Lecture Notes

Reflection Groups

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## CONTENTS

1 Finite Reflection Groups ........................................ 3

2 Root systems ................................................................ 6

3 Generators and Relations ........................................... 14

4 Coxeter group .......................................................... 16

5 Geometric representation of $W(m_{ij})$ ..................... 21

6 Fundamental chamber ................................................ 28

7 Classification .......................................................... 34

8 Crystallographic Coxeter groups ................................. 43

9 Polynomial invariants ............................................... 46

10 Fundamental degrees ............................................ 54

11 Coxeter elements .................................................... 57
1 Finite Reflection Groups

\[ V = (V, \langle \cdot, \cdot \rangle) \] - Euclidean Vector space where \( V \) is a finite dimensional vector space over \( \mathbb{R} \) and \( \langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R} \) is bilinear, symmetric and positiv definit.

**Example:** \( (\mathbb{R}^n, \cdot) : \langle (\alpha_i), (\beta_i) \rangle = \sum_{i=1}^{n} \alpha_i \beta_i \)

Gram-Schmidt theory tells that for all Euclidean vector spaces, there exists an isometry (linear bijective and \( \forall x, y \in V : T(x) \cdot T(y) = \langle x, y \rangle \)) \( T : V \rightarrow \mathbb{R}^n \).

In \((V, \langle \cdot, \cdot \rangle)\) you can

- measure length: \( ||x|| = \sqrt{\langle x, x \rangle} \)
- measure angles: \( \arccos \left( \frac{\langle x, y \rangle}{||x|| \cdot ||y||} \right) \)
- talk about orthogonal transformations

\[ O(V) = \{ T \in \text{GL}(V) : \forall x, y \in V : \langle Tx, Ty \rangle = \langle x, y \rangle \} \leq \text{GL}(V) \]

\( T \in \text{GL}(V) \). Let \( V^T = \{ v \in V : Tv = v \} \) the fixed points of \( T \) or the 1-eigenspace.

**Definition.** \( T \in \text{GL}(V) \) is a reflection if \( T \in O(V) \) and \( \dim V^T = \dim V - 1 \).

**Lemma 1.1.** Let \( T \) be a reflection, \( x \in (V^T)^\perp = \{ v : \forall w \in V^T : \langle v, w \rangle = 0 \}, \ x \neq 0 \). Then

1. \( T(x) = -x \)
2. \( \forall z \in V : T(z) = z - 2 \frac{\langle z, x \rangle}{\langle x, x \rangle} \cdot x \)

**Proof.**

1. Pick \( v \in V^T \implies Tv = v \implies \langle Tx, v \rangle = \langle Tx, Tv \rangle = \langle x, v \rangle = 0 \). Hence \( Tx \in (V^T)^\perp \)
   Since \( \dim (V^T)^\perp = \dim V - \dim V^T = 1 : Tx = \alpha \cdot x \) for some \( \alpha \in \mathbb{R} \). Then
   \[ \alpha^2 \langle x, x \rangle = \langle \alpha x, \alpha x \rangle = \langle Tx, Tx \rangle = \langle x, x \rangle \]
   Since \( \langle x, x \rangle \neq 0 \), \( \alpha^2 = 1 \implies \alpha \in \{-1, 1\} \).
   If \( \alpha = 1 \implies Tx = x \implies x \in V^T \cap (V^T)^\perp \implies x = 0 \) which is a contradiction.
   So \( \alpha = -1 \).

2. \( \langle z - \frac{\langle x, z \rangle}{\langle x, x \rangle} \cdot x, x \rangle = \langle z, x \rangle - \frac{\langle x, z \rangle}{\langle x, x \rangle} \cdot \langle x, x \rangle = 0 \).
   So \( z - \frac{\langle x, z \rangle}{\langle x, x \rangle} \cdot x \in x^\perp = V^T \) and \( T(z) - \frac{\langle x, z \rangle}{\langle x, x \rangle} x = z - \frac{\langle x, z \rangle}{\langle x, x \rangle} x \). Hence
   \[ T(z) = T(z - \frac{\langle x, z \rangle}{\langle x, x \rangle} x) + \frac{\langle x, z \rangle}{\langle x, x \rangle} x = T(z) - \frac{\langle x, z \rangle}{\langle x, x \rangle} x + \frac{\langle x, z \rangle}{\langle x, x \rangle} x \]
   \[ = z - \frac{\langle x, z \rangle}{\langle x, x \rangle} x - \frac{\langle x, z \rangle}{\langle x, x \rangle} x = z - 2 \frac{\langle x, z \rangle}{\langle x, x \rangle} x \]
For each \( x \in V, \ x \neq 0 \) define \( S_x(z) = z - 2\frac{\langle x,z \rangle}{\langle x,x \rangle}x \).

Lemma 1.1 implies that

1. any reflection \( T \) is equal to \( S_x \) for \( x \) determined up to a scalar. Any such \( x \in V \) is called a root of \( T \).

2. \( \forall x \in V, \ x \neq 0 : \ S_x \) is a reflection.

3. any reflection \( T \) satisfies \( T^2 = I \).

**Definition.** A finite reflection group is a pair \((G,V)\) where \( V \) is Euclidean space, \( G \) is a finite subgroup of \( O(V) \) and \( G = \langle \{S_x : S_x \in G\} \rangle \), generated by all reflections in \( G \).

Generation means: if \( G \supseteq X \), then \( X \) generates \( G \) if \( G = \langle X \rangle \) where \( X \) is defined as one of the following equivalent definitions:

1. (semantic) \( \langle X \rangle = \bigcap_{G \supseteq H \supseteq X} H \)

2. (syntactic) \( \langle X \rangle = \{1\} \cup \{a_{i1}^{\pm 1}a_{i2}^{\pm 1} \cdots a_{in}^{\pm 1} : a_i \in X\} \)

**Equivalence:**
\((G_1,V_1) \sim (G_2,V_2)\) if there is an isometry \( \varphi : V_1 \rightarrow V_2 \) s.t. \( \varphi G_1 \varphi^{-1} = G_2 \) and \( \{\varphi \circ T \varphi^{-1} : T \in G_1\} \).

**Problem:** Classify all finite reflection groups \((G,V)\) up to \( \sim \).

**Example:** of finite reflection groups

1. \[
\begin{array}{c}
  x \\
  \downarrow \alpha \\
  y
\end{array}
\]

We want to know \( \langle S_x, S_y \rangle \)
\[
det(S_x, S_y) = 1 \implies S_xS_y \in SO_2(\mathbb{R}) \implies S_xS_y = \text{Rot}_\beta.
\]
Since \( S_x(S_y(y)) = S_x(-y) \implies S_xS_y = 2\alpha \).

- \( \alpha = \frac{\pi}{2} \notin \mathbb{Q} \implies |S_xS_y| = \infty \implies \langle S_x, S_y \rangle \) is not finite
- \( \alpha = \frac{m}{n} \in \mathbb{Q} \) (for \( m,n \) rel. prime) \implies \( |S_xS_y| = |\text{Rot} \frac{2\pi m}{n}| = n \implies \langle S_x, S_y \rangle = D_{2n} \)

Dihedral
\[
I_2(n) = (D_{2n}, \mathbb{R}^2), \ |I_2(n)| = 2n
\]

2. \( S_n \)-symmetric group acting on \( \{1, \ldots, n\} \) extend to action of \( S_n \) on \( \mathbb{R}^n \).

If \( e_i \in \) basis of \( \mathbb{R}^n \), if \( \sigma \in S_n \), then \( T_\sigma : e_i \rightarrow e_{\sigma(i)}, \ T_\sigma \in O_n(\mathbb{R}) \)
4. In Example 3, let $F_0 = \{ \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \}$ has the property that for all $\sigma: T_\sigma(x) = x \implies x \in (\mathbb{R}^n)^S_n$,

$$x^\perp = \{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} : \sum \alpha_i = 0 \}$$

$A_{n-1} = (S_n, x^\perp)$

Reflection since $S_n = \langle (i, j) \rangle$ and $T_{(i, j)} = S_{e_i-e_j}$ because $T_{i,j}(e_i-e_j) = -(e_i-e_j)$ and

$$y = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in (e_i-e_j)^\perp \iff \alpha_i = \alpha_j \iff T_{(i, j)}(y) = y$$

In particular, $|A_n| = (m+1)!$ and $A_2 \sim I_2(3)$

3. $F = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ - field of two elements. Consider action of $S_n$ on $F^n, \varepsilon_1, \ldots, \varepsilon_n$ basis of $F^n$.

$$\forall \sigma \in S_n : t_\sigma(\varepsilon_i) = \varepsilon_{\sigma(i)}$$

Consider semidirect product $S_n \ltimes F^n$. As a set: $S_n \ltimes F^n = S_n \times F^n$, the product is

$$(\sigma, a) \cdot (\tau, b) = (\sigma \tau, t_{-1}(a) + b)$$

$S_n \ltimes F^n$ acts on $\mathbb{R}^n$:

$$T_{(\sigma, a)} : \varepsilon_i \mapsto (-1)^{a_i} \cdot e_{\sigma i}$$

Let us check that this is the action of $S_n \ltimes F^n$:

$$T_{(1, 0)}(T_{(\tau, 0)}(\varepsilon_i)) = T_{(1, 0)}(e_{\tau(i)}) = (-1)^{a_{\tau(i)}} e_{\tau(i)}$$

$$= (-1)^{[t_{-1}(a)]} e_{\tau(i)}$$

$$= T_{(\tau, t_{-1}(a))}(e_i)$$

$$B_n = (S_n \ltimes F^n, \mathbb{R}^n)$$

It is reflection since $S_n \ltimes F^n = \langle ((i, j), 0), (1, \varepsilon_i) \rangle$ and $T_{((i, j), 0)} = S_{e_i-e_j}$, $T_{(1, \varepsilon_i)} = S_{e_i}$.

$|B_n| = n! 2^n$, $B_1 \sim A_1$, $B_2 \sim I_2(4)$

4. In Example 3, let $F_0^n = \{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in F^n : \sum \alpha_i = 0 \}$ codim-1 subspace in $F^n$ or index 2 subgroup.

$\sigma \in S_n, \ a \in F_0^n \implies t_\sigma(a) \in F_0^n$, so $S_n$ acts on $F_0^n$ and

$$D_n = (S_n \ltimes F_0^n, \mathbb{R}^n)$$

It is a reflection since $S_n \ltimes F_0^n = \langle ((i, j), 0), (1, \varepsilon_i + \varepsilon_j) \rangle$,

$$T_{((i, j), 0)} = S_{e_i-e_j}, \ T_{(1, \varepsilon_i+\varepsilon_j)} = S_{e_i+e_j}.$$ 

$$|D_n| = n! \cdot 2^{n-1}, \ D_1 \text{ is trivial } (\{1\}, \mathbb{R}), \ D_2 \sim I_2(2), \ D_3 \sim A_3$$
2 Root systems

Let \((G, V)\) be a finite reflection group. The root system of \((G, V)\) is

\[
\Phi_{(G,V)} := \{ x \in V : \| x \| = 1, \ S_x \in G \}
\]

\(\Phi\) has the following properties:

1. \(x \in \Phi \Rightarrow \mathbb{R}x \cap \Phi = \{ x, -x \}\)
2. \(|\Phi| = 2 \cdot (\text{number of reflections in } G)\)
3. \(T \in G, \ x \in \Phi \Rightarrow T(x) \in \Phi\)

Property 3 follows from

Lemma 2.1. \(T \in O(V), \ x \in V \setminus \{0\} \Rightarrow S_{T(x)} = TS_xT^{-1} \).

Proof. \(\text{RHS: } T(x) \mapsto TS_xT^{-1}Tx = TS_xx = T(-x) = -T(x)\)

\[
\text{RHS} : T(x) \perp y \mapsto TS_x(T^{-1}y) = T(T^{-1}y - 2\frac{\langle x, T^{-1}y \rangle}{\langle x, x \rangle} \cdot x)
\]
\[
= T(T^{-1}y - 2\frac{\langle Tx, y \rangle}{\langle x, x \rangle}x)
\]
\[
= T(T^{-1}y)
\]
\[
= y
\]

Hence \(\text{RHS} = S_{T(x)}. \) \(\square\)

Example: \(I_2(3) \sim A_2\) -dihedral group of order 6, symmetry of regular triangle.

6 roots sitting at the vertices of a regular hexagon.

Definition. A root system is a finite subset \(\Phi \subset V\) s.t.

1. \(0 \notin \Phi\)
2. \(x \in \Phi \Rightarrow \mathbb{R}x \cap \Phi = \{ x, -x \}\)
3. \(x, y \in \Phi \Rightarrow S_x(y) \in \Phi\)
Example:
1. $\Phi_{(G,V)}$ where $(G,V)$ is a finite reflection group
2. $\emptyset = \Phi_{(\{1\},V)}$
3. $\Phi$ is a root system, not $\Phi_{(G,V)}$ because there are vectors of 2 different length. Chopping long vectors by $\sqrt{2}$ gives $\Phi_{B_2}$.

Let $\Phi \subset V$ be a root system. Definition. A simple subsystem is $\Pi \subset \Phi$ s.t.
1. $\Pi$ is linearly independent
2. $\forall x \in \Phi : x = \sum_{y \in \Pi} \alpha_y \cdot y$ where either all $\alpha_y \geq 0$ or all $\alpha_y \leq 0$

Example: In $\alpha \beta \{\alpha, \beta\}$ is a simple system but $\gamma$ is not simple since $\alpha - \beta \in \Phi$.

Lemma 2.2. $\Pi \subset \Phi$ simple system in a root system. Then $\forall x, y \in \Pi, x \neq y \implies \langle x, y \rangle \leq 0$ (angles in a simple system are obtuse)

Proof. Suppose $x \neq y$, $\langle x, y \rangle > 0$.
$\Phi \ni S_x(y) = y - 2\frac{\langle x, y \rangle}{\langle x, x \rangle} x = y + \alpha x$, $\alpha < 0$. Since $\Pi$ is lin. independent and $x \neq y$, $S_x(y) = y + \alpha x$ is the only way to write $S_x(y)$ as a linear combination of elements of $\Pi$ and both positive and negative coefficients are present. This is a contradiction. □

Definition. A total order on $\mathbb{R}$-vector space $V$ is a linear order on $V$ s.t. ($\geq$-order, $x > y$ if $x \geq y$ and $x \neq y$)
1. $x \geq y \implies x + z \geq y + z$
2. $x \geq y$, $\alpha > 0 \implies \alpha x \geq \alpha y$
3. $x \geq y$, $\alpha < 0 \implies \alpha x \leq \alpha y$

Example: Phonebook-order on $\mathbb{R}^n$:

$$
\begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{pmatrix}
> 
\begin{pmatrix}
\beta_1 \\
\vdots \\
\beta_n
\end{pmatrix}
\iff \exists k \in \{1, \ldots, n\} \text{ s.t. } \alpha_i = \beta_i \forall i < k \text{ and } \alpha_k > \beta_k
$$

Given an ordered basis $e_1, \ldots, e_n$ of $V$ we get phonebook order on $V$ by writing $\sum \alpha_i e_i = \begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{pmatrix}$. If $\geq$ is a total order on $V$ then

$$x \in V \setminus \{0\} \implies \text{ either } x > 0 \text{ or } x < 0$$

and $V = V_+ \cup \{0\} \cup V_-$. where
• $V_+ = \{ x \in V : x > 0 \}$
• $V_- = \{ x \in V : x < 0 \}$

**Definition.** A positive system in a root system $\Phi$ is a subset $\Theta \subset \Phi$ s.t. $\exists$ total order on $V$ s.t. $\Theta = \Phi \cap V_+$.

**Example:**

1. **Box** $\alpha \beta$. Let $\geq$ be the total order associated to the ordered basis $\alpha, \beta$. Then

$$2\alpha + \beta > \alpha + \beta > \alpha > \beta > 0 > -\beta > -\alpha - \beta > -2\alpha - \beta$$

So $\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ is a positive system.

2. $I_2(n)$-symmetry of $n$-gon, $n$ reflections, so $2n$ roots, at vertices of regular $2n$-gon.

   $(\alpha, \beta)$ is simple $\iff$ $\Phi \subset$ the cone $\iff$ $\overline{\alpha, \beta} = \frac{(n-1)}{n} \pi$

   - Every root lies in 2 simple systems
   - number of simple systems $= n$
   - positive systems $= \{\text{vectors between some } \alpha \text{ and } \beta \text{ where } \overline{\alpha, \beta} = \frac{(n-1)}{n} \pi\}$

3. $A_n$:
   $$\chi_{T_{(a_1, \ldots, a_k)}}(z) = (z^k - 1)(z - 1)^m$$
   (minimal polynomial of $T_{(a_1, \ldots, a_k)}$ is $(z^k - 1)$).

   $T_\sigma$ reflection $\iff$ $\chi_{T_\sigma} = (1 + z)(1 - z)^m$ $\iff$ $\sigma = (i, j)$

   Reflections are $T_{(i,j)}$, $\exists \frac{n(n-1)}{n}$ of them and $\Phi = \{e_i - e_j : i \neq j \in \{1, \ldots, n+1\}\}$.

