't need to consider elements of the form αe_1

as these have eigenvalues not in $\mathbb{F}_q$ as x is square free. For a fixed value of α then either $\exists\beta \in \mathbb{F}_q^*$ s.t. $\alpha e_1 + \beta e_2$ has eigenvalues $\pm b$ or $\exists\gamma \in \mathbb{F}_q^*$ s.t. $\alpha e_1 + \gamma e_3$ has eigenvalues $\pm b$, both cannot happen at the same time. If we suppose that they both happen at the same time then we get that $b^2 = \alpha^2 x - \beta^2 x$ and $b^2 = \alpha^2 x + \gamma^2$ but if we take one away from the other we get $\gamma^2 + \beta^2 x = 0$ but this is impossible as $q \equiv 1(mod4)$. One of these must happen as follows, we need to cover $\left|\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G\right| = q(q+1)$ elements and we have q choices of α where only one of the following may have a solution $b^2 = \alpha^2 x - \beta^2 x$, $b^2 = \alpha^2 x + \gamma^2$, if one of these has a solution for β or γ then $-\beta$ or $-\gamma$ will also be a solution (we can't have $\beta = 0$ or $\gamma = 0$). So in total we have a maximum of $2q$ possible solutions, but the conjugacy class size means the maximum number of elements we can cover is $2q\frac{q+1}{2}$ but this is the number of elements we need to cover hence for each value of α one of the two options must have a solution. We will now show that for a fixed value of α it doesn't matter which of the two options is true, firstly suppose we have $b^2 = \alpha^2 x - \beta^2 x$ for some $\beta \in \mathbb{F}_q^*$ then $\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]}(Id + z(\alpha e_1 + \beta e_2)) = e^{\frac{2\pi i(x(\alpha-\beta)+(\alpha-\beta)x)}{q}} = e^{\frac{4\pi i\alpha x}{q}}$ but if the second option happens instead i.e. we have $\gamma \in \mathbb{F}_q^*$ s.t. $b^2 = \alpha^2 x + \gamma^2$ we have $\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]}(Id + z(\alpha e_1 + \gamma e_3)) = e^{\frac{4\pi i\alpha x}{q}}$. Now we are in a position to induce the character on these conjugacy classes, so we have:

$$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G(Id + z\mathbf{b}) = \frac{q^3(q-1)}{2q^3}\left(\sum_{\alpha\in\mathbb{F}_q} 2e^{\frac{4\pi i\alpha x}{q}}\right) = 0$$

We will now look at the conjugacy classes in G represented by $Id + z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$ (i.e. row 5). First we will look to see how this conjugacy class splits over our subgroup. As before the conjugacy class in G is simply the elements $Id + z\mathbf{b}$ s.t. $\mathbf{b}$ has eigenvalues $\pm\sqrt{b}$ where $\sqrt{b} \notin \mathbb{F}_q$. Following a similar route to that of the previous row we need to find the set of αe_1, $\alpha e_1 + \beta e_2$ and $\alpha e_1 + \gamma e_3$ s.t. they have eigenvalues $\pm\sqrt{b}$. Looking firstly at the ones of the form αe_1 have the following eigenvalues $\pm\sqrt{\alpha^2 x}$ and letting $\alpha = \pm\kappa_b = \pm\frac{\sqrt{b}}{\sqrt{x}} \in \mathbb{F}_q^*$ we obtain two solutions of the form $\pm\kappa_b e_1$. These are clearly the only solutions of this form. So far we have covered 2 elements and in total we need to cover $q(q-1)$. Now if we look at elements of the form $\alpha e_1 + \beta e_2$ and $\alpha e_1 + \gamma e_3$ to find the rest, we know that $\alpha \neq \pm\kappa_b$ otherwise we have either $\beta \notin \mathbb{F}_q*$ or $\gamma \notin \mathbb{F}_q^*$ so there are at most $q-2$ possibilities for α and as before if for a fixed α $\exists\beta \in \mathbb{F}_q^*$ s.t. $b = (\alpha^2 - \beta^2)x$ then $\nexists\gamma \in \mathbb{F}_q^*$ s.t. $b = \alpha^2 x + \gamma^2$ and visa versa just as before. Hence for any fixed $\alpha \neq \pm\kappa_b$ we have at most two solutions of $b = (\alpha^2 - \beta^2)x$ or $b = \alpha^2 x + \gamma^2$, where the two solutions come form the fact that if β or γ is a solution then so is $-\beta$ or $-\gamma$ (we cannot have $\gamma = 0$ or $\beta = 0$). Now as before we see that the maximum possible number of elements we can cover in this process is $\frac{q+1}{2}2(q-2) = q^2 - q - 2$ (where the $q-2$ is for the choices of $\alpha \neq \pm\kappa$) and if we add this to the two elements we have already covered we see that the maximal number we can cover is $q(q-1)$ which is the size of the conjugacy class in G hence every choice of $\alpha \neq \pm\kappa_b$ must have some pair either $\pm\beta$ s.t. $b = (\alpha^2 - (\pm\beta)^2)x$ or some pair $\pm\gamma$ s.t. $b = \alpha^2 x + (\pm\gamma)^2$ and if we look at the value our cahracter takes on any fixed value of $\alpha \neq \pm\kappa_b$ we see as before it doesn't depend on which form we need (i.e $\alpha e_1 + \beta e_2$ or $\alpha e_1 + \gamma e_3$) hence the induced character is:

$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G(Id + z\mathbf{b}) =$
$q^3(q+1)\left(\frac{e^{\frac{4\pi i\kappa_b}{q}} + e^{\frac{-4\pi i\kappa_b}{q}}}{q^3(q+1)} + \frac{\sum_{\alpha\neq\pm\kappa_b} 2e^{\frac{4\pi ix\alpha}{q}}}{2q^3}\right) = -q\left(e^{\frac{4\pi i\kappa_b}{q}} + e^{\frac{-4\pi i\kappa_b}{q}}\right)$ Where $\mathbf{b} = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$.