   A typical simple system is

   $$e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n$$

   For $i < j$ we have: $e_i - e_j = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \ldots + (e_{j-1} - e_j)$.

**Notation:** $\Phi$ - root system, $\Phi_+ \subset \Phi$ - positive system.

**Definition.** $\Omega \subset \Phi_+$ is a quasisimple system if it satisfies:

1. $\forall x \in \Phi_+ \exists \alpha_t \geq 0$ s.t. $x = \sum_{t \in \Omega} \alpha_t t$

2. No proper subset of $\Omega$ satisfies (1).

**Hint:** simplicity $\iff$ quasisimplicity

**Lemma 2.3.** Let $\Omega \subset \Phi_+$ be a quasisimple system. Then $\forall x, y \in \Omega$, $x \neq y$ $\implies$ $(x, y) \leq 0$.

**Proof.** Suppose $x \neq y \in \Omega$ and $(x, y) > 0$. Then:

$$\Phi \ni S_\lambda(y) = y - 2\frac{(x,y)}{(x,x)}x = y - \lambda x, \ \lambda > 0.$$ Consider two cases:
Case 1 $S_x(y) \in \Phi_+$. By (1): $y - \lambda x = \sum_{t \in \Omega} \lambda_t t$, $\lambda_t \geq 0$. What is $\lambda_y$?

If $\lambda_y < 1$ then $(1 - \lambda_y)y = \lambda x + \sum_{t \neq y} \lambda_t t$. So $y$ is a positive linear combination of elements of $\Omega \setminus \{y\}$. Hence $\Omega \setminus \{y\}$ satisfies (1), contradiction to (2).

Hence $\lambda_y \geq 1$, then $0 = \lambda x + (\lambda_y - 1)y + \sum_{t \neq y} \lambda_t t$, coefficients in the RHS $\geq 0$. Hence RHS $\geq 0$. Since RHS $= 0 \implies \lambda = 0$ which is a contradiction.

Case 2 $S_x(y) \in \Phi_- \implies y - \lambda x = \sum_{t \in \Omega} (-\mu_t) t$, $\mu_t \geq 0$.

$\implies \lambda x - y = \sum_{t \in \Omega} \mu_t t$. Similarly to case 1:

- $\mu_x < \lambda \implies (\lambda - \mu_x)x = y + \sum_{t \neq x} \mu_t t$ contradicts (2)
- $\mu_x \geq \lambda \implies 0 = y + (\mu_x - \lambda)x + \sum_{t \neq x} \mu_t t$. RHS has a positive coefficient, hence $> 0$ which is a contradiction.

$\square$

Theorem 2.4. 1. Every positive system contains a unique simple system.

2. Every simple system is contained in an unique positive system.
Hence there is a natural bijection between

$$\{\Pi \subset \Phi : \Pi \text{ is simple}\} \text{ and } \{\Phi_+ \subset \Phi : \Phi \text{ is positive}\}$$

Proof.

1. Let $\Phi_+ \subset \Phi$ be a positive system. Consider all subsets of $\Phi_+$ that satisfy condition (1) of the definition of a quasisimple system. A minimal subset $\Omega$ among them must satisfy (2), so $\Omega$ is a quasisimple system in $\Phi_+$. To prove that $\Omega$ is a simple system, it suffices to prove that $\Omega$ is linearly independent.

Suppose $\sum_{t \in \Omega} \alpha_t t = 0$. Sorting positive and negative coefficients to two different sides,

$$v = \sum_t \beta_t t = \sum_t \gamma_t t, \quad \beta_t, \gamma_t \geq 0$$

Now we consider

$$\langle v, v \rangle = \left( \sum_t \beta_t t, \sum_t \gamma_t t \right) = \sum_{s,t} \beta_s \gamma_t \langle s, t \rangle \leq 0$$

Hence $v = 0 \implies \beta_t = 0 = \gamma_t \implies \alpha_t = 0$. Hence $\Omega$ is linearly independent and simple.

Let $\Pi, \Pi' \subset \Phi$ be two distinct simple systems. Wlog $\exists x \in \Pi' \setminus \Pi$.

Since $x \in \Phi_+$, $x = \sum_{y \in \Pi} \alpha_y y$, $\alpha_y \geq 0$. We know

$$\Pi \ni y = \sum_{z \in \Pi'} \beta_y^z z, \quad \beta_y^z \geq 0$$
\[ x = \sum_{z \in \Pi'} \left( \sum_{y \in \Pi} \alpha_y \beta_z y \right) z \]

\[ \implies \exists y_0 \in \Pi \setminus \Pi' : \alpha_{y_0} \neq 0 \implies \exists \text{ at least two } z_1, z_2 \text{ s.t. } \beta_{y_0}^{z_1} \neq 0, \beta_{y_0}^{z_2} \neq 0 \text{ which is a contradiction with the linearity independece of } \Pi'. \]

2. Let \( \Pi \subset \Phi \) be a simple system. \( \Pi \) is linearly independent \( \implies \) we can extend \( \Pi \) to a basis \( B \) of \( V \). Choose an order on \( B \) and consider the phonebook total order on \( V \). Clearly, \( \Pi \subset V_+ \implies \Pi \subset \Phi_+ = V_+ \cap \Phi \). Uniqueness follows from the definition of a simple system: Every \( x \in \Phi \) is a nonnegative or nonpositive linear combination of elements of \( \Pi \implies \) nonnegative linear combinations in \( V_+ \), nonpositive linear combinations in \( V_- \). Hence \( \Phi_+ = \{ \sum_{t \in \Pi} \alpha_t t : \alpha_t \geq 0 \} \).

\[ \square \]

**Proposition 2.5.** Let \( \Phi \supset \Phi_+ \supset \Pi \) be a root system, positive system, simple system. Then \( \forall x \in \Pi, \forall y \in \Phi_+, x \neq y \):

\[ S_x(y) \in \Phi_+ \]

**Note:** \( x \neq y \) is essential since \( S_x(x) = -x \in \Phi_- \).

**Proof.** Let \( y = \sum_{t \in \Pi} \alpha_t t, \alpha_t \geq 0. y \neq x \implies \exists t_0 \in \Pi \text{ s.t. } t_0 \neq x, \alpha_{t_0} > 0. \)

Hence \( S_x(y) = \sum_{t \in \Pi} \alpha_t t - \lambda x \) with \( \alpha_{t_0} > 0. \) Hence \( S_x(y) \in \Phi_+ \).

\[ \square \]

**Example:** Box: simple:

\[ \text{simple: } \]

\( \forall g \in G, \) let \( L(g) = \{ x \in \Phi_+ : g(x) \in \Phi_- \} = \Phi_+ \cap g^{-1}(\Phi_-). \) We know that \( L(1) = \emptyset, L(S_x) = \sum_{x \in \Pi} \{ x \}. \)

Define function \( l : G \longrightarrow \mathbb{Z}_{\geq 0} \) called length by \( l(g) = |L(g)|. \) In particular \( l(1) = 0, l(S_x) = 1. \)

**Proposition 2.6.** The following statements about \( l( ) \) hold:

1. \( l(g) = l(g^{-1}) \)

2. \( l(S_x g) = \begin{cases} 
   l(g) + 1 & \text{if } g^{-1}(x) \in \Phi_+ \\
   l(g) - 1 & \text{if } g^{-1}(x) \in \Phi_- 
\end{cases} \)

3. \( l(g S_x) = \begin{cases} 
   l(g) + 1 & \text{if } g(x) \in \Phi_+ \\
   l(g) - 1 & \text{if } g(x) \in \Phi_- 
\end{cases} \)

**Proof.**

1. \( x \mapsto -g(x) \) is a bijection between \( L(g) \) and \( L(g^{-1}) \)

\[ x \in \Phi_+ \iff -x = g^{-1}(-g(x)) \in \Phi_- \]

\[ g(x) \in \Phi_- \iff -g(x) \in \Phi_+ \]

10
2. By 2.5, $\pm x$ is the only root that changes the sign under $S_x$.

$$g^{-1}(x) \xrightarrow{g} x \xrightarrow{S_x} -x$$

if $g^{-1}(x) \in \Phi_+$ $\implies$ $g^{-1}(x) \notin L(g)$ but $g^{-1}(x) \in L(S_xg)$. So $L(S_xg) = L(g) \cup \{g^{-1}(x)\}$.

if $g^{-1}(x) \in \Phi_-$ $\implies$ $-g^{-1}(x) \in L(g)$ but not in $L(S_xg)$ and $L(S_xg) = L(g) \setminus \{-g^{-1}(x)\}$.

$G$ acts on $V, \Phi, \{\Pi \subset \Phi : \Pi$ is simple $\}$.

**Theorem 2.7.** Let $\Pi, \Pi' \subset \Phi$ be two simple systems. Then $\exists \ g \in G$ s.t.

$$\Pi' = g(\Pi) = \{gx : \ x \in \Pi\}$$

**Note:** such $g$ is unique but we need some more tools to prove it.

**Proof.** Let $\Phi_+, \Phi'_+$ be the corresponding positive systems, $\Phi_-, \Phi'_-$ the corresponding negative systems. Proceed by induction on $|\Phi_+ \cap \Phi'_-|$ (distance between $\Pi$ and $\Pi'$).

If $|\Phi_+ \cap \Phi'_-| = 0 \implies \Phi_+ = \Phi'_+ \implies \Pi = \Pi'$.

$|\Phi_+ \cap \Phi'_-| = k$ suppose proved.

$|\Phi_+ \cap \Phi'_-| = k + 1$. Since $k + 1 \geq 1 \implies \Pi \neq \Pi' \implies \Pi \not\subset \Phi'_+ \implies \exists \ x \in \Pi \cap \Phi'_-$. Consider $\tilde{\Pi} = S_x(\Pi)$ corresponding positive system is

$$\tilde{\Phi}_+ = (\Phi_+ \setminus \{x\}) \cup \{-x\}$$

Hence $\tilde{\Phi}_+ \cap \Phi'_+ = (\Phi_+ \cap \Phi'_+) \setminus \{x\}$. In particular $|\tilde{\Phi}_+ \cap \Phi'_+| = k$ and by induction assumption there is a $g$ s.t. $\Pi' = g(\tilde{\Pi})$, hence

$$\Pi' = g(S_x(\Pi)) = gS_x(\Pi)$$

Define a height function $h : \Phi \to \mathbb{R}$ by

$$h \left( \sum_{t \in \Pi} \alpha_t t \right) = \sum_{t \in \Pi} \alpha_t$$

**Theorem 2.8.** $G, \Phi \supset \Pi$ as before.

1. $\forall \ x \in \Phi \ \exists y \in \Pi \ \exists g \in G$ s.t. $x = gy$.

2. $G = \langle S_x \rangle_{x \in \Pi}$

**Proof.** Let $H = \langle S_x \rangle_{x \in \Pi} \leq G$. Pick any $t \in \Phi$ let

$$\Sigma_t = Ht \cap \Phi_+$$
By 2.4:

Finally we have shown that all reflections \( S_x \) are in \( H \), so \( H = G \).

(1) follows because if \( x \) is positive this proof shows that \( x \in G \Pi \), if \( x \) is negative we use \( t = -x \) to show that \( S_x \in H, t = -x \in G \Pi \). Hence \( x = S_x(-x) \in G \Pi \).

Finally we have shown that all reflections \( S_x \) are in \( H \), so \( H = G \).

\( l : G \longrightarrow \mathbb{Z}_{\geq 0} \). Now since \( G = \langle S_x \rangle_{x \in \Pi} \), is \( l \) related to group theoretic length in these generators?

**Theorem 2.9.** YES

\[ \forall g \in G : l(g) = \min \{ k : \exists x_1, \ldots, x_k \in \Pi : g = S_{x_1}S_{x_2} \cdots S_{x_k} \} \]

**Proof.** 2.6: \( l(S_x h) \leq l(h) + 1 \) \( \implies \) if \( g = S_{x_1} \cdots S_{x_k} \) then \( l(g) \leq k \). Hence \( l(g) \leq RHS \).

Note: the opposite direction is based on deletion principle.

Assume that \( g = S_{x_1} \cdots S_{x_k}, x_i \in \Pi \) but \( l(g) < k \). Hence \( \exists j \) s.t.

\[ l(S_{x_1} \cdots S_{x_j}) = j \text{ but } l(S_{x_1} \cdots S_{x_{j+1}}) = j - 1 \]

(we pick the position where the length goes down for the first time)

\[ \implies S_{x_1} \cdots S_{x_j}(x_{j+1}) \in \Phi_- \text{ and } x_{j+1} \in \Pi \subset \Phi_+. \] Find largest \( i \) s.t. \( i \leq j \) and

\[ S_{x_i}(y) = S_{x_i} \cdots S_{x_j}(x_{j+1}) \in \Phi_-, \text{ } y = S_{x_{i+1}} \cdots S_{x_j}(x_{j+1}) \in \Phi_+ \]

\[ \implies y = x_i S_{x_{i+1}} \cdots S_{x_j}(x_{j+1}) = T(x_{j+1}). \] Hence \( S_{x_i} = TS_{x_{j+1}}T^{-1} \). Then

\[ g = S_{x_1} \cdots S_{x_i} \cdots S_{x_j} \cdots S_{x_k} = S_{x_1} \cdots S_{x_{i-1}}S_{x_i}TS_{x_{j+1}} \cdots S_{x_k} \]

\[ = S_{x_1} \cdots S_{x_{i-1}}(TS_{x_{j+1}}T^{-1}TS_{x_{j+1}})S_{x_{j+2}} \cdots S_{x_k} = S_{x_1} \cdots S_{x_{i-1}}TS_{x_{j+2}} \cdots S_{x_k} \]

\[ \square \]
Theorem 2.10. If \( \Pi, \Pi' \subset \Phi \) are two simple systems then \( \exists! g \in G \) s.t. \( \Pi' = g(\Pi) \).

**Proof.** Existence is Theorem 2.7. Suppose \( g, h \in G \) satisfy \( \Pi' = g(\Pi) = h(\Pi) \). Then
\[
\Pi = h^{-1}g\Pi \implies \Pi_+ = h^{-1}g(\Phi_+) \text{ hence } l(h^{-1}g) = 0 \implies h^{-1}g = 1.
\]

Corollary 2.11. Consider action of \( G \) and \( X = \{ \Pi \subset \Phi : \Pi \text{ is simple} \} \). For all \( \Pi \in X \), the orbit map
\[
g_{\Pi} : G \to X : g \mapsto g(\Pi)
\]

is a bijection.

**Proof.** See orbit-stabiliser Theorem. \( \square \)
3 Generators and Relations

- \langle X \rangle\text{-free group on a set } X
- \langle \emptyset \rangle = C_1
- \langle x \rangle \cong \mathbb{Z}\text{-infinite cyclic group}
- \langle X \rangle = \{ \text{all finite words (including } \emptyset \text{) in alphabet } x^+, x^- \text{ as } x \in X \text{ without subworts } x^+x^- \text{ or } x^-x^+ \}\}
- v \cdot w = \text{Reduction of concatenated word } vw
- \text{It is a Theorem that } \cdot \text{ is well-defined.}
- Universal property:

\[
\begin{array}{c}
\xymatrix{
X \ar[r]^f & G \\
\langle X \rangle \ar[u]^\exists \varphi
}
\end{array}
\]

\forall \text{ groups } G \forall \text{ functions } f : X \to G \exists! \text{ group homomorphism } \\
\varphi : \langle X \rangle \to G \text{ s.t. } \forall a \in X : \varphi(a) = f(a)

- \langle X \mid R \rangle : R \text{ is a set of relations i.e. either words in alphabet } x^\pm : x \in X \text{ or } u = v \text{ where } u, v \text{ are two words. Relation between words and equations:}

\[
\begin{align*}
u &\to u = 1, \ uv^{-1} \to uv^{-1} = 1 &\iff u = v
\end{align*}
\]

\text{E.g. dihedral group } I_2(n) = \left\langle a^{\text{rot}}, b^{\text{refl}} : b^2, a^n, bab^{-1} = a^{-1} \right\rangle.

\textbf{Definition. } \langle X \mid R \rangle = \langle X \rangle / \overline{R}. \text{ If the relations are words } u_1, u_2, \ldots \text{ then } \overline{R} = \bigcap_{\langle X \rangle \subseteq H, \text{ all } u_i \in H} H.

\text{Universal property of } \langle X \mid R \rangle:

\[
\begin{array}{c}
\xymatrix{
\langle X \mid R \rangle \ar[r]^\exists \varphi & G \\
X \ar[u]^q \ar[r]^f
}
\end{array}
\]

\forall \text{ functions } f : X \to G \text{ s.t. relations in } R \text{ hold in } G \text{ for elements } f(x), x \in X \text{ then } \exists! \text{ group homomorphism } \varphi : \langle X \mid R \rangle \to G \text{ s.t. } \forall x \in X : \varphi(x\overline{R}) = f(x).
Example: \( B(2, 5) = \langle x, y \mid w^5 \rangle \) where \( w \) is any word in the alphabet \( x^{\pm 1}, y^{\pm 1} \).

Universal property: \( \forall \) groups \( G \) s.t. \( \forall x \in G : \ x^5 = 1, \ \forall a, b \in G \ \exists! \ \varphi : B(2, 5) \rightarrow G \) s.t. \( \varphi(x) = a, \ \varphi(y) = b. \)

Known: \( B(2, 5) \) is either infinite or of order \( 5^{34} \).
4 COXETER GROUP

A Coxeter graph is a non-oriented graph without loops or double edges s.t. each edge is marked by a symbol \( m \in \mathbb{Z}_{\geq 3} \cup \{\infty\} \).

**Convention:** Draw \( \circ \quad \circ \) instead of \( \circ \quad 3 \quad \circ \)
We will study only graphs with finitely many vertices.

**Definition.** A Coxeter matrix (on set \( X \)) is an \( X \times X \) matrix \((m_{ij})_{i,j \in X}\) s.t.
- \( m_{ij} \in \mathbb{Z}_{\geq 1} \cup \{\infty\} \)
- \( m_{ij} = m_{ji} \)
- \( m_{ij} = 1 \iff i = j \)

There is a 1:1 correspondence between Coxeter graphs and Coxeter matrices

\[
\text{Graph} \quad \leftrightarrow \quad \text{Matrix} \quad (c_{ij})
\]

\( X = \text{vertices of the graph} \)

\[
i \quad \xrightarrow{m_{ij}} \quad j \quad \mapsto \quad c_{ij} = m_{ij}
\]

\[
i \quad \xrightarrow{} \quad j \quad \mapsto \quad c_{ij} = 2
\]

**Example:**

\[
\begin{array}{ccc}
1 & \rightarrow \quad 2 & \rightarrow \\
\infty & \rightarrow \quad 77 & \rightarrow \\
3 & \rightarrow \quad 4
\end{array}
\mapsto
\begin{pmatrix}
1 & 3 & \infty & 2 \\
3 & 1 & 3 & 77 \\
\infty & 3 & 1 & 2 \\
2 & 77 & 1 & 1
\end{pmatrix}
\]

**Definition 4.1.** The Coxeter group of a Coxeter graph (or corresponding \( X \times X\)-matrix \((m_{ij})\)) is

\[W = W(\text{graph}) = (X \mid (ab)^{m_{ab}} = 1)\]

**Remark 4.2.**
1. \( m_{ab} = \infty \implies \text{no relation} \)
2. \( a = b, m_{aa} = 1 \implies (aa)^1 = 1 \implies a^2 = 1 \)
3. \( \text{no edge} \iff m_{ab} = 2 \implies (ab)^2 = 1 \iff ab = b^{-1}a^{-1} = ba \) Therefore:
   \( \text{no edge} \iff a, b \text{ commute} \)

**Lemma 4.3.** Let \( G \) be a group generated by \( a_1, \ldots, a_n \), all of order 2. Then \( G \) is a quotient of \( W(m_{ij} = [a_i, a_j]) \).
Proof. \( f: \{1, \ldots, n\} \rightarrow G \) where \( f(i) = a_i \) is a function. Then \( f(i)^2 = a_i^2 = 1 \) and \( (f(i)f(j))^{m_{ij}} = (a_ia_j)^{|a_i a_j|} = 1 \). So \( f(i) \) satisfy relations of the Coxeter group and \( \exists! \) group homomorphism \( \varphi: W(m_{ij}) \rightarrow G \) s.t. \( \varphi(i) = a_i \) (by the universal property). \( \text{Im } \varphi \ni a_i \Rightarrow \varphi \) is surjective.

Big monster \( M \) is the largest sporadic simple group,
\[
M \sim 10^{53}, \quad M \cong \frac{W}{H}
\]

Theorem 4.4. \( G \) is a finite reflection group \( \Pi \subset \Phi \)-simple system in the root system. Then the natural surjection for \( \Pi \times \Pi \)-matrix \( m_{x,y} = |S_x S_y| \), \( W((m_{x,y})) \rightarrow G: x \mapsto S_x \) is an isomorphism.

Example: \( O_2(\mathbb{R}) \geq I_2(\infty) = \left\langle \left\{ \begin{array}{c}
S_\alpha, \\
\text{order } 2 \text{ in } \Phi
\end{array} \right\}, \alpha \not\in \mathbb{Q}\right\rangle \cong C_2 \times C_\infty \)
\[ W\left( \begin{array}{c}
\infty \\
\end{array} \right) \rightarrow \langle S_a, S_b \rangle \text{ where } \frac{ab}{\pi} \not\in \mathbb{Q}, \qquad \cong \langle S_a, \text{Rot}_\alpha \rangle \]

action of \( W\left( \begin{array}{c}
\infty \\
\end{array} \right) \) is \( I_2(\infty) \) or \( \mathbb{R}^2 \) is a mess.

But \( I_2(\infty) \) has action on \( \mathbb{R}^1 \) with geometric meaning.
\[
\begin{array}{ccc}
\text{ } & S_0 & S_1 \\
-1 & 0 & 1
\end{array}
\]

\( S_0(x) = x, \ S_1(x) = -2-x, \ S_1S_0(x) = 2 + x \rightarrow \text{translation by } 2. \) So \( \langle S_1, S_0 \rangle \) is \( W\left( \begin{array}{c}
\infty \\
\end{array} \right) \) inside the group of motions on \( \mathbb{R} \). \( C_\infty \) is a subgroup of translations \( (x \mapsto x + 2n) \). As \( C_2 \leq W\left( \begin{array}{c}
\infty \\
\end{array} \right) \) you can choose any \( \{1, S_n\} \) where \( S_n(x) = 2n - x \). The fundamental domain is \([0,1], \ I_2(\infty) \) tesselates \( \mathbb{R}^1 \) by interval \([0,1]\).

Example: \( W\left( \begin{array}{c}
\end{array} \right) \cong \langle \text{siderelections across the sides of a triangle with angles } \frac{\pi}{\alpha}, \frac{\pi}{\beta}, \frac{\pi}{\gamma} \rangle \).

where \( \alpha = \frac{a}{\pi}, \ \beta = \frac{b}{\pi}, \ \gamma = \frac{c}{\pi} \).
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1 \text{ spherical} \\
= 1 \text{ euclidean} \\
< 1 \text{ hyperbolic}
\]

17
→ tesselation of the space.

In particular, \( W\left(\begin{array}{c} a \\ b \\ c \end{array}\right) \) \(\leftrightarrow\) triangles in tesselation

**Lemma 4.5.** (Deletion Condition)

\( G \) - finite reflection group, \( \Phi \) - root system, \( \Phi \supset \Pi \) - simple system.

If \( x = S_{a_1} \cdot S_{a_n}, \ a_i \in \Pi \) and \( l(x) < n \) then \( \exists i < j \) s.t.

\[
x = S_{a_1} \cdots \widehat{S}_{a_i} \cdot S_{a_j} \cdot S_{a_n}
\]

**Proof.** already proved.

**Proof.** of Theorem 4.3

\[
\langle \Pi \rangle \xrightarrow{\varphi} W(m_{ab}) \xrightarrow{\psi} G
\]

\[
a \mapsto \varphi(a) \mapsto S_a
\]

even we will work in here

It suffices to show that \( \psi \) is injective.

Let \( w = a_1 a_2 \cdots a_n \in \text{Ker } \psi \) \((\alpha_1^2 = 1)\). Then \( \psi(w) = S_{a_1} \cdot S_{a_n} = 1 \). Since \( \det S_{a_i} = -1 \), \((-1)^n = 1 \rightarrow n \) is even, write \( n = 2k \).

Proceed by induction on \( k = \frac{n}{2} \)

- \( k = 1 \quad n = 2 \quad w = a_1 a_2 \)

\[
1 = \psi(w) = S_{a_1} S_{a_2} \rightarrow S_{a_1} = S_{a_2} \rightarrow a_1 = \pm a_2 \quad \implies \quad a_1 = a_2 \rightarrow w = \alpha_1^2 = 1
\]

- \( k \leq m - 1 \): done (induction assumption)

- \( k = m \Rightarrow n = 2m \), \( w = \alpha_1 \cdots \alpha_{2m} \), \( 1 = \psi(w) = S_{a_1} \cdots S_{a_{2m}} \). Consider

\[
w_1 = \alpha_1^{-1} w \alpha_1 = \alpha_1 w \alpha_1 = \alpha_2 a_3 \cdots a_{2m} \alpha_1
\]

Repeating conjugation w.r.t. the first symbol, we can consider two cases:

**Case 1** \( w = \bar{a} \bar{b} \bar{a} \bar{b} \cdots \bar{a} \bar{b} \) for some \( a, b \in \Pi \), \( w = (\alpha \beta)^m \).

\[
1 = \psi(w) = (S_{a} S_{b})^m \implies |S_{a} S_{b}| = m_{ab} m \implies w = (\alpha \beta)^m = 1 \in W
\]

**Case 2** there is a \( v \in W \) s.t. \( w_2 = v^{-1} w v = b_1 b_2 b_3 \cdots b_{2m} \) with all \( b_i \in \Pi \) and \( b_1 \neq b_3 \).

Let us play the trick: Observe that

\[
x = S_{b_1} \cdots S_{b_{m+1}} = S_{b_2 m} \cdots S_{b_{m+2}} = y
\]

Indeed \( y^{-1} = S_{b_{m+2}} \cdots S_{b_{2m}} \) and \( x y^{-1} = S_{b_1} \cdots S_{b_{2m}} = \psi(w) = 1 \). Hence \( x = y \).

By deletion principle (note that \( l(x) \leq m - 1 \)) there are \( 1 \leq i < l \leq m + 1 \) s.t.

\[
x = S_{b_1} \cdots \widehat{S}_{b_l} \cdots S_{b_j} \cdots S_{b_{m+1}}
\]

Consider two cases:
Case 2.1 \( (i,j) \neq (1, m + 1) \)

\[
S_{b_1} \cdots S_{b_j} = S_{b_{j+1}} \cdots S_{b_{j-1}} \\
S_{b_1} \cdots S_{b_j} S_{b_{j-1}} \cdots S_{b_{j+1}} = 1
\]

Hence \( w_3 = \overline{b_1} \cdots \overline{b_j} \overline{b_{j-1}} \cdots \overline{b_{i+1}} \in \ker \psi \). It has < 2m terms hence by induction assumption \( w_3 = 1 \). Hence

\[
\overline{b_1} \cdots \overline{b_j} = \overline{b_{i+1}} \cdots \overline{b_{j-1}}
\]

So \( w_2 = \overline{b_1} \cdots \hat{\overline{b_i}} \cdots \hat{\overline{b_j}} \cdots \overline{b_{2m}} \). By induction assumption \( w_2 = 1 \) and therefore

\[
w = vv^{-1} = vv^{-1} = 1
\]

Case 2.2 \( (i,j) = (1, m + 1) \)

Then \( w_3 = \overline{b_1} \overline{b_2} \cdots \overline{b_{m+1}} \overline{b_m} \cdots \overline{b_2} \in \text{Ker } \psi \) has length \( 2m \) and we seem to be stuck. (Hint: eventual contradiction is with \( b_1 \neq b_3 \).

Let \( w_4 = \overline{b_1}^{-1} w_2 \overline{b_1} = \overline{b_2} \cdots \overline{b_4} \overline{b_1} \in \ker \psi \). Let us play the trick on

\[
x_1 = S_{b_2} \cdots S_{b_{m+1}} S_{b_{m+2}} = S_{b_1} \cdots S_{b_{m+3}}
\]

Deletion principle \( \rightarrow x_1 = S_{b_2} \cdots \hat{S}_{b_{i'}} \cdots \hat{S}_{b_{j'}} \cdots S_{b_{m+3}} \). Get 2 cases:

Case 2.2.1 \( (i', j') \neq (2, m + 2) \)

\[
S_{b_{i'}} \cdots S_{b_{j'}} = S_{b_{i'+1}} \cdots S_{b_{j'-1}} \\
S_{b_{i'}} \cdots S_{b_{j'}} S_{b_{j'-1}} \cdots S_{b_{i'+1}} = 1
\]

\( w_5 = \overline{b_{i'}} \cdots \overline{b_{j'}} \overline{b_{j'-1}} \cdots \overline{b_{i'+1}} \in \ker \psi \). By induction assumption \( w_5 = 1 \)

\[
\overline{b_{i'}} \cdots \overline{b_{j'}} = \overline{b_{i'+1}} \cdots \overline{b_{j'-1}}
\]

and \( w_2 = \overline{b_1} \cdots \hat{\overline{b_{i'}}} \cdots \hat{\overline{b_{j'}}} \cdots \overline{b_{2m}} \). By induction assumption \( w_2 = 1 \implies w = vv^{-1} = 1 \).

Case 2.2.2 \( (i', j') = (2, m + 2) \)

\[
w_5 = \overline{b_2} \cdots \overline{b_{m+1} b_{m+2} b_{m+1}} \overline{b_m} \cdots \overline{b_3} \in \ker \psi
\]

but of length \( n \). Let \( w_6 = \overline{b_3} w_5 b_3^{-1} = \overline{b_3 b_2 b_3} \cdots \overline{b_{m+2} b_{m+1}} \cdots \overline{b_4} \in \ker \psi \)

and repeat the trick on \( \psi(w_6) = 1 \).

\[
x_2 = S_{b_3} S_{b_2} S_{b_3} \cdots S_{b_{m+1}} = S_{b_4} \cdots S_{b_{m+2}}
\]

can delete at \( i'', j'' \). Get 2 cases:

Case 2.2.2.1 \( (i'', j'') \neq (1st \text{} incidence \ of \ 3, m + 1) \)

\[
w_7 = b_{i''} \cdots b_{j''} b_{j''-1} \cdots b_{i''+1} \in \ker \psi
\]

Induction assumption \( \implies w_7 = 1 \implies w_6 = 1 \implies w_5 = 1 \implies w_2 = 1 \implies w = 1 \).
Case 2.2.2.2 \((i'', j'') = (\text{first } 3, m + 1)\)

\[ w_7 = b_3 b_2 b_3 \cdots b_{m+1} b_m \cdots b_2 \in \ker \psi \]

Remember \(w_3 = b_1 b_2 b_3 \cdot b_{m+1} b_m \cdots b_2\). Then

\[ w_7 = \overline{b_3 b_1 w_3} \mapsto \overline{b_3 b_1} = w_7 w_3^{-1} \in \ker \psi \]

\[ S_{b_3} S_{b_1} = 1 \implies S_{b_3} = S_{b_1} \implies b_3 = \pm b_1 \implies b_3 = b_1 \]

But this is a contradiction!
5 Geometric representation of $W(m_{ij})$

Recall some geometry of a vector space $V$ with symmetric bilinear form $\langle \cdot, \cdot \rangle$. (the field is $\mathbb{R}$, dim $V$ - any, $\langle \cdot, \cdot \rangle$ - any)

Still $\forall W \leq V$ we have

- $W^\perp = \{ x \in V : \forall a \in W : \langle a, w \rangle = 0 \}$
- $(W^\perp)^\perp \supset W$

Example: $\mathbb{R}^2$, $\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \rangle = xa - yb$

$W = \mathbb{R} : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies W^\perp = W \implies W + W^\perp = W \neq \mathbb{R}^2$.

Vectors $v$ with $\langle x, x \rangle = 0$ are called isotropic. If $\langle x, x \rangle \neq 0$ (x is not isotropic) then

$$V = \mathbb{R}x \oplus x^\perp$$

and

$$S_x : V \longrightarrow V, \ S_x(a) = a - 2\frac{\langle a, x \rangle}{\langle x, x \rangle}x$$

is well-defined and has usual properties

- $S_x(x) = -x$
- $S_x|_{x^\perp} = id$
- $S_x^2 = id$
- $S_x \in O(V, \langle \cdot, \cdot \rangle)$

Start with $X \times X$ Coxeter matrix $(m_{ab})$ or its Coxeter graph. Let

$$V(m_{ab}) = \{ \sum_{a \in X} \alpha_a e_a : \alpha_a \in \mathbb{R}, \text{finitely many } \alpha_a \neq 0 \}$$

$e_a, a \in X$ form a basis of $V(m_{ab})$. Symmetric bilinear form $\langle \cdot, \cdot \rangle : V(m_{ab}) \times V(m_{ab}) \longrightarrow \mathbb{R}$ is defined on the basis by

$$\langle e_a, e_b \rangle = -\cos \left( \frac{\pi}{m_{ab}} \right)$$

- $a = b \implies m_{ab} = 1 \implies \langle e_a, e_a \rangle = 1$
- $m_{ab} = 2 \implies \langle e_a, e_b \rangle = 0$
- $m_{ab} = \infty \implies \langle e_a, e_b \rangle = -\cos \left( \frac{\pi}{\infty} \right) = -\cos 0 = -1$
Lemma 5.1. $V(\frac{m}{\pi})$ is euclidean $\iff m \neq \infty$.