Next we will look at the conjugacy class represented by $Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$ (row 7 in the table) from before we know that $Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G = \left\{Id + z\begin{pmatrix} -ac\varepsilon & a^2\varepsilon \\ -c^2\varepsilon & ca\varepsilon \end{pmatrix}\middle| (a,c) \in \mathbb{F}_q^2 \setminus \{0\}\right\}$ from this we see that the top right entry of the matrix and the bottom left are both square free (as $q \equiv 1(mod4)$) or zero, never both zero. It is not possible for an element of the form $Id + z(\alpha e_1 + \gamma e_3)$ to satisfy this as it has either 2 zeros along the reverse diagonal or a square free element, hence all conjugacy class reps in $G_{[0,x,1]}$ must be of the form $Id + z(\alpha e_1 + \beta e_2)$, so for an element of this form to be in our conjugacy class in G we see that either a or c, but not both, must be zero. So we are look at either $Id + z\begin{pmatrix} 0 & a^2\varepsilon \\ 0 & 0 \end{pmatrix}$ (*) or $Id + z\begin{pmatrix} 0 & 0 \\ -c^2\varepsilon & 0 \end{pmatrix}$ (**). There are $\frac{q-1}{2}$ of each type and to get each one we simply need to take $\alpha = \beta = \frac{a^2\varepsilon}{2}$ for type (*) and $\alpha = -\beta = \frac{-c^2\varepsilon}{2x}$ for type(**), in total the conjugacy classes in $G_{[0,x,1]}$ which these elements represent account for $2\frac{q-1}{2}\frac{q+1}{2} = \frac{q^2-1}{2}$, which is the size of the conjugacy class in G. Thus using the formula for induced characters we have the following result:

$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G(Id + z\mathbf{b}) = 2q^4\left(\frac{\sum_{\alpha\in\mathbb{F}_q^*} e^{\frac{4\pi i\alpha x}{q}}}{2q^3}\right) = -q$, where $\mathbf{b} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$

For row (9) we have an identical result which follows from an identical method giving us:
$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G(Id + z\mathbf{b}) = -q$, where $\mathbf{b} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Now we need to turn to look at how $\xi^n \left(Id + z \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right)^G$ splits over $G_{[0,\mu^2\varepsilon,1]}$ for $1 \leq n \leq \frac{q-1}{2}$. First we will look into the relationship between ξ which is a fixed generator of $\left\{ \begin{pmatrix} \alpha & \beta \\ \beta\varepsilon & \alpha \end{pmatrix} \middle| \alpha^2 - \beta^2\varepsilon = 1 \right\} \cong C_{q+1}$ and $H_{[0,\mu^2\varepsilon,1]} = \left\{ \begin{pmatrix} a & b \\ b\mu^2\varepsilon & a \end{pmatrix} \middle| a^2 - (\mu b)^2\varepsilon = 1 \right\}$ if $\xi = \begin{pmatrix} \alpha & \beta \\ \beta\varepsilon & \alpha \end{pmatrix}$ then we will show that it is conjugate to two elements of $H_{[0,\mu^2\varepsilon,1]}$ in $SL_2(\mathbb{F}_q)$. First consider the eigenvalues of ξ in $\mathbb{F}_{q^2}$ the quadratic extension of $\mathbb{F}_q$ we get that $\xi \begin{pmatrix} 1 \\ \sqrt{\varepsilon} \end{pmatrix} = (\alpha + \beta\sqrt{\varepsilon}) \begin{pmatrix} 1 \\ \sqrt{\varepsilon} \end{pmatrix}$ and $\xi \begin{pmatrix} 1 \\ -\sqrt{\varepsilon} \end{pmatrix} = (\alpha - \beta\sqrt{\varepsilon}) \begin{pmatrix} 1 \\ -\sqrt{\varepsilon} \end{pmatrix}$ similarly for a $\begin{pmatrix} a & b \\ b\mu^2\varepsilon & a \end{pmatrix} \in H_{[0,\mu^2\varepsilon,1]}$ we get eigenvalues $a + b\mu\sqrt{\varepsilon}$ and $a - b\mu\sqrt{\varepsilon}$. So we see that if this elements are going to be conjugate in G we must have $a = \alpha$ and $b\mu = \pm\beta$ due to the fact that $1, \sqrt{\varepsilon}$ form a basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Claim that ξ is conjugate to $\begin{pmatrix} \alpha & \pm\frac{\beta}{\mu} \\ \pm\frac{\beta x}{\mu} & \alpha \end{pmatrix}$. This is where the argument splits into two subcases:

If μ is square then we have $\begin{pmatrix} \frac{1}{\sqrt{\pm\mu}} & 0 \\ 0 & \sqrt{\pm\mu} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta\varepsilon & \alpha \end{pmatrix} \begin{pmatrix} \sqrt{\pm\mu} & 0 \\ 0 & \frac{1}{\sqrt{\pm\mu}} \end{pmatrix} = \begin{pmatrix} \alpha & \frac{\pm\beta}{\mu} \\ \pm\mu\varepsilon\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & \frac{\pm\beta}{\mu} \\ \frac{\pm x\beta}{\mu} & \alpha \end{pmatrix}$ using the fact that $\mu^2\varepsilon = x$. As conjugate elements in $SL_2(\mathbb{F}_q)$ must have the same order we see that since $|\xi| = q + 1$ we have $\left| \begin{pmatrix} \alpha & \frac{\beta}{\mu} \\ \frac{x\beta}{\mu} & \alpha \end{pmatrix} \right| = q + 1$ and hence is a cyclic generator of $H_{[0,x,1]}$ so from Lemma 2.2 we define $\eta = \begin{pmatrix} \alpha & \frac{\beta}{\mu} \\ \frac{x\beta}{\mu} & \alpha \end{pmatrix}$ and so $H_{[0,x,1]} = \langle \eta \rangle$. This results in the following:
$\begin{pmatrix} \frac{1}{\sqrt{\pm\mu}} & 0 \\ 0 & \sqrt{\pm\mu} \end{pmatrix} \xi^n \left(Id + z \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \sqrt{\pm\mu} & 0 \\ 0 & \frac{1}{\sqrt{\pm\mu}} \end{pmatrix} = \eta^{\pm n} \left(Id + z \begin{pmatrix} 0 & \frac{\pm b}{\mu} \\ 0 & 0 \end{pmatrix} \right)$
I claim $\xi^n \left(Id + z \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right)^G \cap G_{[0,\mu^2\varepsilon,1]} = \eta^n \left(Id + z \begin{pmatrix} 0 & \frac{b}{\mu} \\ 0 & 0 \end{pmatrix} \right)^{G_{[0,\mu^2\varepsilon,1]}} \cup \eta^{-n} \left(Id + z \begin{pmatrix} 0 & \frac{-b}{\mu} \\ 0 & 0 \end{pmatrix} \right)^{G_{[0,\mu^2\varepsilon,1]}}$, for $1 \leq n \leq \frac{q-1}{2}$ and $b \in \mathbb{F}_q$, this follows by considering how many elements we need to cover in $G_{[0,\mu^2\varepsilon,1]}$, we need to cover $(q-1)q^3$ elements with the classes above, i.e. all the elements we haven't previously covered, but for each $\left| \eta^n \left(Id + z \begin{pmatrix} 0 & \frac{b}{\mu} \\ 0 & 0 \end{pmatrix} \right)^{G_{[0,\mu^2\varepsilon,1]}} \cup \eta^{-n} \left(Id + z \begin{pmatrix} 0 & \frac{-b}{\mu} \\ 0 & 0 \end{pmatrix} \right)^{G_{[0,\mu^2\varepsilon,1]}} \right| = 2q^2$ and we have $\frac{q-1}{2}$ choices of n and for each choice of n we have a further q choices of b hence we have covered the required number of elements so we are done. And hence by the formula for induced characters we have $\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G(\xi^n(Id + z\mathbf{b})) = \frac{q(q+1)}{q(q+1)} \left(e^{\frac{2\pi inl}{q+1}} e^{\frac{2\pi i\varepsilon\mu b}{q}} + e^{\frac{-2\pi inl}{q+1}} e^{\frac{-2\pi i\varepsilon\mu b}{q}} \right)$, where $\mathbf{b} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$.

If μ is square free we follow a very similar line of argument where we find ξ is conjugate to $\begin{pmatrix} \alpha & \frac{\pm\beta}{\mu} \\ \frac{\pm\beta x}{\mu} & \alpha \end{pmatrix}$ where we conjugate by $\begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}^i \begin{pmatrix} 0 & \sqrt{\frac{-1}{\mu\varepsilon}} \\ \varepsilon\mu\sqrt{\frac{-1}{\mu\varepsilon}} & 0 \end{pmatrix}$ (where $i = 0$ gives the $+$ and $i = 1$ gives $-$ in the $\pm$) hence we can define $\eta = \begin{pmatrix} \alpha & \frac{\beta}{\mu} \\ \frac{\beta x}{\mu} & \alpha \end{pmatrix}$. Then we find that

$$\xi^n \left(Id + z \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right)^G \cap G_{[0,\mu^2\varepsilon,1]} = \eta^n \left(Id + z \begin{pmatrix} 0 & 0 \\ \varepsilon\mu\beta & 0 \end{pmatrix} \right)^{G_{[0,\mu^2\varepsilon,1]}} \cup \eta^{-n} \left(Id + z \begin{pmatrix} 0 & 0 \\ -\varepsilon\mu\beta & 0 \end{pmatrix} \right)^{G_{[0,\mu^2\varepsilon,1]}}$$

and we get the induced character from the formula as required. □

Hence by Theorem 2.1 we have completed the character table for $q \equiv 1(mod4)$.