Proof. $(e_1, e_2) = -\cos \frac{\pi}{m}$. If $x = \alpha e_1 + \beta e_2$ then
\[
\langle x, x \rangle = \alpha^2 \langle e_1, e_1 \rangle + 2\alpha\beta \langle e_1, e_2 \rangle + \beta^2 \langle e_2, e_2 \rangle
= \alpha^2 + \beta^2 - 2\alpha\beta \cos \left(\frac{\pi}{m}\right)
= (\alpha + \beta)^2 - 2\alpha\beta(1 + \cos \left(\frac{\pi}{m}\right))
\geq 0
\]
and
\[
\langle x, x \rangle = 0 \iff \alpha = \beta = 0 OR m = \infty, \alpha = \beta
\]

\[\square\]

Lemma 5.2. $(V, \langle \cdot, \cdot \rangle) - \mathbb{R}$-vector space with symmetric bilinear form. Let $U \subset V$ be a finite dimensional subspace s.t. $U \cap U^\perp = 0$. Then
\[V = U \oplus U^\perp\]

Note: it breaks down if dim $U = \infty$.

Example: $(V, \langle \cdot, \cdot \rangle)$ Hilbert space with Hilbert base $e_1, \ldots, U = \text{span}(e_i) \subset V$ dense. Then $U^\perp = 0$, so $U \cap U^\perp = 0$ but $U \oplus U^\perp = U \neq V$.

Proof. Pick $v \in V$, basis $e_1, \ldots, e_n$ of $U$.

It suffices to find $x_1, \ldots, x_n \in \mathbb{R}$ s.t.
\[
v - \sum x_i e_i \in U^\perp \iff \forall j: \langle v - \sum x_i e_i, e_j \rangle = 0 \iff \langle v, e_j \rangle - \sum_i x_i \langle e_i, e_j \rangle = 0
\]

The system of $n$ equations $\sum_i x_i \langle e_i, e_j \rangle = \langle v, e_i \rangle$, $j = 1, \ldots, n$ has a solution. It holds true because $U \cap U^\perp = 0$ which implies that $\langle \cdot, \cdot \rangle|_U$ is non-degenerate which is equivalent to $\text{det} \langle e_i, e_j \rangle \neq 0$.

\[\square\]

Coxeter graph (matrix) $\Gamma = (m_{ab})_{a,b \in X} \rightarrow$ Coxeter group $W(\Gamma) = \langle X : (ab)^{m_{ab}} = 1 \rangle$

$\rightarrow$ vector space with bilinear form
\[V(\Gamma) = \bigoplus_{a \in X} \mathbb{R}e_a, \langle e_a, e_b \rangle = -\cos \left(\frac{\pi}{m_{ab}}\right)\]

Operators
\[\rho_a = S_{e_a} : V(\Gamma) \rightarrow V(\Gamma) : x \mapsto x - 2 \langle e_a, x \rangle e_a\]

Since $\rho : a(e_a) = -e_a, \rho_a|_{e_b} = I, |\rho_a| = 2$.

Proposition 5.3.
\[|\rho_a\rho_b| = m_{ab}\]
Proof.

Case 1 \( m_{ab} \neq \infty \). \( 5.1 \) Let \( U = \mathbb{R}e_a \oplus \mathbb{R}e_b \) be a Euclidean vector space w.r.t \( \langle , \rangle \). In particular \( U \cap U^\perp = 0 \) if and only if \( V(\Gamma) = U \oplus U^\perp \). Since \( U^\perp \subset e_a^\perp, \rho_a|_{U^\perp} = I \). Similarly, \( \rho_b|_{U^\perp} = I \) since \( \rho_b(U^\perp) \subset U^\perp \), \( \rho_a|_{U^\perp} \cdot \rho_b|_{U^\perp} = \rho_a|_{U^\perp} \cdot \rho_b|_{U^\perp} = I \cdot I = I \). On \( U \), we can do explicit calculation.

Hence \( \rho_a|_U \cdot \rho_b|_U = \text{Rot} by \ \frac{2\pi}{m_{ab}} \) and \( |\rho_a\rho_b|_U = |\rho_a\rho_b|_U = m_{ab} \).

Case 2 Exercise: show that in \( e_a, e_b \) basis

\[
\rho_a|_U \cdot \rho_b|_U = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

so \( |\rho_a\rho_b|_U = \infty \rightarrow |\rho_a\rho_b|_U = \infty \).

Hence we have a group homomorphism

\[
\rho : W(\Gamma) \rightarrow GL(V(\Gamma)) : (a \in X) \mapsto \rho_a
\]

\( \rho \) is called the geometric representation of \( W(\Gamma) \).

Suppose \( \Gamma = \Gamma_1 \cup \Gamma_2 \) (disjoint union of 2 graphs). On the level of Coxeter matrices

\[
X = X_1 \cup X_2, \ X_1 \cap X_2 = \emptyset
\]

and for all \( a \in X_1, \ b \in X_2 : m_{ab} = 2 \).

**Proposition 5.4.** \( W(\Gamma) \cong W(\Gamma_1) \times W(\Gamma_2) \)

**Proof.** Construct inverse group homomorphism \( \psi : W(\Gamma) \rightarrow W(\Gamma_1) \times W(\Gamma_2), \ \varphi : W(\Gamma_1) \times W(\Gamma_2) \rightarrow W(\Gamma) \).

If \( a \in X_1 \), let

\[
\psi(\bar{a}) = \begin{cases} (\pi, 1) & \text{if } a \in X_1 \\ (1, \bar{a}) & \text{if } a \in X_2 \end{cases}
\]

Since all relations of \( W(\Gamma) \) hold in \( W(\Gamma_1) \times W(\Gamma_2) \) such \( \psi \) uniquely extends to a group homomorphism.

If \( a \in X_i \subset X, \ \varphi_i(\bar{a}) = \bar{a} \) extends to a group homomorphism \( \varphi_i : W(\Gamma_i) \rightarrow W(\Gamma) \) (all relations of \( W(\Gamma_i) \) hold in \( W(\Gamma) \) )

Notice that generators of the image of \( \varphi_1 \) commute with generators of \( Im \varphi_2 \). Hence subgroups \( Im \varphi_1, \ Im \varphi_2 \) commute. Hence \( \varphi(x, y) = \varphi_1(x)\varphi_2(y) \) is a well-defined group homomorphism.

Since \( \forall \ a \in W(\Gamma), \ \varphi(\psi(a)) = a, \ \varphi\psi = I_{W(\Gamma)} \). Similarly \( \psi\varphi = I_{W(\Gamma_1) \times W(\Gamma_2)} \).
\[\Gamma\text{ is connected if }\Gamma = \Gamma_1 \cup \Gamma_2 \implies \Gamma_1 = \emptyset \text{ or } \Gamma_2 = \emptyset.\]

Each \(\Gamma\) has connected components \(\Gamma_1, \Gamma_2, \ldots\) s.t.
\[\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \ldots\]

**Corollary 5.5.** *In this case* \(W(\Gamma) \cong W(\Gamma_1) \times W(\Gamma_2) \times \ldots\) and \(V(\Gamma) = V(\Gamma_1) \oplus V(\Gamma_2) \oplus \ldots\) with \(V(\Gamma_i) \perp V(\Gamma_j)\) for \(i \neq j\).

**Proof.** Exercise (use transfinite induction).

A representation of a group \(G\) is the pair \((V, \rho)\), where \(V\) is a vector space and \(\rho : G \to GL(V)\).

**Example:** Take a Reflection group \((G, V)\), then \((V, i : G \to GL(V))\) is a representation.

If \(\Gamma\) is a Coxeter graph, then \((V(\Gamma), \rho)\) is a representation of \(W(\Gamma)\).

A subrepresentation of \((V, \rho)\) is a subspace \(U \subset V\) s.t. \(\forall g \in G : \rho(g)(U) \subset U\). So \((U, \rho^{GL(U)})\) is a representation.

\(V\) is irreducible if \(V \neq 0\) and \(0, V\) are the only subrepresentations. \(V\) is called completely irreducible if for all subrepresentations \(U \subset V\) there exists a subrepresentation \(W \subset V\) s.t. \(V = U \oplus W\).

Convention: The Coxeter graph \(\emptyset\) is not connected.

**Proposition 5.6.** If \(\Gamma\) is connected, then any proper \(W(\Gamma)\)-subrepresentation of \(V(\Gamma)\) lies in \((V(\Gamma))^{\perp}\).

Exercise: \((V(\Gamma))^{\perp}\) is a subrepresentation of \(V(\Gamma)\).

**Proof.** Let \(U \subset V = V(\Gamma)\) be a subrepresentation. Then \(\forall a \in X (\Gamma = (m_{ab})_{a,b \in X} ), \rho_a(U) \subset U\). Observe that \(e_a \notin U\). Otherwise: \(\forall b \in X, m_{ab} \neq 2\) holds:
\[U \ni \rho_b(e_a) = e_a + 2 \cos \frac{\pi}{m_{ab}} e_b \implies e_b \in U\]

Since \(\Gamma\) is connected, this would imply that \(e_b \in U\) for all \(b \in X\). So \(U = V\) and not a proper subrepresentation which is a contradiction.

By 5.7, \(V = \mathbb{R}e_a \oplus \mathbb{R}e_a^{\perp}\). Pick any \(z \in U\) and write \(z = \alpha e_a + z'\) where \(z' \in \mathbb{R}e_a^{\perp}\). Then
\[U \ni \rho_a(z) = z - 2 \langle e_a, z \rangle e_a = \alpha e_a + z' - 2 \alpha \langle e_a, e_a \rangle e_a = 2 \langle e_a, z' \rangle e_a = -\alpha e_a + z'\]

Hence \(z' = \frac{1}{2}(z + \rho_a(z)) \in U\) and \(2\alpha e_a \in U\) \(\implies \alpha = 0\). Hence \(z = z' \in \mathbb{R}e_a^{\perp}\) and therefore \(U \subset \mathbb{R}e_a^{\perp}\) and \(U \cap \bigcap \mathbb{R}e_a^{\perp} = V^{\perp}\).

**Corollary 5.7.** If \(\Gamma\) is connected and \(V(\Gamma)\) nonsingular (\(V(\Gamma)^{\perp} = 0\)), then \(V(\Gamma)\) is irreducible.

**Theorem 5.8.** *(Maschke)*

If \(G\) is a finite group and \(\mathbb{F}\) a field s.t. char \(\mathbb{F}\) \(\nmid |G|\) and \(V\) is a representation of \(G\) over \(\mathbb{F}\), then \(V\) is completely irreducible.
Proof. Pick a subrepresentation \( U \subset V \).

Observe that direct sum representations \( V = U \oplus W \) are in bijection with linear operators \( p \in \text{End}_F(V) \) s.t. \( p^2 = p \) and \( \text{Im}(p) = U \):

\[
V = U \oplus W \rightarrow p(v) = u \quad \text{where} \quad v = u + w \quad \text{is the unique decomposition with} \quad u \in U, w \in W
\]

\[
V = \text{Im}(p) \oplus \ker(p) \leftarrow p
\]

Pick any vector space decomposition and let \( p \) be the corresponding operator, \( \rho : G \rightarrow \text{GL}(X) \).

Define \( \tilde{p} = \frac{1}{|G|} \sum_{x \in G} \rho(x)p\rho(x^{-1}) \)

- if \( t \in U \)

\[
\tilde{p}(t) = \frac{1}{|G|} \sum_{x \in G} \rho(x)(p\rho(x^{-1})(t)) = \frac{1}{|G|} \sum_{x \in G} \rho(x)\rho(x^{-1})(t) = t
\]

So \( \tilde{p}|_U = I \).

- since \( \tilde{p}(t) = t \) for \( t \in U \), \( \text{Im}(\tilde{p}) \supset U \)

- by definition of \( \tilde{p} \), \( \text{Im}(\tilde{p}) \subset \text{Im}(p) \subset U \). Hence \( \text{Im}(\tilde{p}) = U \implies \tilde{p}^2 = \tilde{p} \)

- \( \tilde{p} \) is a homomorphism of representations (commutes with any \( \rho(x), x \in G \)):

\[
\rho(x)\tilde{p} = \frac{1}{|G|} \sum_{y \in G} \rho(x)\rho(y)p\rho(y^{-1})
\]

\[
= \frac{1}{|G|} \sum_{y \in G} \rho(xy)p\rho((xy)^{-1})\rho(x)
\]

\[
= \frac{1}{|G|} \sum_{z \in G} \rho(z)p\rho(z^{-1})\rho(x)
\]

\[
= \tilde{p}\rho(x)
\]

Hence \( \ker \tilde{p} \) is a \( G \)-subrepresentation and \( V = U \oplus \ker \tilde{p} \).

\[\square\]

**Corollary 5.9.** If \( \Gamma \) is connected and \( |W(\Gamma)| < \infty \) then \( V(\Gamma) \) is nonsingular (w.r.t \( \langle , \rangle \)) and irreducible (as \( W(\Gamma) \)-representation).

**Proof.** Irreducibility follows from nonsingularity by Corollary 5.?. Suppose \( V^\perp \neq 0 \) (\( V = V(\Gamma) \)). By 5.8, \( V = V^\perp \oplus U \) for some \( G \)-subrepresentation \( U \). By 5.6, \( U \subset V^\perp \Rightarrow U = 0 \Rightarrow V = V^\perp \Rightarrow \langle , \rangle = 0 \). Nonsense since \( \langle e_a, e_a \rangle = 1 \) and \( \Gamma \) is non-empty.

\[\square\]

**Proposition 5.10.** (Schur’s Lemma)

If \( \Gamma \) is connected and \( |W(\Gamma)| < \infty \) then \( \text{End}_{W(\Gamma)}V(\Gamma) = \mathbb{R} \). 

\[25\]
Hence \( \langle e_a, z \rangle \Theta(e_a) = \langle e_a, \Theta(z) \rangle e_a \) by definition.

Hence all \( \Theta \) is symmetric since \( 2 \Theta(u, v) = \langle u, v \rangle + \langle v, u \rangle \).

By 5.9, \( \ker \Theta - \lambda I = 0 \) is a subrepresentation of \( V(G) \).

\( \Theta \) is euclidean.

\[ \langle u, v \rangle_2 = \sum_{x \in \Gamma} \langle \rho_x(u), \rho_x(v) \rangle_1 \]

- \( \langle \cdot, \cdot \rangle_2 \) is symmetric since \( \langle \cdot, \cdot \rangle_1 \) is.
- \( u \neq 0, \langle u, u \rangle_2 = \sum_{x \neq 0} \langle \rho_x(u), \rho_x(u) \rangle_1 \) hence \( \langle \cdot, \cdot \rangle_2 \) is euclidean.
- \( \forall y \in \Gamma: \]

\[ \langle \rho_y(u), \rho_y(v) \rangle_2 = \sum_x \langle \rho_x \rho_y(u), \rho_x \rho_y(v) \rangle_1 \]

\[ = \sum_x \langle \rho_{xy}(u), \rho_{xy}(v) \rangle_1 \]

\[ = \sum_x \langle \rho_x(u), \rho_x(v) \rangle_1 \]

\[ = \langle u, v \rangle_2 \]

Hence all \( \rho_y \in O(V(G), \langle \cdot, \cdot \rangle_2) \). Remember that all \( \rho_a \in O(V(G), \langle \cdot, \cdot \rangle) \) and consider \( \Theta : V \to V \) defined by \( \langle \Theta(u), v \rangle = \langle u, v \rangle \) for all \( u \neq 0 \).

\( \Theta \) is euclidean.

\[ \langle \rho_x(\Theta(u)), v \rangle_2 = \langle \Theta(u), \rho_x^{-1}(v) \rangle_2 \]

\[ = \langle u, \rho_x^{-1}(v) \rangle_2 \]

\[ = \langle \rho_x(u), v \rangle \]

\[ = \langle \Theta(\rho_x(u)), v \rangle_2 \]

\[ \Theta \in End_{W(G)} V(G) \]
This implies $\rho_x \Theta = \Theta \rho_x$. By 5.10, $\Theta = \lambda I$ for some $\lambda \in \mathbb{R}$, so for all $u, v$:

$$
\lambda \langle u, v \rangle_2 = \langle u, v \rangle
$$

Since $\langle e_a, e_a \rangle = 1 = \lambda \langle e_a, e_a \rangle_2 \implies \lambda = \frac{1}{\langle e_a, e_a \rangle_2} > 0.

$$
\square
$$

Section 4 $\implies$ Finite Reflection group $\cong$ Coxeter group
Section 5 $\implies$ Finite Coxeter group $\longrightarrow$ Reflection group $(V(\Gamma), \rho(W(\Gamma)))$
Problem: $\rho$ could have a kernel. Section 6 rules this out.

Example: Exercise: $D_{4n} \cong C_2 \times D_{2n}$ when $n$ is odd.

Hence distinct reflection groups $I_2(2n) = W \begin{array} {c} 2n \end{array}$ and $A_1 \times I_2(n) = W \begin{array} {c} n \end{array}$ are isomorphic as groups.
6 Fundamental chamber

Aim: \( \rho : W(\Gamma) \rightarrow Im(\rho) \) is isomorphic (at least for finite Coxeter groups)
Define \( l : W(\Gamma) \rightarrow \mathbb{Z}_{\geq 0} : l(x) = \min \{ n : x = \bar{a}_1 \cdots \bar{a}_n, \ a_i \in X(\Gamma) \} \).
In particular: \( l(1) = 0, \ l(\bar{a}_i) = 1 \).

**Lemma 6.1.** For all \( a \in X, \ g \in W \) holds: \( l(\bar{a}g) \) is either \( l(g) + 1 \) or \( l(g) - 1 \).

**Proof.** \( g = \bar{a}_1 \cdots \bar{a}_n \) with \( l(g) = n \), then \( \bar{a}g = \bar{a}_1 \cdots \bar{a}_n \) so \( l(\bar{a}g) \leq l(g) + 1 \). Since \( g = \bar{a}(\bar{a}g), \ l(g) \leq l(\bar{a}g) + 1 \) so \( l(\bar{a}g) \geq l(g) - 1 \).

Need to rule out \( l(\bar{a}g) = l(g) \). Consider sign representation:

\[ sgn : W \rightarrow GL_1(\mathbb{R}) : sgn(\bar{a}) = (-1) \]

Since relations of \( W \) hold in \( GL_1(\mathbb{R}) \) \( ((\bar{a}b)m_{ab} = 1 \rightarrow ((-1)(-1))m_{ab} = 1 \) ok \) \( sgn \) is well-defined.

If \( l(x) = n \) then \( x = \bar{a}_1 \cdots \bar{a}_n \) and \( sgn(x) = sgn(\bar{a}_1) \cdots sgn(\bar{a}_n) = (-1)^n = (-1)^{l(x)} \).

If \( l(\bar{a}g) = l(g) \) then \( sgn(\bar{a}g) = sgn(g), \ sgn(\bar{a})sgn(g) = sgn(g) \Rightarrow sgn(\bar{a}) = 1 \) which is a contradiction. \( \square \)

For each \( a \in X(\Gamma) \) consider half spaces

- \( H^+_a = \{ u \in V(\Gamma) : \langle u, e_a \rangle > 0 \} \)
- \( H^-_a = \{ u \in V(\Gamma) : \langle u, e_a \rangle < 0 \} \)

the dividing hyperplane \( H_a = \{ u \in V(\Gamma) : \langle u, e_a \rangle = 0 \} \)
and the cone \( C = \bigcap_{a \in X(\Gamma)} H^+_a \) which we call the fundamental chamber.

Idea: Let \( x \in W : \)
Lemma 6.2.

Proof. If \( \Gamma = (m_{ab})_{a,b \in X} \) and denote \( W = W(\Gamma), V = V(\Gamma) \)
\( |W| < \infty, a, b \in X \). Then \( \forall g \in \langle \bar{a}, \bar{b} \rangle \leq W : \) either

- \( g(H^+_a \cap H^+_b) \subset H^+_a \) OR
- \( g(H^+_a \cap H^+_b) \subset H^-_a \) and \( l(\pi g) = l(g) - 1 \)

Pick \( z \in g(H^+_a \cap H^+_b) \cap H_a \). Then \( z \in H_a \implies \langle z, e_a \rangle = 0 \)
\( z \in g(H^+_a \cap H^+_b) \implies g^{-1}(z) \in H^+_a \cap H^+_b \implies \langle g^{-1}(z), e_a \rangle > 0, \langle g^{-1}(z), e_b \rangle > 0 \)

On the other hand \( \langle g^{-1}(z), g^{-1}(e_a) \rangle = \langle z, e_a \rangle = 0 \).

As \( g \in \langle \bar{a}, \bar{b} \rangle \):
\[ g^{-1}(e_a) = \alpha e_a + \beta e_b \quad \alpha, \beta \in \mathbb{R} \]

We denote by \( \Phi(\Gamma) \) a root system of \( \rho(W) \in V(\Gamma) \) and since \( e_a, e_b \) form a simple system of \( \Phi(\Gamma) \cap \text{span}(e_a, e_b) \), either

- \( \alpha \geq 0, \beta \geq 0 \) s.t. not both are 0 and then
\[ \langle g^{-1}(z), g^{-1}(e_a) \rangle = \langle g^{-1}(z), \alpha e_a + \beta e_b \rangle = \alpha \langle g^{-1}(z), e_a \rangle + \beta \langle g^{-1}(z), e_b \rangle > 0 \]
which is a contradiction

- OR \( \alpha \leq 0, \beta \leq 0 \) s.t. not both are 0 and therefore
\[ \langle g^{-1}(z), g^{-1}(e_a) \rangle < 0 \]
which is a contradiction too.

It remains to prove the length statement if \( g(H^+_a \cap H^+_b) \subset H^-_a \). Inspect \( U = \text{span}(e_a, e_b) \) and \( L = U \cap H^+_a \cap H^+_b \). This is a dihedral group of order \( 2m_{ab} \).

![Diagram](image)

\[ gL \subset H^+_a \cap U \iff g = a \text{ or } g = ab \text{ or } \ldots \text{ or } g = abab\ldots b^{m_{ab}} \]

All such \( g \) admit a reduced expression starting with \( a \). Hence \( l(ag) = l(g) - 1 \). \( \square \)
Remark 6.3. There are 2 finiteness assumptions:

1. $W$ is finite
2. $\Gamma$ is finite, i.e. $|X| < \infty$ and $\forall a, b \in X : m_{ab} < \infty$.

Exercise: $W$ is finite $\implies \Gamma$ is finite.

Hint: $a \neq b \in X \implies \rho_a \neq \rho_b \in GL(V(\Gamma)) \implies \pi \neq \overline{\tau} \in W(\Gamma)$. Then one needs to show that $m_{ab} < \infty$. This follows from $\rho_a \rho_b \mapsto \text{span}(e_a, e_b) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Note: Lemma 6.2 holds under assumption that $\Gamma$ is finite.

Proposition 6.4. Let $\Gamma$ be finite and let $C$ be the fundamental chamber. Then:

1. For all $w \in W$ and for all $u \in X$: either
   - $w(C) \subset H_a^+$ OR
   - $w(C) \subset H_a^-$ and $l(aw) = l(w) - 1$

2. for all $w \in W$ and for all $a \neq b \in X$ there is a $g \in \langle \pi, \overline{\tau} \rangle$ s.t.
   $$w(C) \subset g(H_a^+ \cap H_b^+) \text{ and } l(w) = l(g) + l(g^{-1}w)$$

Proof. Simultaneous induction on $l(w)$:

Base $l(w) = 0 \implies w = 1 \implies w(C) = C \subset H_a^+$ and $g = 1$ works for (2).

Assumption if $l(w) = k - 1$ then (1) and (2) are assumed to have been proved

Step Let $l(w) = k$.

1. Write $w = dw', d \in X, l(w') = k - 1$. Consider two cases:
   - Case 1: $a = d$.
     Statement (1) for $w'$ gives either $w'(C) \subset H_a^+$ or $w'(C) \subset H_a^-$. 
     - $w'(C) \subset H_a^- \implies w(C) = \overline{a}(w'(C)) \subset \overline{a}(H_a^-) = H_a^+$
     - $w'(C) \subset H_a^+ \implies w(C) = \overline{a}(w'(C)) \subset \overline{a}(H_a^+) = H_a^-$
   - Case 2: $a \neq d$.
     By (2) for $w'$ there is a $g \in \langle \pi, \overline{\tau} \rangle$ s.t. $w'(C) \subset g(H_a^+ \cap H_b^+)$ and $l(w') = l(g) + l(g^{-1}w)$. Then
     $$w(C) = \overline{a}(w'(C)) \subset dg(H_a^+ \cap H_b^+)$$

By Lemma 6.2 either $dg(H_a^+ \cap H_b^+) \subset H_a^+$ ($\implies w(C) \subset H_a^+$) or 
$$dg(H_a^+ \cap H_b^+) \subset H_a^- \text{ and } l(adg) = l(dg) - 1.$$ 

Then $w(C) \subset dg(H_a^+ \cap H_b^+) \subset H_a^-$ and
$$l(aw) = l(adg^{-1}w) \leq l(adg) + l(g^{-1}w') \leq l(g) + l(g^{-1}w') = l(w') = k - 1$$
$$\implies l(aw) = l(w) - 1$$
2. Now assume (1) and (2) for \( l(w) = k - 1 \) and note that (1) for \( l(w) = k \) is established.

Consider two Cases:

**Case 1:** \( wC \subset H^+_a \cap H^+_b \). Then \( g = 1 \) satisfies (2).

**Case 2:** \( wC \not\subset H^+_a \cap H^+_b \).

Observe that \( wC \cap H_a = \emptyset \). \( wC \subset w(H^+_a \cap H^+_b) \) is in either \( H^+_a \) or \( H^-_a \) by 6.2.

Wlog \( wC \subset H^-_a \).

By part (1): \( l(aw) = l(w) - 1 \).

Use part (2) on \( w' = aw \) (by induction assumption) then there is \( g' \in \langle a, b \rangle \) s.t. \( w'C \subset g'(H^+_a \cap H^+_b) \) and \( l(w') = l(g') + l((g')^{-1}w) \).

Claim: \( g = \pi g' \) satisfies (2) for \( w \).

\[
\begin{array}{c|c|c|c|c}
H^-_a \cap H^+_b & H^+_a \cap H^+_b & H^-_b & H^+_b \cap H^-_a & H_a \\
\hline
H^-_a \cap H^-_b & H^+_a \cap H^-_b & & & \\
\end{array}
\]

Hence \( l(w) = l(g) + l(g^{-1}w) \).

**Theorem 6.5.**

1. \( W \) is finite \( \implies V = \bigcup_{g \in W} gC \)

2. \( \Gamma \) is finite, then: \( \forall \: g, x \in W : gC \cap xC \neq \emptyset \implies g = x \)

**Note:** If \( W \) is infinite \( \bigcup_{g \in W} gC \) is called Tits cone which is a proper subset of \( V \).

**Proof.**

1. \( \overline{C} = \{ \alpha \in W : \forall \: a \in X : \langle \alpha, a \rangle \geq 0 \} \).

Recall the height function \( h : V \to \mathbb{R} : v = \sum_{a \in X} \beta_a e_a \mapsto \sum_{a \in X} \beta_a \).

Pick any \( v \in V \). Since \( W \) is finite, the height achieves a maximum on the set \( Wv = \{ g(v) : g \in W \} \). Let \( z = g(v) \) be such a maximum.
Claim: \( z \in C \) and this implies that \( v = g^{-1}(z) \in g^{-1}(C) \) and \( V = \bigcup gC \).

Indeed, otherwise there is an \( a \in X \) s.t. \( \langle z, e_a \rangle < 0 \). Then:

\[
\rho_a(z) = h(z) - 2 \langle z, e_a \rangle e_a
\]

has height

\[
h(\rho_a(z)) = h(z) - 2 \langle z, e_a \rangle > h(z)
\]

which is a contradiction to the maximality.

2. Assume \( gC \cap xC \neq \emptyset \). Then \( C \cap g^{-1}xC \neq \emptyset \). If \( g^{-1}x = 1 \) then \( g = x \) and we are done. If \( g^{-1}x \neq 1 \) then there is an \( a \in X \) s.t. \( l(ag^{-1}x) = l(g^{-1}x) - 1 \).

Set \( w = ag^{-1}x \), then \( l(aw) = l(w) + 1 \). By 6.3 \( ag^{-1}xC = wC \subset H_a^+ \implies g^{-1}xC \subset H_a^- \).

Then \( C \cap g^{-1}xC \subset H_a^+ \cap H_a^- = \emptyset \) which is a contradiction.

\[ \square \]

**Corollary 6.6.** \( \Gamma \) is finite. Then \( \rho: W \to GL(V) \) is injective.

**Proof.** Suppose \( \rho(g) = \rho(w) \). Then \( gC = wC \). Hence \( g = w \) by 6.4. \[ \square \]

Let \((G, V)\) be a reflection group, \( G \leq O(V) \), \( V \) is Euclidean. Let \( V_1 = \text{span}(x : x \in \Phi(G)) \).

Say \((G, V)\) is essential if \( V_1 = V \).

**Corollary 6.7.**

\[ \Gamma \leftrightarrow (W(\Gamma), V(\Gamma)) \]

is a bijection between isomorphism classes of Coxeter graphs with finite \( W(\Gamma) \) and equivalence classes of essential reflection groups.

**Proof.**

\[ \Gamma \to (W(\Gamma), V(\Gamma)) \]

\[ X = \Pi \text{ is a simple system in } \Phi(G), \ m_{ab} = |S_a S_b| \leftrightarrow (G, V) \]

By definition, \((W(\Gamma), V(\Gamma))\) is essential. Going \( \Gamma \to (W(\Gamma), V(\Gamma)) \to (X \text{ with } m_{ab} = |S_a S_b|) \) gives \( \Gamma \) back by 6.5 and 4.3 (?).

\((G, V) \to \Gamma = (X, m_{ab}) \to (W(\Gamma), V(\Gamma)) \).

Then

\[
V(\Gamma) \to V
\]

\[
\sum_{x \in X} \alpha_x e_x \leftrightarrow \sum_{x \in X} \alpha_x x
\]

\[
W(\Gamma) \to G
\]

\[
\pi_1 \pi_2 \cdots \pi_n \leftrightarrow S_{a_1} \cdots S_{a_n}
\]

This is an isomorphism by 4.3 and 6.5. \[ \square \]

Some fun bits:

- \( \mathbb{R}^n \setminus \mathbb{R}^{n-1} \): disjoint union of 2 contractible half-spaces
\( \mathbb{C}^n \setminus \mathbb{C}^{n-1} \): is connected with fundamental group \( \mathbb{Z} \)

Try \( \mathbb{C}C = \mathbb{C} \otimes_R V \supset \mathbb{C}H^a = \mathbb{C} \otimes_R H^a \). Then \( \mathbb{C}V^0 = \mathbb{C}V \setminus \bigcup_{g \in W, a \in X} g(\mathbb{C}H^a) \) is connected.

\( W \) acts freely on \( \mathbb{C}V^0 \) and the universal cover of \( \mathbb{C}V^0 \) is simply connected (if \( W \) is finite).

One can draw the fundamental group of \( \mathbb{C}V^0/W \):

\[
\begin{array}{c}
\bullet \rho_a(x) \\
\bullet x \\
\hline
T_a \\
\hline
H_a
\end{array}
\]

\[ C \subset \mathbb{C} \mathbb{V}^0 \]

\([x] = [\rho_a(x)] \in \mathbb{C}V^0/W \).

\( T_a \) is a loop in \( \mathbb{C}V^0/W \). In fact:

\[
B(\Gamma) = \pi_1(\mathbb{C}V^0/W) = \left\langle T_a, a \in X : \underbrace{T_aT_b\cdots}_m = \underbrace{T_bT_a\cdots}_m \right\rangle
\]

\( V^0 \rightarrow V^0/W \) gives

\[
1 \rightarrow \underbrace{PB(\Gamma)}_{\text{pure Braid group}} \leftrightarrow B(\Gamma) \rightarrow W \rightarrow 1
\]

33
7 Classification

Let \((V, \langle \cdot, \cdot \rangle)\) be an Euclidean vector space. Therefore it has a norm

\[ \|v\| = \sqrt{\langle v, v \rangle} \]

It induces a norm on \(\text{Lin}_R(V, V)\):

\[ \|T\| = \sup_{v \in V \setminus \{0\}} \frac{\|Tv\|}{\|v\|} \]

1. \(\|Tx\| \leq \|T\| \cdot \|x\|\)
2. \(\|T\| \geq 0\) and \(\|T\| = 0 \iff T = 0\]
3. \(\|T + S\| \leq \|T\| + \|S\|\)
4. \(T \in O(V) \implies \|Tv\| = \|v\| \implies \|T\| = 1\)
5. \(O(V)\) is a closed bounded subset of \(\text{Lin}_R(V, V)\).

**Theorem 7.1.** Let \(\Gamma\) be finite. Then \(W(\Gamma)\) is finite \(\iff V(\Gamma)\) is Euclidean.

**Proof.**

"\(\Rightarrow\)" Theorem 5.9

"\(\Leftarrow\)" short version: By 6.4 \(W(\Gamma)\) acts discretely on the unit sphere in \(V(\Gamma)\). Hence it must be finite.

Long version: pick \(v \in C\) s.t. \(\|v\| = 1\). Pick \(\varepsilon > 0\) s.t. \(B_\varepsilon(v) \subset C\).

\[ g \neq x \in W \implies gC \cap xC = \emptyset \implies B_\varepsilon(gv) \cap B_\varepsilon(xv) = \emptyset \]

\[ \implies \|gv - xv\| > \varepsilon \implies \|\rho(g) - \rho(x)\| > \varepsilon \]

By fact 5, \(O(V)\) is compact. Suppose \(W\) is infinite, \(W \subset O(V)\) compact. Then there is a convergent sequence \(T_1, T_2, \ldots, T_n\) in \(W\) s.t. \(T_i \neq T_j\) for \(i \neq j\).