Theorem 3.10. *For $q \equiv 3(mod4)$ the induced character of $\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G$ is the same as when $q \equiv 1(mod4)$.*

Proof. Unfortunately as in the case where $q \equiv 1(mod4)$ we cannot find explicit conjugacy class representatives of our group $G_{[0,\mu^2\varepsilon,1]}$ quite so straight forwardly so we shall take a slightly indirect route to inducing the character. First we will induce the character on most of the conjugacy classes of G with representatives of the form $Id + z\mathbf{b}$ where $\mathbf{b} \in sl_2(\mathbb{F}_q)$. In order to understand how these conjugacy classes split over our subgroup as before we only need to consider the action of $H_{[0,\mu^2\varepsilon,1]}$ on $\{Id + z\mathbf{b} | \mathbf{b} \in sl_2(\mathbb{F}_q)\}$ by conjugation. As before this gives us a representation of $H_{[0,\mu^2\varepsilon,1]}$ over $sl_2(\mathbb{F}_q)$, which can be split into a direct sum of subrepresentations exactly as before using the following basis of $sl_2(\mathbb{F}_q)$, $e_1 = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ -x & 0 \end{pmatrix}$ and $e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we have $sl_2(\mathbb{F}_q) = \langle e_1 \rangle \oplus \langle e_2, e_3 \rangle$ as representations, where e_1 is stable under the action of $H_{[0,\mu^2\varepsilon,1]}$. Since we know that e_1 is stable under the action we know we will have conjugacy class representatives of the form $\alpha e_1 + \mathbf{b}$ where $\mathbf{b} \in \langle e_2, e_3 \rangle$. So we will concentrate on trying to find the conjugacy classes of $\langle e_2, e_3 \rangle$. First if we have $\alpha e_2 + \beta e_3$ ($(\alpha, \beta) \neq (0,0)$) then we have

$$|Stab_{H_{[0,x,1]}}(Id + z(\alpha e_2 + \beta e_3))| = \left| \left\{ \begin{pmatrix} a & b \\ bx & a \end{pmatrix} \in H_{[0,x,1]} \middle| \begin{pmatrix} a & b \\ bx & a \end{pmatrix} (\alpha e_2 + \beta e_3) = (\alpha e_2 + \beta e_3) \begin{pmatrix} a & b \\ bx & a \end{pmatrix} \right\} \right| =$$

$$\left| \left\{ \begin{pmatrix} a & b \\ bx & a \end{pmatrix} \in H_{[0,x,1]} \middle| \begin{pmatrix} a\beta - b\alpha x & a\alpha - b\beta \\ b\beta x - a\alpha x & bx\alpha - a\beta \end{pmatrix} = \begin{pmatrix} \beta a + \alpha bx & \beta b + a\alpha \\ -\alpha a x - \beta bx & -\alpha bx - \beta a \end{pmatrix} \right\} \right| = \left| \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in H_{[0,x,1]} \right\} \right|$$

So we have $|Stab_{H_{[0,x,1]}}(Id + z(\alpha e_2 + \beta e_3))| = 2$ and so $|Orb_{H_{[0,x,1]}}(Id + z(\alpha e_2 + \beta e_3))| = \frac{q+1}{2}$. Looking at our subrepresentation of $sl_2(\mathbb{F}_q)$ w.r.t. our basis e_2, e_3 the conjugation action of $\begin{pmatrix} a & b \\ bx & a \end{pmatrix}$ is given by the matrix $\begin{pmatrix} b^2x + a^2 & -2ab \\ -2abx & b^2x + a^2 \end{pmatrix}$. Now if we will look at the set of lines in $\langle e_2, e_3 \rangle$, of which there are $q+1$, where these lines will be deonted by $L_1 = [e_3], L_2 = [e_2], L_3 = [e_2 + e_3], \cdots, L_{q+1} = [e_2 + (q-1)e_3]$, we want to see what the stabiliser of a line is under our action. Firstly we will look to see for what values of a, b we can diagonalize our action, i.e. diagonalize the matrix $\mathbf{A} = \begin{pmatrix} b^2x + a^2 & -2ab \\ -2abx & b^2x + a^2 \end{pmatrix}$, we see that the $tr(\mathbf{A}) = 2(b^2x + a^2)$ and $det(\mathbf{A}) = (a^2 + b^2x)^2 - 4a^2b^2x$ and hence the characteristic polynomial $C_{\mathbf{A}}(\lambda) = \lambda^2 - tr(\mathbf{A})\lambda + det(\mathbf{A})$ and looking at the discriminant D of this equation we get $D = tr(\mathbf{A})^2 - 4det(\mathbf{A}) = 16a^2b^2x$ which is square free unless either $b = 0$ or $a = 0$, so from this we can see if $a, b \neq 0$ then our line will be mapped off itself. If we let $b = 0$ then we have $a = \pm 1$ as $a^2 - b^2x = 1$ and so our action is given by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which stabilises all lines, and if we let $a = 0$ we get $-b^2x = 1$ which has 2 solutions $\pm c$ for some $c \in \mathbb{F}_q^*$ since $q \equiv 3(mod4)$ and so our action is given by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ which again stabilises all the lines, hence we have $|Stab_{H_{[0,x,1]}}(L_i)| = 4$. Now if we look at the orbit of a fixed line L_i, this will lead to $\frac{q-1}{2}$ orbits in $\langle e_2, e_3 \rangle$ as we can only map $\alpha_2 e_2 + \alpha_3 e_3$ to $-(\alpha_2 e_2 + \alpha_3 e_3)$ within L_i, and each line has $q-1$ non zero points. Now if we have a point $\alpha_2 e_2 + \alpha_3 e_3$ on our line L_i and it is mapped to another line $L_j = [\beta_2 e_2 + \beta_3 e_3]$ then it is straightforward to see that every point on L_j will be covered by mapping from the points on L_i. So given a line L_i this gives rise to $\frac{q-1}{2}$ conjugacy classes of size $\frac{q+1}{2}$ so a given line leads to us covering $\frac{q^2-1}{4}$ elements in $\langle e_2, e_3 \rangle$, and so to cover the $q^2 - 1$ non zero elements in $\langle e_2, e_3 \rangle$ we simply need to select 4 lines which are not mapped onto each other by our action. Only needing to pick 4 lines which all have orbits of the same size tells us that each line maps to $\frac{q+1}{4}$ other lines (including the original line).

We cannot explicitly pick the 4 lines but if we have a line $L_i = [\alpha_2 e_2 + \alpha_3 e_3]$ and we look at any two points on our line, $p_i = \lambda_i(\alpha_2 e_2 + \alpha_3 e_3)$ we see that $det(p_i) = \lambda_i^2 det(\alpha_2 e_2 + \alpha_3 e_3)$, so we can say a line is square or square free depending on whether the determinant of a representative of the line is square or square free and since our action is by conjugation and so preserves determinant we see that a square(free) line can only be mapped to square(free) lines. Hence we have split the 4 lines we need to find into two sets, square and square free, next is to show that these two sets of lines must both be of size 2. If we fix the representatives of our lines as the following $L_1 = [e_3], L_2 = [e_2], L_3 = [e_2 + e_3], \cdots, L_{q+1} = [e_2 + (q-1)e_3]$ then these can be expressed as L_1, L_2 and $L_\alpha = [e_2 + \alpha e_3]$ where $\alpha \in \mathbb{F}_q^*$, then we get $det(L_1) = -1$ which is square free, $det(L_2) = x$ which is square free and $det(L_\alpha) = -\alpha^2 + x$. In order to find the number of $\alpha \in \mathbb{F}_q^*$ s.t. $-\alpha^2 + x$ is square or square free is just an application of theorem 1.1, as follows, suppose $-\alpha^2 + x$ is sqaure so we have $-\alpha^2 + x = \beta^2$ where β must be non zero (a zero solution is impossible anyway so this is not a restriction) then rearanging gives us a nondegenerate bilinear form $\alpha^2 + \beta^2 = x$ and so this has $q+1$ solutions by the theorem, we know $\alpha \neq 0 \neq \beta$ so for any fixed α which we obtain a solution this gives us two solutions $\pm\beta$ and so the number of α s.t. there exists a β s.t. $-\alpha^2 + x = \beta^2$ is

$\frac{q+1}{2}$ hence we have $\frac{q+1}{2}$ lines which are square and $2+\frac{q-3}{2}=\frac{q+1}{2}$ lines which are square free. So since a line will map to $\frac{q+1}{4}$ other lines then 2 of the lines we pick must be square and 2 must be square free in order to ensure all the lines are covered. Now let $\tilde{L_1}, \tilde{L_2}, \tilde{L_3}, \tilde{L_4}$ be our line representatives where 1 and 2 are square and 3 and 4 are square free. If we let f_1, f_2, f_3, f_4 be some fixed representatives of our lines we now have the following conjugacy class representatives of $\{Id + z\mathbf{b} | \mathbf{b} \in sl_2(\mathbb{F}_q)\}$ in $G_{[0,x,1]}$,

$Class\, representative$	$Variables$	$Class\, size$	λ^2
$Id + z\alpha e_1$	$\alpha \in \mathbb{F}_q$	1	$\alpha^2 x$
$Id + z(\alpha e_1 + \beta f_i)$	$\alpha \in \mathbb{F}_q, 1 \leq \beta \leq \frac{q-1}{2}, 1 \leq i \leq 4$	$\frac{q+1}{2}$	$\alpha^2 x - \beta^2 det(f_i)$

Where λ are the eigenvalues of $\mathbf{b}$ in $Id + z\mathbf{b}$.

We are now in a position to induce our characters. Observe that $\chi_{[0,x,1]}(\alpha e_1 + \beta f_i) = \chi_{[0,x,1]}(\alpha e_1)$.