Hence \(T_i \to T \in O(V)\), so there is a \(N\) s.t. \(\forall\ i, j > N: \|T_i - T_j\| < \varepsilon\).

But \(T_i, T_j \in \rho(W)\) with \(T_i \neq T_j\), so this is a contradiction.

\(\square\)

\(\Gamma\) is positive definite if \(V(\Gamma)\) is Euclidean. This is equivalent to \(W(\Gamma)\) being finite. From what we proved, we may conclude that any finite reflection group has a form

\[ (W(\Gamma_1) \times \ldots \times W(\Gamma_n), V(\Gamma_1) \oplus \ldots \oplus V(\Gamma_n) \oplus \mathbb{R}^k) \]

where the \(\Gamma_i\) are connected positive definite Coxeter graphs and each \(W(\Gamma_i)\) acting trivially on \(V(\Gamma_j)\) with \(i \neq j\) and \(\mathbb{R}^k\).

**Problem:** What are the connected positive definite Coxeter graphs?
Proposition 7.2. $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ symmetric matrix over $\mathbb{R}$.

$A$ is positive definite $\iff \forall j \in \{1, \ldots, n\} : \det \begin{pmatrix} a_{11} & \cdots & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jj} \end{pmatrix} > 0$.

Proof. Induction on $n$:

$n=1$ obvious

$n-1$ we assume to be done

$n \implies n$ $A = \begin{pmatrix} B \\ \vdots \\ a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ where $B$ is $(n-1) \times (n-1)$ matrix.

$B$ represents the same bilinear form restricted to $U = \left\{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} : \alpha_n = 0 \right\}$ in the standard basis of $U$.

In particular: for all $x \in U \setminus \{0\} : x^T B x > 0$. By induction assumption:

$\begin{pmatrix} a_{11} & \cdots & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jj} \end{pmatrix} > 0$

for $j < n$. Since there exists an orthogonal $Q$ s.t.

$QAQ^{-1} = \begin{pmatrix} \lambda_1 & \cdots \\ \vdots & \ddots \\ \lambda_n \end{pmatrix}$

and $A$ is positive definite $\iff$ all $\lambda_i > 0$, we conclude that $\det(A) = \lambda_1 \cdots \lambda_n > 0$.

$n \implies$ By induction assumption $B$ is positive definite, so $(U, \langle x, y \rangle = x^T B y)$ is Euclidean. By Lemma ??:

$\mathbb{R}^n = U \oplus \overset{\text{under } x^T A y}{U^\perp}$

Any $x \in \mathbb{R}^n$ can be written as $x = y + z$, $y \in U$, $z \in U^\perp$ and

$\langle x, x \rangle = \langle y + z, y + z \rangle = \langle y, y \rangle + \langle z, z \rangle$

So it suffices to check that for $z \in U^\perp$, $z \neq 0$ and $\langle z, z \rangle > 0$.

Let $f_1, \ldots, f_{n-1}$ be an orthonormal basis of $U$, $f_n \in U^\perp$, $f_n \neq 0$. $x^T A y$ in the basis
Let \( \Gamma = (X, m_{ab}) \). We say that \( \Gamma' = (Y, n_{ab}) \) is a subgraph of \( \Gamma \) if

- \( Y \subset X \)
- \( \forall a, b \in X : n_{ab} \leq m_{ab} \)

**Lemma 7.3.** If \( \Gamma' \) is a Coxeter subgraph of \( \Gamma \) and \( \Gamma \) is positive definite then \( \Gamma' \) is positive definite.

**Proof.**

\[
2 \leq n_{ab} \leq m_{ab} \implies -\cos \frac{\pi}{n_{ab}} \geq -\cos \frac{\pi}{m_{ab}}
\]

Pick any \( v = \sum_{a \in Y} \alpha_a e_a \in V(\Gamma') \) and suppose \( \langle v, v \rangle \leq 0 \). Let \( w = \sum_{a \in Y} |\alpha_a| e_a \). Then:

\[
\langle w, w \rangle_{V(\Gamma)} = \sum_{a, b \in Y} |\alpha_a| \cdot |\alpha_b| (-\cos \frac{\pi}{m_{ab}}) \\
\leq \sum_{a, b \in Y} |\alpha_a| \cdot |\alpha_b| (-\cos \frac{\pi}{n_{ab}}) \\
\leq \sum_{a, b \in Y} \alpha_a \alpha_b (-\cos \frac{\pi}{n_{ab}}) \\
\leq 0
\]

Since \( V(\Gamma) \) is Euclidean, \( w = 0 \). Then all \( \alpha_a = 0 \) and \( v = 0 \).

Let \( d(\Gamma) = \det(2 \langle e_a, e_n \rangle)_{X \times X} = \det(-2 \cos \frac{\pi}{m_{ab}})_{a, b \in X} \).

**Note:** \( d(\Gamma) \) is independent of the order on \( X \) one chooses to write a matrix.

**Lemma 7.4.** Let \( \Gamma \) be of the following form:

Then \( d(\Gamma) = 2d(\Gamma_1) - d(\Gamma_2) \).
Proof. Let $|X| = n$, call the two vertices $n$ and $n - 1$. Then the matrix $C(\Gamma)$ looks like
\[
\begin{pmatrix}
* & 0 \\
\vdots & \vdots \\
C(\Gamma_2) & \vdots \\
* & 0 \\
* \cdots * & 2 \ -1 \\
0 \cdots 0 & -1 \ 2
\end{pmatrix}
\]
Expanding the last row:
\[
d(\Gamma) = 2d(\Gamma_1) + (-1)^{n+1}(n-1)(-1)det \begin{pmatrix}
0 \\
\vdots \\
C(\Gamma_2) \\
0 \\
*\cdots* & -1
\end{pmatrix} = 2d(\Gamma_1) - d(\Gamma_2)
\]
\hfill \square

Lemma 7.5. The following Coxeter graphs $\Gamma$ have $d(\Gamma) > 0$:

\begin{itemize}
  \item $A_n \quad \circ \circ \cdots \circ \circ \circ$
  \item $B_n \quad \circ \quad 4 \circ \cdots \circ \circ$
  \item $D_n \quad \circ \cdots \circ \circ \circ \\
      \quad \circ$
  \item $E_6 \quad \circ \circ \circ \circ \circ \circ \\
      \quad \circ$
  \item $E_7 \quad \circ \circ \circ \circ \circ \circ \circ \\
      \quad \circ$
  \item $E_8 \quad \circ \circ \circ \circ \circ \circ \circ \circ \circ \\
      \quad \circ$
  \item $F_4 \quad \circ \circ \quad 4 \circ \circ
  \item $H_2 \quad \circ \quad 5 \circ$
  \item $H_3 \quad \circ \quad 5 \circ \circ$
  \item $H_4 \quad \circ \quad 5 \circ \circ \circ
  \item $I_2(m) \quad \circ \quad m \circ$
\end{itemize}

Note: $A_2 = I_2(3), B_2 = I_2(4), H_2 = I_2(5)$. Sometimes $G_2 = I_2(6)$ is used.

Proof. Let $x_n$ denote $d(X_n)$.
\begin{itemize}
  \item $i_2(m) = det \begin{pmatrix}
2 & -2 \cos \frac{\pi}{m} \\
-2 \cos \frac{\pi}{m} & 2
\end{pmatrix} = 4 - 4(\cos \frac{\pi}{m})^2 > 0$
\end{itemize}
• $a_1 = \det(2) = 2$
  $a_2 = i_2(3) = 4 - 4(\cos \frac{\pi}{3})^2 = 3$

• Lemma 7.4 allows us to compute $a_n$ by

\[
\begin{array}{c}
A_n \\
\circ \\
\bigcirc \\
\circ \\
\cdots \\
\bigcirc \\
A_{n-2} \\
\bigcirc \\
A_{n-1} \\
\bigcirc
\end{array}
\]

Then $a_n = 2(a_{n-1}) - a_{n-2}$ and by induction $a_n = n + 1$.

• $b_2 = i_2(4) = 2$

\[
\begin{array}{c}
B_3 \\
\circ \\
\bigcirc \\
\circ \\
\bigcirc \\
4 \\
\bigcirc \\
A_1 \\
\bigcirc \\
B_2 \\
\bigcirc
\end{array}
\]

$b_3 = 2b_2 - a_1 = 2 \cdot 2 - 2 = 2$
And for $B_n$:

\[
\begin{array}{c}
B_n \\
\circ \\
\bigcirc \\
\circ \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\cdots \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
B_{n-2} \\
\bigcirc \\
\bigcirc \\
B_{n-1} \\
\bigcirc
\end{array}
\]

$\iff b_n = 2b_{n-1} - b_{n-2} \iff b_n = 2$

• Before computing $D_n$ notice that:

\[
d(\Gamma_1 \cup \Gamma_2) = d(\Gamma_1) \cdot d(\Gamma_2)
\]

The decomposition for $D_n$ is:

\[
\begin{array}{c}
\circ \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\cdots \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc \\
A_1 \cup A_{n-3} \\
\bigcirc \\
A_{n-1} \\
\bigcirc
\end{array}
\]

Therefore $d_n = 2a_{n-1} - a_1a_{n-3} = 2n - 2(n - 3 + 1) = 4$.  

Similarly: $e_n = 2a_{n-1} - a_2a_{n-4} = 9 - n$. Hence $e_n$ is positive for $n \leq 8$.

- $h_2 = i_2(5) = 4 - 4\cos^2 \frac{\pi}{5} = \frac{5 - \sqrt{5}}{2}$
- $h_3 = 2h_2 - a_1 = 2\frac{5 - \sqrt{5}}{2} = 3 - \sqrt{5} > 0$
- $h_4 = 3h_3 - h_2 = 2(3 - \sqrt{5}) - \frac{5 - \sqrt{5}}{2} = \frac{7 - 3\sqrt{5}}{2} > 0$
- Notice that $h_5 = 3h_4 - h_3 = 5 - 2\sqrt{5} < 0$.

\[\square\]

**Lemma 7.6.** The following Coxeter graphs have $d(\Gamma) < 0$:

- $\tilde{A}_n$
- $\tilde{B}_n$
- $\tilde{C}_n$
- $\tilde{B}_n$
- $\tilde{E}_6$
- $\tilde{E}_7$
- $\tilde{E}_8$
\[
\begin{align*}
\tilde{F}_4 & \quad \circ \quad \circ \quad \circ \quad \circ \quad 4 \quad \circ \quad \circ \quad \circ \\
\tilde{G}_2 & \quad \circ \quad \circ \quad \circ \quad \circ \quad 6 \\
\tilde{H}_3 & \quad \circ \quad \circ \quad \circ \quad \circ \quad 5 \\
\tilde{H}_4 & = H_5 \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \\
\end{align*}
\]

**Proof.**

- We have computed \(\tilde{c}_8 = \tilde{e}_9 = 9 - 9 = 0\).
- \(\tilde{h}_4 = h_5 = 4 - 2\sqrt{5} < 0\)
- \(\tilde{a}_n = \text{det} \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 2 \end{pmatrix} = 0\) because the sum of the rows is zero
- \(\tilde{b}_n = 2b_n - a_1b_{n-2} = 4 - 4 = 0\)
- \(\tilde{d}_n = 2d_n - a_1d_{n-2} = 2 \cdot 4 - 2 \cdot 4 = 0\)
- \(\tilde{e}_6 = 2e_6 - a_5 = 0\)
- \(\tilde{e}_7 = 2e_7 - d_6 = 0\)
- \(\tilde{f}_4 = 2f_4 - b_3 = 0\)
- \(\tilde{g}_2 = 2i_2(6) - a_1 = 0\)
- \(\tilde{h}_3 = 2h_3 - a_2 = 3 - 2\sqrt{5} < 0\)
- \(c(\tilde{C}_n)\) looks like \[
\begin{pmatrix}
\ast & 0 \\
\ast & 0 \\
\ast & 2 & -2 \\
0 & -2 & 2
\end{pmatrix}
\]

Similarly to Lemma 7.4, expanding the last row gives
\[
d(\tilde{C}_n) = 2d(B_n) - 2d(B_{n-1}) = 2 \cdot 2 - 2 \cdot 2 = 0
\]

\[
\square
\]
**Theorem 7.7.** The connected Coxeter graphs $\Gamma$ which give a finite $W(\Gamma)$ are precisely the graphs listed in Lemma 7.5.

**Proof.** The graphs on 7.5 are closed under taking connected Coxeter subgraphs. Therefore 7.2 tells us that all $c(\Gamma)$ are positive definite in 7.5 and 7.1 tells us that all $W(\Gamma)$ are finite in 7.5.

Now let $\Gamma$ be a connected graph with finite $W(\Gamma)$. Then $c(\Gamma)$ is positive definite by 7.1. In particular, $\Gamma$ contains no subgraphs in 7.6

- No $\tilde{A}_n \implies \Gamma$ has no cycles
- No $\tilde{D}_4 \implies \Gamma$ has no vertex with $\geq 4$ edges
- No $\tilde{D}_n$, $n \geq 5 \implies \Gamma$ contains at most one vertex with 3 edges.

**Case 1** No vertex has 3 edges, hence $\Gamma$ looks like this:

$$
\circ \overset{x_1}{\cdots} \overset{x_m}{\circ}
$$

If $\Gamma$ has $\leq 2$ vertices it is either $A_1$ or $I_2(m)$, both in Lemma 7.5.

Let $\Gamma$ contain $\geq 3$ vertices ($m \geq 2$).

- No $\tilde{G}_3 \implies$ edges may have multiplicity at most 5
- No $\tilde{C}_n \implies$ at most one edge has multiplicity $\geq 4$.

**Case 1.1** No edge of multiplicity $\geq 4 \implies \Gamma = A_n$

**Case 1.2** There is one edge of multiplicity 4:

- if it is on a side, $\Gamma = B_n$
- if it is in the middle then no $\tilde{F}_4 \implies \Gamma = F_4$

**Case 1.3** There is one edge of multiplicity 5:

- No $\tilde{H}_3 \implies$ the multiple edge is on a side and no $\tilde{H}_4 \implies \Gamma = H_3$ or $H_4$

**Case 2** there is a vertex with 3 edges. Then $\Gamma$ look like this:

$$
\circ \\
| \\
\circ \\
\cdots \circ \overset{b}{\cdots} \overset{a}{\circ} \overset{c}{\cdots} \overset{\circ}{\cdots} \\
\circ \\
$$

Let $b$ denote the vertices going up, $a$ the vertices going left and $c$ the vertices going right. Then we have $a + b + c + 1$ vertices in total and wlog $a \geq b \geq c$.

- No $\tilde{B}_n \implies \Gamma$ has no multiple edges.
- No $\tilde{E}_6 \implies c = 1$
- No $\tilde{E}_7 \implies b \leq 2$
- if $b = c = 1 \implies \Gamma = D_{a+3}$
- if $b = 2$, $c = 1$, then no $\tilde{E}_8 \implies a < 5$. As $a \geq b = 2$ there are 3 cases:
  1. $a = 2 \implies \Gamma = E_6$
2. $a = 3 \implies \Gamma = E_7$
3. $a = 4 \implies \Gamma = E_8$

<table>
<thead>
<tr>
<th>Crystallographic</th>
<th>NON Crystallographic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n, B_n, D_n, F_4, E_n, G_2 = I_2(6)$</td>
<td>$I_2(m)$ for $m = 5, m \geq 7, H_3, H_4$</td>
</tr>
<tr>
<td>$d(\Gamma) \in \mathbb{Z}$</td>
<td>$d(\Gamma) \notin \mathbb{Z}$</td>
</tr>
<tr>
<td>The character table of $\Gamma$ is integral</td>
<td>Non-integral</td>
</tr>
<tr>
<td>All but one representation of $W(\Gamma)$ can be realised by integral matrices</td>
<td></td>
</tr>
</tbody>
</table>
8 Crystallographic Coxeter Groups

Let $V$ be a vector space over $\mathbb{R}$. A lattice in $V$ is an abelian subgroup $L$ of $V$ s.t. $L = \langle e_1, \ldots, e_n \rangle$ where $e_1, \ldots, e_n$ is a basis of $V$. Informally, $L$ is $\mathbb{Z}^n$ inside $\mathbb{R}^n$.

**Definition.** $W(\Gamma)$ is crystallographic if there is a lattice $L \subset V(\Gamma)$ s.t.

$$\forall g \in W(\Gamma) : g(L) \subset L$$

Informally this means that there is a basis $e_1, \ldots, e_n$ of $V(\Gamma)$ s.t. $\rho(g)$ is an integer matrix for each $g \in W(\Gamma)$.