Firstly we see that the induced characters of $\pm Id$ will be exactly the same as for $q \equiv 1(mod 4)$, so the first conjugacy classes to look at are $\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G = \{Id + z\mathbf{b} |\, eigenvalues = \pm b\}$, where $1 \leq b \leq \frac{q-1}{2}$. Now we see how one of these classes splits over $G_{[0,x,1]}$. It is clear that none of the classes of the form $Id + ze_1$ are required as these have eigenvalues not in $\mathbb{F}_q$, so we know that all the classes that make up the $q(q+1)$ elements in $\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G$ have size $\frac{q+1}{2}$ hence we need a total of $2q$ classes in $G_{[0,x,1]}$. Consider a fixed $\alpha \in \mathbb{F}_q$ then of the 4 equations $\alpha^2 x - \beta^2 det(f_i) = b^2$ we can have at most 2 which have solutions, this can easily be shown by repeating the following argument for all the possible combinations, suppose we have $\alpha^2 x - \beta^2 det(f_1) = b^2$ and $\alpha^2 x - \beta_3^2 det(f_3) = b^2$ or $\alpha^2 x - \beta_4^2 det(f_4) = b^2$, then this implies $\beta^2 det(f_1) = \beta_3^2 det(f_3)$ or $\beta^2 det(f_1) = \beta_4^2 det(f_4)$ but both of these options are impossible as $det(f_1)$ is square whereas $det(f_3)$ and $det(f_4)$ are square free. So for each choice of α we have at most 2 conjugacy classes that are in $\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G$ (note we don't get 4 as if $\alpha^2 x - \beta^2 det(f_i) = b^2$ only one of the $\pm\beta$ solutions is needed as they lie in the same class), and since we need a total of 2q conjugacy classes of this form we have that for any fixed α either $Id+z(\alpha e_1+\beta_1 f_1)^{G_{[0,x,1]}}$ and $Id+z(\alpha e_1+\beta_2 f_2)^{G_{[0,x,1]}}$ or $Id+z(\alpha e_1+\beta_3 f_3)^{G_{[0,x,1]}}$ and $Id+z(\alpha e_1+\beta_4 f_4)^{G_{[0,x,1]}}$ are in $\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G$, never both of the options but always one. Now if for a fixed α we have $Id+z(\alpha e_1+\beta_1 f_1)^{G_{[0,x,1]}}, Id+z(\alpha e_1+\beta_2 f_2)^{G_{[0,x,1]}} \subset \left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G$ then $\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]}(Id + z(\alpha e_1 + \beta_1 f_1)) = \tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]}(Id + z(\alpha e_1 + \beta_2 f_2)) = e^{\frac{4\pi i\alpha x}{q}}$, if instead for our fixed α we have $Id + z(\alpha e_1 + \beta_3 f_3)^{G_{[0,x,1]}}, Id + z(\alpha e_1 + \beta_4 f_4)^{G_{[0,x,1]}} \subset \left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right)^G$, then we get $\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]}(Id + z(\alpha e_1 + \beta_3 f_3)) = \tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]}(Id + z(\alpha e_1 + \beta_4 f_4)) = e^{\frac{4\pi i\alpha x}{q}}$. So of the two options which can happen we get the same character and hence we have

$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G\left(Id + z\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\right) = (q-1)\sum_{\alpha\in\mathbb{F}_q} 2e^{\frac{4\pi i\alpha x}{q}} = 0$

Now to induce our character on classes of the form $Id + z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$ where b is some square free. So as before we need to see how one of these conjugacy classes splits over our subgroup. These conjugacy classes are just the set of elements of the form $Id+z\mathbf{b}$ where $\mathbf{b}$ has eigenvalues $\pm\sqrt{b}$, which has size $q(q-1)$. First we see that we have look at the following elements , $Id+z\alpha e_1$ we see that if $\alpha = \pm\sqrt{\frac{b}{x}} = \kappa_b$ then αe_1 has the required eigenvalues and hence we have $Id \pm z\kappa_b e_1^{G_{[0,x,1]}} \subset Id+z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}^G$. Now if we consider $\alpha \neq \pm\kappa_b$ and follow the procedure of the previous type of conjugacy class we see that for a given $\alpha \neq \pm\kappa_b$ we have either $Id+z(\alpha e_1+\beta_1 f_1)^{G_{[0,x,1]}}, Id+z(\alpha e_1+\beta_2 f_2)^{G_{[0,x,1]}} \subset \left(Id + z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right)^G$ or $Id+z(\alpha e_1+\beta_3 f_3)^{G_{[0,x,1]}}, Id+z(\alpha e_1+\beta_4 f_4)^{G_{[0,x,1]}} \subset \left(Id + z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right)^G$ but never both (where the β_i ensure correct eigenvalues), and since the character is not effected by which of the options is true for a given α we get that

$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G\left(Id + z\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\right) = q^3(q+1)\left(\frac{e^{\frac{4\pi i\kappa_b}{q}} + e^{\frac{-4\pi i\kappa_b}{q}}}{q^3(q+1)} + \frac{\sum_{\alpha\neq\pm\kappa_b} 2e^{\frac{4\pi ix\alpha}{q}}}{2q^3}\right) = -q\left(e^{\frac{4\pi i\kappa_b}{q}} + e^{\frac{-4\pi i\kappa_b}{q}}\right)$