**Lemma 8.1.** Let $|X| < \infty$. If $W(\Gamma)$ is crystallographic then $\forall a, b \in X$:

$$m_{ab} \in \{2, 3, 4, 6, \infty\}$$

**Proof.** Assume $W(\Gamma)$ is crystallographic. Then $\forall g \in W : \rho(g)$ can be written as an integer matrix, in particular $Tr(\rho(\Gamma)) \in \mathbb{Z}$.

If $m_{ab} = \infty$ then we are done.

Suppose $m_{ab} = \infty \implies U = \text{span}(e_a, e_b)$ is Euclidean $\implies W(\Gamma) = U \oplus U^\perp$.

In the basis $e_a, e_b$, any basis of $U^\perp$:

$$\rho(S_aS_b) = \begin{pmatrix} \cos \frac{2\pi}{m_{ab}} & \pm \sin \frac{2\pi}{m_{ab}} & 0 \\ \mp \sin \frac{2\pi}{m_{ab}} & \cos \frac{2\pi}{m_{ab}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Tr(\rho(S_aS_b)) = |X| - 2 + 2 \cos \frac{2\pi}{m_{ab}} \text{ hence}$$

$$2 \cos \frac{2\pi}{m_{ab}} \in \mathbb{Z} \iff \cos \frac{2\pi}{m_{ab}} \in \frac{1}{2} \mathbb{Z} \iff m_{ab} \in \{2, 3, 4, 6\}$$

**Note:** $\iff$ is true but the proof is harder unless $W(\Gamma)$ is finite.

**Theorem 8.2.** Let $W(\Gamma)$ be a finite Coxeter group. Then $W(\Gamma)$ is crystallographic $\iff \forall a, b \in X : m_{ab} \in \{2, 3, 4, 6\}$.

**Proof.**

"$\Rightarrow$" Lemma 8.1
Let us choose a new basis $\alpha_a = c_a e_a$, $a \in X$, $c_a \in \mathbb{R} \setminus \{0\}$. Then

$$\rho_a(\alpha_b) = \alpha_b - 2 \langle \alpha_b, e_a \rangle e_a = \alpha_b - 2 \frac{\langle \alpha_a, \alpha_b \rangle}{\langle \alpha_a, \alpha_a \rangle} \alpha_a$$

Since $\rho_a = S_{c_a} = S_{\alpha_a}$ and $\rho(\pi_1 \cdots \pi_n) = \rho_{a_1} \cdots \rho_{a_n}$ it suffices to ensure that for all $a, b \in X$:

$$2 \frac{\langle \alpha_a, \alpha_b \rangle}{\langle \alpha_a, \alpha_a \rangle} \in \mathbb{Z}$$

Wlog $\Gamma$ is connected.

- $m_{ab} = 2 \implies \langle \alpha_a, \alpha_b \rangle = 0 \implies 2 \frac{\langle \alpha_a, \alpha_b \rangle}{\langle \alpha_a, \alpha_a \rangle} = 0$
- $m_{ab} = 3 \implies \langle \alpha_a, \alpha_b \rangle = c_a c_b (c_a, c_b) = c_a c_a - \cos \frac{\pi}{m_{ab}} = -\frac{c_a c_b}{2}$
  $$\implies 2 \frac{\langle \alpha_a, \alpha_b \rangle}{\langle \alpha_a, \alpha_a \rangle} = -\frac{c_a c_b}{c_a} = -\frac{c_b}{c_a}$$

Need in this case $c_a = \pm c_b$

- $m_{ab} = 4 \implies \langle \alpha_a, \alpha_b \rangle = c_a c_b (-\cos \frac{\pi}{4}) = -\frac{c_a c_b}{\sqrt{2}}$
  $$\implies 2 \frac{\langle \alpha_a, \alpha_b \rangle}{\langle \alpha_a, \alpha_a \rangle} = -\sqrt{2} \frac{c_b}{c_a}$$

Need $c_a = \sqrt{2} c_b$ or $c_b = \sqrt{2} c_a$

- $m_{ab} = 6 \implies \langle \alpha_a, \alpha_b \rangle = -\sqrt{3} c_a c_b$
  $$\implies 2 \frac{\langle \alpha_a, \alpha_b \rangle}{\langle \alpha_a, \alpha_a \rangle} = 2 \frac{-c_a c_b \sqrt{3}}{c_a c_a} = -\sqrt{3} \frac{c_b}{c_a}$$

Need $c_a = \pm \sqrt{3} c_b$ or $c_b = \pm \sqrt{3} c_a$

In each component of $\Gamma$ we choose

**Case 1** $A_n, D_n, E_n$: choose all $c_a = 1$.

**Case 2** $B_n, F_4$

$$\begin{array}{c}
\Gamma_1 \\
\text{choose } c_a = 1
\end{array} \quad \begin{array}{c}
4 \\
\text{choose } c_a = \sqrt{2}
\end{array}$$

**Case 3** $G_2 = I_2(6)$:

$$a \quad 6 \quad b$$

$\alpha_a = \sqrt{3} e_a$, $\alpha_b = e_b$

□

**Corollary 8.3.** The finite crystallographic reflection groups are $W(\Gamma)$ where $\Gamma$ is either
• $A_n$, $n \geq 1$
• $B_n$, $n \geq 2$
• $D_n$, $n \geq 4$
• $E_n$, $6 \leq n \leq 8$
• $F_4$
• $G_2$

**Note:** In $B_n$, $n \geq 3$ there are 2 non-isomorphic choices of crystallographic structure

\[
\begin{array}{cccccccc}
1 & 4 & 2 & 3 & \cdots & \cdots & \cdots & n
\end{array}
\]

• $\alpha_1 = e_1$, $\alpha_2 = \sqrt{2}e_2, \ldots, \alpha_k = \sqrt{2}e_k$ which is called the crystallographic reflection group $B_n$

• $\alpha_1 = \sqrt{2}e_1$, $\alpha_2 = e_2, \ldots, \alpha_k = e_k$ which is called the crystallographic reflection group $C_n$

We have seen $A_n, B_n, D_n, I_2(m)$ explicitly.
Can we compute $|W(\Gamma)|, |\Phi(\Gamma)|$ for all $\Gamma$?
9 Polynomial invariants

Let $G$ be a finite group acting on $V$, $\dim V < \infty$. Let $S = \mathbb{R}[V]$, algebra of all polynomial functions on $V$. Let $e_1, \ldots, e_n$ be a basis of $V$ and $X_i : V \to \mathbb{R}$ the linear maps which form the dual basis. Then $X_i(e_j) = \delta_{ij}$. In fact, $S = \mathbb{R}[X_1, \ldots, X_n]$. $G$ acts on the algebra $S$: $g \in G, F \in S, v \in V : g \cdot F(v) = F(g^{-1}v)$.

Exercise: check that $G$ acts on $S$ by algebra automorphisms.

Note: for $F \in S$ the following are equivalent:

1. $\forall \ g \in G: gF = F$
2. $F$ is constant on $G$-orbits in $V$.

We call such functions invariant and they form a subalgebra $S^G$.

We will prove that $S^G \cong \mathbb{R}[z_1, \ldots, z_k]$ (when $G$ is a reflection group). In fact

$$S^G \cong \mathbb{R}[z_1, \ldots, z_k] \iff G \text{ is a reflection group}$$

(no proof)

$S$ is a graded algebra, i.e. $S = \bigoplus_{k=0}^{\infty} S_k$ where $S_k$ consists of polynomials where each monomial has degree $k$.

Example:

- $2010 \in S_0$
- $x_1^2 + x_2^2 + x_1x_3 \in S_2$
- $1 + x_1 \in S_0 \oplus S_1$ but not in any $S_i$

If $F \in S_i$ we say $\deg F = i$.

**Lemma 9.1.** Let $F \in S^G$, $F = \sum_{j=0}^{m} F_j$, $F_j \in S_j$. Then $\forall j$: $F_j \in S^G$.

**Proof.** $G \times \mathbb{R}^*$ acts on $V$: $(g, \alpha)v = g(\alpha v) = \alpha g(v)$. Hence $G \times \mathbb{R}^*$ also acts on $S$:

$$(g, \alpha) \cdot F : v \mapsto F(g^{-1}\alpha^{-1}v)$$

In particular, $f \in S_j$: $(1, \alpha) \cdot f = \alpha^{-j} \cdot f$.

Since actions of $G$ and $\mathbb{R}^*$ commutes, $\forall \alpha \in \mathbb{R}^*$ :

$$(1, \alpha)F = \sum_{j=0}^{m} \alpha^{-j}F_j \in S^G$$

$1, \ldots, m+1 : m+1$ different real numbers
1. \( F = F_0 + \ldots + F_{m+1} \in S^G \)
2. \( F = F_1 + \frac{1}{2} F_1 + \ldots + \frac{1}{2^{m+1}} F_{m+1} \in S^G \)

\[ F = F_0 + \ldots + F_{m+1} \in S^G \]

\[ m+1 \cdot F = F_0 + \frac{1}{m+1} F_1 + \ldots + \frac{1}{(m+1)^{m+1}} F_{m+1} \in S^G \]

If \( M = \begin{pmatrix}
1 & \ldots & 1 \\
\frac{1}{2} & \ldots & \left(\frac{1}{2}\right)^{m+1} \\
\vdots & \vdots & \vdots \\
\frac{1}{m+1} & \ldots & \left(\frac{1}{m+1}\right)^{m+1}
\end{pmatrix} \in \mathbb{R}^{(m+1)\times (m+1)} \) is a VanderMonde matrix then

\[ M \cdot \begin{pmatrix}
F_0 \\
\vdots \\
F_m
\end{pmatrix} \in (S^G)^{m+1} \]

Since \( M \) is invertible, \( \begin{pmatrix}
F_0 \\
\vdots \\
F_m
\end{pmatrix} = M^{-1} \begin{pmatrix}
H_0 \\
\vdots \\
H_m
\end{pmatrix} \) where \( H_i \in S^G \).

Hence all \( F_i \in S^G \).

Lemma 9.1 means that \( S^G \) is a graded subalgebra of \( S \), i.e.

\[ S^G = \bigoplus_{k=0}^{\infty} (S^G \cap S_k) \quad \text{and} \quad S_k^G = S^G \cap S_k \]

Let \( S_k^G = \bigoplus_{k=1}^{\infty} S_k^G \) be the invariants of positive degree. Let \( I = (S^G_+) \) be the ideal of \( S \) generated by \( S^G_+ \). By Hilbert’s Basis Theorem, any ideal in \( S \) is generated by finitely many elements.

Say \( I = (f_1, \ldots, f_k)_S \), \( f_i \in S^G \). Write \( f_i = \sum_j f_{ij} \) where \( f_{ij} \in S_j \). With Lemma 9.1:

\( f_{ij} \in S_j \), \( j < 0 \).

Hence \( I = (I_1, \ldots, I_r)_S \) where \( I_j \in S^G_j \). If for any \( k \): \( I_k \in (I_1, \ldots, I_{k-1}, I_{k+1}, \ldots, I_r)_S \) we throw \( I_k \) away. Wlog, \( I_1, \ldots, I_r \) is a minimal system of such generators.

**Definition.** \( I_1, \ldots, I_r \) are called fundamental invariants if they are a minimal system of homogeneous invariant generators of \((S^G_+)_S\).

- Coinvariants: \( S_G = S / I \)
- Integral: \( \int : S \rightarrow S^G \)

\[ \int F = \frac{1}{|G|} \sum_{g \in G} gF \]
Note that $F \in S^G \implies gF = F \implies \int_G F = F$ and $\forall H \in S:\$

$$\int_G (FH) = \frac{1}{|G|} \sum_g (gF)(gH) = F \frac{1}{|G|} \sum_g gH = F \int_G H$$

So $\int$ is an $S^G$-linear map $S \to S^G$.

**Proposition 9.2.** Let $G$ be any finite group acting on $V$, $I_1, \ldots, I_r \in S^G = \mathbb{R}[V]^G$ fundamental invariants. Then:

$$\forall F \in S^G \exists p(z_1, \ldots, z_r) \in \mathbb{R}[z_1, \ldots, z_r] \text{ s.t. } F = p(I_1, \ldots, I_r)$$

**Proof.** Lemma 9.1: we may restrict to homogeneous $F$. We do induction on $d = \deg(F)$.

$d = 0$: $\implies F = \text{const} \implies p = \text{const} \cdot 1$ does the job.

$d < N$: is done

$d = N > 0$: $F \in S^G \implies F \in (S^G)_S \implies F = \sum_{j=1}^r H_jI_j, \ H_j \in S$. Since $d = \deg(F)$, $d_j = \deg(I_j)$:

$$F = \sum_{j=1}^r (H_j)^{d-d_j}I_j$$

and therefore

$$F = \int_G F = \sum_{j=1}^r I_j \int_G (H_j)^{d-d_j}$$

By induction assumption $\forall j \exists p_j \in \mathbb{R}[z_1, \ldots, z_r] \text{ s.t. } \int_G (H_j)^{d-d_j} = p_j(I_1, \ldots, I_r)$. Hence

$$p = \sum_{j=1}^r z_jp_j$$

does the job.

So for any real representation $V$ of any finite group $G$ the algebra map

$$\mathbb{R}[z_1, \ldots, z_r] \to S^G, \ z_i \mapsto I_i$$

is surjective.

1. Injectivity holds only for reflection groups
2. Hilbert’s 14-th problem is that $S^G$ is a finitely generated algebra for any group
3. $S^G_0 = \mathbb{R} \cdot 1$ constant functions
4. \( f \in S^G_1, f \neq 0 \implies V = \text{Ker}(f) \oplus \mathbb{R} \). Hence \( S^G_1 \neq 0 \iff gV \) has a trivial constituent

5. if \( \langle , \rangle \) is a Euclidean form on \( V \) then \([x,y] = \int G \langle x,y \rangle \) makes \( G \) orthogonal. Hence \( F(x) = [x,x] \) defined \( F \in S^G_2 \).

**Lemma 9.3.** Let \( P \in S, L \in S_1 \setminus \{0\} \), if \( \forall x \in V: L(x) = 0 \implies P(x) = 0 \) then \( L|P \).

**Proof.** \( S = \mathbb{R}[x_1, \ldots, x_n] \). Wlog \( L = \alpha x_n + b(x_1, \ldots, x_{n-1}) \) with \( \alpha \neq 0 \). Let us divide \( P \) by \( L \) with remainder by replacing every \( x_n \) with \(-\frac{1}{\alpha} b(x_1, \ldots, x_{n-1}). \) \( x_n = \frac{1}{\alpha} L - \frac{1}{\alpha} b(x_1, \ldots, x_{n-1}). \)

**Example:** \( L = x_1 - x_2 \)

\[
P = x_1^2 = x_1 \cdot x_1 = x_1(L + x_2)
\]

\[
= x_1 L + x_1 x_2 = x_1 L + (L + x_2)x_2
\]

\[
= (x_1 + x_2) L + x_2^2
\]

\( P = AL + B \), \( A \in \mathbb{R}[x_1, \ldots, x_n], B \in \mathbb{R}[x_1, \ldots, x_{n-1}] \). If \( B = 0 \) we are done. If \( B \neq 0 \) pick \( a_1, \ldots, a_{n-1} \in \mathbb{R} \) s.t. \( B(a_1, \ldots, a_{n-1}) \neq 0 \).

Let \( a_n = -\frac{1}{\alpha} b(a_1, \ldots, a_{n-1}) \implies L(a_1, \ldots, a_n) = 0 \implies P(a_1, \ldots, a_n) = 0. \)

But \( P(a_1, \ldots, a_n) = B(a_1, \ldots, a_{n-1}) \neq 0 \) which is a contradiction. \( \square \)

**Proposition 9.4.** Let \( G = W \) be a finite reflection group. Let \( J_1, \ldots, J_k \in S^W \) s.t. \( J_1 \notin (J_2, \ldots, J_k)_S \). If \( P_1 \in S_h \) s.t. \( \sum_{i=1}^k P_i J_i = 0 \) then \( P_1 \in (S^W_+) \).

**Proof.** Observe that \( J_1 \notin (J_2, \ldots, J_k)_S \): if \( J_1 = \sum_{i=2}^k H_i J_i \) with \( H_i \in S \) then

\[
J_1 = \int J_1 = \sum_{i=2}^k \int H_i \in (J_2, \ldots, J_k)_S
\]

Proceed by induction on \( \text{deg}(P_1) \):

\[
\text{deg}(P_1) = 0: \implies P_1 = c \cdot 1 \implies c \cdot J_1 = \sum_{i=2}^k (-P_i) J_i \implies c = 0 \text{ because } J_1 \notin (J_2, \ldots, J_k)_S \implies P_1 = 0 \in (S^W_+) \]

\( \text{deg}(P_1) < N \): done

\( \text{deg}(P_1) = N \): Let \( \Phi \subset W \) be the root system of \( G \). \( \forall \alpha \in V \) let \( L_\alpha \in S_1 \) be \( L_\alpha(v) = \langle \alpha, v \rangle \).