Next we will induce our character on elements of the form $\xi^n\left(Id + z\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right)$ where $1 \leq n \leq \frac{q-1}{2}$, $b \in \mathbb{F}_q$. First observe that in the preamble to Lemma 3.8 to find the conjugacy classes of $G_{[0,x,1]}$ in the discussion of the last row of the table we don't assume anything about the values of $q(mod4)$ hence the bottom row of the table of conjugacy classes is still valid for $q \equiv 3(mod4)$. First we see that if $\xi = \begin{pmatrix} \alpha & \beta \\ \beta\varepsilon & \alpha \end{pmatrix}$ then if μ is square we have $\xi^n \left(Id + z\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right)$ is conjugate in G to $\begin{pmatrix} \alpha & \frac{\beta}{\mu} \\ \frac{\beta x}{\mu} & \alpha \end{pmatrix}^n \left(Id + z\begin{pmatrix} 0 & \frac{b}{\mu} \\ 0 & 0 \end{pmatrix}\right)$ and $\begin{pmatrix} \alpha & \frac{-\beta}{\mu} \\ \frac{-\beta x}{\mu} & \alpha \end{pmatrix}^n \left(Id + z\begin{pmatrix} \frac{abcx}{\mu} & \frac{a^2b}{\mu} \\ \frac{-c^2x^2b}{\mu} & \frac{-abcx}{\mu} \end{pmatrix}\right)$ where conjugation is by $\begin{pmatrix} \frac{1}{\sqrt{\mu}} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix}$ and $\begin{pmatrix} a & c \\ -cx & -a \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{\mu}} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix}$ respectively, where $-a^2 + c^2x = 1$ (there are such matrices by considering theorem 1.1). From identical reasoning about the numbers of elements as in the proof of theorem 3.9 we have $\xi^n \left(Id + z\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right)^G \cap G_{[0,\mu^2\varepsilon,1]} = \eta^n \left(Id + z\begin{pmatrix} 0 & \frac{b}{\mu} \\ 0 & 0 \end{pmatrix}\right)^{G_{[0,\mu^2\varepsilon,1]}} \cup \eta^{-n} \left(Id + z\begin{pmatrix} \frac{abcx}{\mu} & \frac{a^2b}{\mu} \\ \frac{-c^2x^2b}{\mu} & \frac{-abcx}{\mu} \end{pmatrix}\right)^{G_{[0,\mu^2\varepsilon,1]}}$ where as in the proof of theorem 3.9 we let $\eta = \begin{pmatrix} \alpha & \frac{\beta}{\mu} \\ \frac{\beta x}{\mu} & \alpha \end{pmatrix}$ be the cyclic generator of $H_{[0,x,1]}$. Hence when μ is square free we have the following

$$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G\left(\xi^n \left(Id + z\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right)\right) = e^{\frac{2\pi inl}{q+1}} e^{\frac{2\pi ibx}{\mu q}} + e^{\frac{-2\pi inl}{q+1}} e^{\frac{2\pi i(-c^2x+a^2)bx}{\mu q}}$$

$$= e^{\frac{2\pi inl}{q+1}} e^{\frac{2\pi ibx}{\mu q}} + e^{\frac{-2\pi inl}{q+1}} e^{\frac{-2\pi ibx}{\mu q}} = e^{\frac{2\pi inl}{q+1}} e^{\frac{2\pi ib\varepsilon\mu}{q}} + e^{\frac{-2\pi inl}{q+1}} e^{\frac{-2\pi ib\varepsilon\mu}{q}}$$

as $-a^2 + c^2x = 1$ and $x = \mu^2\varepsilon$.

If μ is square free we find $\xi^n \left(Id + z\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right)$ is conjugate in G to $\begin{pmatrix} \alpha & \frac{-\beta}{\mu} \\ \frac{-\beta x}{\mu} & \alpha \end{pmatrix}^n \left(Id + z\begin{pmatrix} 0 & \frac{-b}{\mu} \\ 0 & 0 \end{pmatrix}\right)$ and $\begin{pmatrix} \alpha & \frac{\beta}{\mu} \\ \frac{\beta x}{\mu} & \alpha \end{pmatrix}^n \left(Id + z\begin{pmatrix} \frac{-abcx}{\mu} & \frac{-a^2b}{\mu} \\ \frac{c^2x^2b}{\mu} & \frac{abcx}{\mu} \end{pmatrix}\right)$ where conjugation is by $\begin{pmatrix} \frac{1}{\sqrt{-\mu}} & 0 \\ 0 & \sqrt{-\mu} \end{pmatrix}$ and $\begin{pmatrix} a & c \\ -cx & -a \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{-\mu}} & 0 \\ 0 & \sqrt{-\mu} \end{pmatrix}$ respectively, where $-a^2 + c^2x = 1$. Following the process as for μ square we get

$$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G\left(\xi^n \left(Id + z\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right)\right) = e^{\frac{2\pi inl}{q+1}} e^{\frac{2\pi ib\varepsilon\mu}{q}} + e^{\frac{-2\pi inl}{q+1}} e^{\frac{-2\pi ib\varepsilon\mu}{q}}$$

Now all is left is to look at the conjugacy classes in G given by $Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$. The first thing to note is that as $q \equiv 3(mod4)$ we will have that $Id + z\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \in Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G$ and hence we must have for any character $\chi\left(Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\right) = \overline{\chi\left(Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)}$. All the elements in these conjugacy classes in G have zero eigeinvalues, in the z coefficient and the z coefficients are all non zero. So the only way we can acheive zero eigenvalues with non zero matrices from our conjugacy class representatives is by considering elements of the form $\alpha e_1 + \beta f_3$ and $\alpha e_1 + \beta f_4$ where $1 \leq \beta \leq \frac{q-1}{2}$(this ensures the choice of β is unique as we know $\alpha e_1 + \beta f_i$ is in the same conjugacy class as $\alpha e_1 - \beta f_i$) and $\alpha \neq 0$ (as we require $\alpha^2 x - \beta^2 det(f_i) = 0$ or $\alpha^2 x = 0$) . For a fixed value of $\alpha \neq 0$ then we get one conjugacy class of each of the two types (with either f_3 or f_4) available to us with zero eigenvalues and non zero, hence we have a total of $2(q-1)$ conjugacy classes which have z coefficients with zero eiginvalues but which are themselves non-zero. By considering the size of each of these conjugacy classes we see that we need $q-1$ of them to make up each of $Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G$ and $Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^G$. So we have that $Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G$ is some union $q-1$ conjugacy classes, define $\mathbf{S}_1$ to be the set of $q-1$ conjugacy class representatives of the form $\alpha e_1 + \beta f_3$ and $\alpha e_1 + \beta f_4$ ($\alpha \neq 0$ and $1 \leq \beta \leq \frac{q-1}{2}$) s.t. $Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^G = \cup_{S \in \mathbf{S}_1} S^{G_{[0,x,1]}}$ and $\mathbf{S}_2$ similarly s.t. $Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}^G = \cup_{S \in \mathbf{S}_2} S^{G_{[0,x,1]}}$. Now we can define $\phi : \mathbf{S}_1 \cup \mathbf{S}_2 \to \mathbb{F}_q^*$ by $\phi(S) = \phi(\alpha e_1 + \beta f_i) = \alpha$ and we can observe that since for each $\alpha \neq 0$ we get two conjugacy class representatives we have that $\sum_{S\in\mathbf{S}_1\cup\mathbf{S}_2} e^{\frac{2\pi i\phi(S)}{q}} = \sum_{\alpha\in\mathbb{F}_q^*} 2e^{\frac{2\pi i\alpha}{q}} = -2 = \sum_{\alpha\in\mathbb{F}_q^*} 2e^{\frac{4\pi i\alpha}{q}} = \sum_{S\in\mathbf{S}_1\cup\mathbf{S}_2} e^{\frac{4\pi i\phi(S)}{q}}$. Now it is important to notice that $\chi_{[0,x,1]}(S) = e^{\frac{4\pi i\phi(S)}{q}}$ and hence we have by the induced character formula

$$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G\left(Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \frac{\frac{2q^4(q^2-1}{q^2-1}}{\frac{2q^3(q+1)}{q+1}} \sum_{S\in\mathbf{S}_1} e^{\frac{4\pi i\phi(S)}{q}} = q\sum_{S\in\mathbf{S}_1} e^{\frac{4\pi i\phi(S)}{q}}$$

$$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G\left(Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\right) = q\sum_{S\in\mathbf{S}_2} e^{\frac{4\pi i\phi(S)}{q}}$$

and so we see that

$$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l}\otimes\chi_{[0,\mu^2\varepsilon,1]} \uparrow G\left(Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)+\tilde{\rho}_{[0,\mu^2\varepsilon,1],l}\otimes\chi_{[0,\mu^2\varepsilon,1]} \uparrow G\left(Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\right) = q\sum_{S\in\mathbf{S}_1\cup\mathbf{S}_2} e^{\frac{4\pi i\phi(S)}{q}} = -2q$$

We will abbreviate some of the notation for the next section, let $\chi = \tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G$ and denote $W = \sum_{S\in\mathbf{S}_1} e^{\frac{4\pi i\phi(S)}{q}}$ so we have $W+\overline{W} = -2$,since $\chi\left(Id + z\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\right)+\chi\left(Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \overline{\chi\left(Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)}+\chi\left(Id + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = q(\overline{W} + W) = -2q$. Now all is left to show is that $W = -1$, we will do this by taking the inner product of our character with itself, since we know it is irreducible we must have $\langle\chi,\chi\rangle = 1$. We get that

$q^4(q^2-1) = |G|\langle\chi,\chi\rangle = 2q^2(q-1)^2 + 2q(q-1)\sum_{\sqrt{b}\notin\mathbb{F}_q} q^2(e^{\frac{4\pi i\kappa_b}{q}} + e^{\frac{-4\pi i\kappa_b}{q}})^2 + \frac{q^2-1}{2}(4q^2W\overline{W})$
$+q^3(q-1)\sum_{b\in\mathbb{F}_q}\sum_{1\leq n\leq\frac{q-1}{2}}(e^{\frac{2\pi inl}{q+1}}e^{\frac{2\pi i\varepsilon b}{q}} + e^{\frac{-2\pi inl}{q+1}}e^{\frac{-2\pi i\varepsilon b}{q}})^2 = q^6 - 3q^4 + 2q^2 + 2(q^2-1)q^2W\overline{W}$

Hence we must have $W\overline{W} = 1$, combining this with $W + \overline{W} = -2$ we get $W^2 + 1 = -2W$ which is $(W+1)^2 = 0$ and hence $W = -1$ thus by Theorem 2.1 completing the character table and varifying that the character is the same for $q \equiv 3(mod 4)$. □

So to recap we have from Theorem 2.1 the irreducible characters of $SL_2(\mathbb{F}_q[z]/\langle z^2\rangle)$ are the following

Chracter	*Variables*
$\tilde{\rho}_{[I,0,0],l} \otimes \chi_{[I,0,0]} \uparrow G$	$1 \leq I \leq \frac{q-1}{2}, 1 \leq l \leq q-1$
$\tilde{\rho}_{[0,1,0],l_1,l_2} \otimes \chi_{[0,1,0]} \uparrow G$	$1 \leq l_1 \leq 2, 1 \leq l_2 \leq q$
$\tilde{\rho}_{[0,\varepsilon,0],l_1,l_2} \otimes \chi_{[0,\varepsilon,0]} \uparrow G$	$1 \leq l_1 \leq 2, 1 \leq l_2 \leq q$
$\tilde{\rho}_{[0,\mu^2\varepsilon,1],l} \otimes \chi_{[0,\mu^2\varepsilon,1]} \uparrow G$	$1 \leq \mu \leq \frac{q-1}{2}, 1 \leq l \leq q+1$

together with the irreducible characters of $SL_2(\mathbb{F}_q)$ which have all been described in this last chapter.

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