\( L_\alpha = 0 \implies \alpha \perp v \implies S_\alpha(v) = v \)

\[
\implies \forall j: (S_\alpha P_j - P_j)(v) = P_j(S_\alpha^{-1} v) - P_j(v) = 0
\]

With Lemma 9.3: \( S_\alpha P_j - P_j = L_\alpha \overline{P_j} \) for some \( \overline{P_j} \in S \).

Note that \( \text{deg}(\overline{P_j}) < \text{deg}(P_j) \)

- \( P_1 J_1 + \ldots + P_k J_k = 0 \)
- \( (S_\alpha P_1) J_1 + \ldots + (S_\alpha P_k) J_k = 0 \)

49
\[ (P_1 - S_a P_1)J_1 + \ldots + (P_k - S_a P_k)J_k = 0 \]

Hence \( \alpha_0 (\mathcal{P}_1 J_1 + \ldots + \mathcal{P}_k J_k) = 0 \). Since \( S \) is a domain \( \mathcal{P}_1 J_1 + \ldots \mathcal{P}_k J_k = 0 \). By induction assumption, \( \mathcal{P}_1 \in (S^W_+) \). Hence \( S_a P_1 - P_1 = L_a \mathcal{P}_1 \in (S^W_+) \).

For each polynomial \( F \), denote \( \hat{F} = F + (S^W_+)S \in S_W \). Hence \( \forall \alpha \in \Phi : S_a \hat{P}_1 = \hat{P}_1 \).

Since \( W \) is generated by \( S_a \), \( \forall g : g \hat{P}_1 = \hat{P}_1 \). Hence \( \int_W \hat{P}_1 = \hat{P}_1 \) (\( W \) acts on \( S_W \) and \( \int \hat{F} = \int F \))

So \( P_1 \in \int P_1 + (S^W_+)S \subset (S^W_+) \).

\[ \square \]

**Lemma 9.5.** (Euler’s formula)

\( F \in S_k = \mathbb{R}[x_1, \ldots, x_n]_k \), then

\[
\sum_{j=1}^{n} x_j \frac{\partial F}{\partial x_j} = kF
\]

**Proof.** Both sides are linear in \( F \), hence it suffices to verify the formula on monomials. Let \( F = x_1^{a_1} \ldots x_n^{a_n}, \sum a_i = k \). Then

\[
x_j \frac{\partial F}{\partial x_j} = a_j F
\]

Hence \( \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} = (\sum a_i) F = kF \).

\[ \square \]

**Proposition 9.6.** \( I_1, \ldots, I_r \) fundamental invariants for a (finite) reflection group \( (W, V) \). If \( P(I_1, \ldots, I_r) = 0 \) for some \( P \in \mathbb{R}[z_1, \ldots, z_r] \) then \( P = 0 \).

**Note:** \( I_1, \ldots, I_r \) are algebraically independent.

**Proof.** Consider a grading on \( \mathbb{R}[z_1, \ldots, z_r] \) with \( \deg z_i = d_i = \deg I_i \). Let \( P = \sum P_j, P_j \) homogeneous of degree \( j \). Then

\[ P(I_1, \ldots, I_r) = \sum_{j \in S_j} P_j(I_1, \ldots, I_r) = 0 \]

Hence \( P_j(I_1, \ldots, I_r) = 0 \). WLOG, we assume \( P \) is homogeneous.

Let \( P_{ij} = \frac{\partial P}{\partial z_j} \) and \( f_j = P_{ij}(I_1, \ldots, I_r) \in S^W \) is homogeneous. Let us fiddle around with the ideal

\[ K = (f_1, \ldots, f_r)S_W \triangleleft S^W \]

WLOG, \( f_1, \ldots, f_m \) is a minimal system of generators of \( K, m \leq r \). Note that \( \deg f_i = \deg P - \deg I_i \) and \( f_i \) is homogeneous. If \( i \geq m + 1: f_i \in K = (f_1, \ldots, f_m)S_W \) so there are \( Q_{ij} \in S^W, Q_{ij} \) homogeneous of degree \( \deg I_i - \deg I_j \) s.t. \( f_i = \sum_{j=1}^{m} Q_{ij} f_j \).

\[
P(I_1, \ldots, I_r) = 0 \implies 0 = \frac{\partial P(I_1, \ldots, I_r)}{\partial x_k} = \sum_{i=1}^{r} \frac{\partial P}{\partial z_i}(I_1, \ldots, I_r) \frac{\partial I_i}{\partial x_k} \forall k \in \{1, \ldots, n\}
\]
Hence:

\[ \forall k : \sum_{i=1}^{m} f_i \frac{\partial I_i}{\partial x_k} + \sum_{i=m+1}^{r} \sum_{j=1}^{m} Q_{ij} f_j \frac{\partial I_i}{\partial x_k} = 0 \]

\[ \forall k : \sum_{i=1}^{m} f_i \left( \frac{\partial I_i}{\partial x_k} + \sum_{j=m+1}^{r} Q_{ji} \frac{\partial I_j}{\partial x_k} \right) = 0 \]

By 9.4, for all \( i \in \{1, \ldots, m\} \):

\[ \frac{\partial I_i}{\partial x_k} + \sum_{j=m+1}^{r} Q_{ji} \frac{\partial I_j}{\partial x_k} \in (S^W \cap S) \]

Hence for some \( H^k_i \in S \):

\[ \forall i, k : \frac{\partial I_i}{\partial x_k} + \sum_{j=m+1}^{r} Q_{ji} \frac{\partial I_j}{\partial x_k} = \sum_{t=1}^{r} I_t H^k_i \]

Multiply each equality by \( x_k \) and sum over \( k \). By Euler’s formula:

\[ \forall i : d_i I_i + \sum_{j=m+1}^{r} d_j Q_{ji} I_j = \sum_{t=1}^{r} I_t \tilde{H}_t \]

where \( \tilde{H}_t \) have no free terms (\( \tilde{H}_t = \sum x_k H^k_i \)). \( I_i \) enters both sides of the equality. If we consider the homogeneous degree \( d_i \) parts of both sides, then

\[ d_i I_i + \sum_{j=m+1}^{r} c_j I_j = \sum_{t=1}^{r} I_t b_t \]

where \( b_t \) has to be zero because \( \tilde{H}_t \) has no free terms. Therefore

\[ I_i = \sum_{j \neq i} I_j G_j \implies I_i \in (I_1, \ldots, I_{i-1}, I_{i+1}, \ldots, I_r) \]

which is a contradiction. Therefore all \( \frac{\partial P}{\partial x_k} = 0 \implies P = 0. \]

\[ \square \]

Example: \( S_n \) acting on \( \mathbb{R}^n \) by permutations.

\[ \mathbb{R}^n = U \oplus \mathbb{R} \text{ where } U = \left\{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} : \sum \alpha_i = 0 \right\} \text{ and } \mathbb{R} = \mathbb{R} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \]

\((S_n, U)\) is then a type \( A_{n-1} \) essential reflection group with elementary symmetric functions:

\[ \sigma_1(x_1, \ldots, x_n) = x_1 + \ldots + x_n \]
\[ \sigma_2(x_1, \ldots, x_n) = x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n \]
\[ \ldots \]
\[ \sigma_n(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n \]
is a system of fundamental invariants. The power functions $\pi_j = x_1^j + \ldots + x_n^j$, $j = 1, \ldots, n$ is another fundamental system.

$A_{n-1}$: $\sigma_1|_U = 0$
$\sigma_2|_U, \ldots, \sigma_n|_U$ are fundamental invariants with fundamental degrees $2, 3, \ldots, n$.

$B_n$: group has more transformations $x_i \mapsto \pm x_i$, $S^W(B_n) = (S^S_n)^C_2 = \mathbb{R}[\sigma_1, \ldots, \sigma_n]^C_2$.
Hence the fundamental invariants are $\sigma_j(x_1^2, \ldots, x_n^2)$ with degrees $2 \cdot 1, 2 \cdot 2, \ldots, 2 \cdot n$.

$D_n$: in $D_n$, instead of the full group $C_2^n$, it is a subgroup of index 2. So $\sigma_n = x_1 \cdots x_n \in S^W(D_n)$.
So the fundamental invariants are $\sigma_1(x_1^2), \ldots, \sigma_{n-1}(x_1^2), \sigma_n(x_i)$ with degrees $2 \cdot 1, 2 \cdot 2, \ldots, 2 \cdot (n-1), n$.

**Theorem 9.7.** Let $(G, V)$ be a finite reflection group, $n = \dim V$, $S = \mathbb{R}[V]$, $I_1, \ldots, I_r$ fundamental invariants. Then

$$S^G = \mathbb{R}[I_1, \ldots, I_r] \quad \text{and} \quad r = n$$

**Proof.** The natural algebra homomorphism

$$\psi : \mathbb{R}[z_1, \ldots, z_r] \longrightarrow S^G, \ F(z_1, \ldots, z_r) \mapsto F(I_1, \ldots, I_r)$$

is surjective by 9.2 and injective by 9.6. Hence $\psi$ is an isomorphism. It remains to show that $n = r$.
Consider the field of rational functions

$$F = \mathbb{R}(x_1, \ldots, x_n) = \{ \frac{f}{g} : f, g \in S, \ g \neq 0 \}$$

Transcendence degree is $\text{trdeg}_\mathbb{R} F = n$. The smaller field $A = \mathbb{R}(I_1, \ldots, I_r) \leq F$, $\text{trdeg}_\mathbb{R} A = r$.
From Galois theory, $\text{trdeg}_A F = \text{trdeg}_\mathbb{R} F - \text{trdeg}_\mathbb{R} A = n - r$. It suffices to show that $\text{trdeg}_A F = 0$, i.e. every element of $F$ is algebraic over $A$.
Pick $f \in F$, $G$ acts on $F$ with $F^G = A$. Consider $h(z) \in F[z]$ defined by

$$h(z) = \prod_{g \in G} (z - g.f)$$

Since \( \forall t \in G, t.h = \prod_{g \in G} (z - tg.f) = h \) \( \implies h \in A[z] \).

$$h(f) = \prod_{g \in G} (f - g.f) = (f - f) \prod_{g \neq 1} (f - g.f) = 0$$

Hence $f$ is algebraic over $A$. \( \square \)

Let $d_1, \ldots, d_n$ with $d_i = \deg I_i$ be the fundamental degrees.

**Theorem 9.8.** If $I'_1, \ldots, I'_n$ is another system of fundamental invariants with degrees $d'_1, \ldots, d'_n$, $d'_j = \deg I'_j$ then $\exists \ \sigma \in S_n$ s.t.

$$\forall j : d_j = d'_{\sigma(j)}$$

52
Proof. By 9.7 there are $F_1, \ldots, F_n, G_1, \ldots, G_n \in \mathbb{R}[z_1, \ldots, z_n]$ s.t.

$$I_j = F_j(I_1', \ldots, I_n') \quad \text{and} \quad I_j' = G_j(I_1, \ldots, I_n) \quad \forall \ j$$

Then $\forall \ j: \ I_j = F_j(G_1(I_1, \ldots, I_n), \ldots, G_n(I_1, \ldots, I_n))$. Hence, since $I_1, \ldots, I_n$ are algebraically independent:

$$z_j = F_j(G_1(z_1, \ldots, z_n), \ldots, G_n(z_1, \ldots, z_n))$$

and therefore

$$\frac{\partial z_j}{\partial z_k} = \sum_{s=1}^{n} \frac{\partial F_j}{\partial z_s}(G_1(z_1, \ldots, z_n), \ldots, G_n(z_1, \ldots, z_n)) \frac{\partial G_s}{\partial z_k}(z_1, \ldots, z_n)$$

And on matrix level:

$$I_n = \left( \frac{\partial F_i}{\partial z_s} \right)_{I_1', \ldots, I_n'} \left( \frac{\partial G_s}{\partial z_k} \right)_{I_1, \ldots, I_n} \in Mat_n(S)$$

Hence $\det \left( \frac{\partial F_i}{\partial z_s} \right)$ is invertible in $S = \mathbb{R}[x_1, \ldots, x_n]$. Hence $\exists \ \sigma \in S_n$ s.t.

$$\frac{\partial F_1}{\partial z_{\sigma(1)}} \cdots \frac{\partial F_n}{\partial z_{\sigma(n)}} = c \cdot 1 + \text{higher degree terms, } c \neq 0$$

Since $\deg I'_i = \deg z_i = d'_i$, $\deg F_i = \deg I_i = d_i$:

$$\forall \ i: \ d_i = d'_i$$

\[\square\]
10 Fundamental Degrees

Discuss the role of \(d_1, \ldots, d_n\).

**Lemma 10.1.** A finite group \(G\) acts in a finite dimensional vectorspace over \(\mathbb{C}\) (by \(\rho : G \rightarrow GL_n(\mathbb{C})\)). Then \(\forall x \in G: \rho(x)\) is diagonalisable with eigenvalues of absolute value 1.

**Proof.** \(|G| = n < \infty \implies x^n = 1 \implies \rho(x)^n = I_n\).

Hence the minimal polynomial of \(\rho(x)\), \(\mu_x(z)\) divides \(z^n - 1\). Therefore eigenvalues \(\lambda\) satisfy \(\lambda^n - 1 = 0\), hence \(\lambda^n = 1 \implies |\lambda|^n = 1 \implies |\lambda| = 1\).

Further, since \(z^n - 1\) has no multiple roots, \(\mu_x(z)\) has no multiple roots and all Jordan blocks of \(\rho(x)\) have size 1.

Let \(\mathbb{C}[z] = \{ \sum_{n=0}^{\infty} \alpha_n z^n \}\) be the ring of formal power series. If \(V = \bigoplus_{n=0}^{\infty} V_n\) is a graded vector space with \(\dim V_n < \infty\), the "right" notion of dimension of \(V\) is the Poincaré polynomial

\[ p_V(t) = \sum_{n=0}^{\infty} \dim V_n \cdot t^n \]

**Example:**

- \(p_{\mathbb{C}[z]}(t) = 1 + t^d + t^{2d} + t^{3d} + \ldots = \frac{1}{1-t^d}\) where \(\deg z = d\).
- \(p_{V \oplus W}(t) = p_V(t) + p_W(t)\)
- \(p_{V \otimes W}(t) = p_V(t)p_W(t)\)
- \(p_{\mathbb{C}[z_1, \ldots, z_n]} = \prod_j p_{\mathbb{C}[z_j]}(t) = \frac{1}{1-t^{d_1}} \cdots \frac{1}{1-t^{d_n}}\) where \(\deg z_i = d_i\), \(\mathbb{C}[z_1, \ldots, z_n] = \mathbb{C}[z_1] \otimes \ldots \otimes \mathbb{C}[z_n]\).

If \(d_1 = \ldots = d_n = 1\) then \(p_{\mathbb{C}[z_1, \ldots, z_n]}(t) = \frac{1}{(1-t)^n}\).

**Example:** If \(G\) acts on \(V\), then \(\forall x \in G\) let

\[ \chi_x(t) = \det(I_V - t\rho(x))^{-1} = \prod_j \frac{1}{1-\lambda_j t} = \prod_j (1 + \lambda_j t + \lambda_j^2 t^2 + \ldots) = \sum_{j=0}^{\infty} t^j \left( \sum_{a_1 + \ldots + a_n = j} \lambda_1^{a_1} \cdots \lambda_n^{a_n} \right) \]

where \(\lambda_1, \ldots, \lambda_n\) are the eigenvalues of \(\rho(x)\).

**Theorem 10.2.** Let \((G, V)\) be a finite reflection group, \(d_1, \ldots, d_n\) the fundamental degrees. Then

\[ \frac{1}{|G|} \sum_{x \in G} \chi_x(t) = \prod_{j=1}^{\infty} \frac{1}{1-t^{d_j}} \in \mathbb{C}[[t]] \]
Proof. By 9.7 and the Example before, the RHS is \( p_{SG}(t) \).

\( x|_{S_1} \) belongs to \( \begin{pmatrix} \lambda_1 & 0 \\ . & . \\ 0 & \lambda_n \end{pmatrix} \) in some basis \( e_1, \ldots, e_n \) of \( S_1 \). Then \( e_1^{a_1} \cdots e_n^{a_n} \) with \( a_1 + \ldots + a_n = j \) form a basis of \( S_j \) and

\[
x e_1^{a_1} \cdots e_n^{a_n} = \lambda_1^{a_1} \cdots \lambda_n^{a_n} e_1^{a_1} \cdots e_n^{a_n}
\]

hence

\[
\text{tr}(x|_{S_j}) = \sum_{a_1 + \ldots + a_n = j} \lambda_1^{a_1} \cdots \lambda_n^{a_n}
\]

Hence

\[
\text{LHS} = \frac{1}{|G|} \sum_{x \in G, j=0}^{\infty} \text{tr}(x|_{S_j}) t^j
\]

Remember \( \pi_j = \int_G : S_j \rightarrow S_j^G \), \( \pi_j(v) = \frac{1}{|G|} \sum_{g \in G} gv \). In particular \( \text{LHS} = \sum_{j=0}^{\infty} \text{tr}(\pi_j) t^j \).

\[
\pi_j^2 = \pi_j \implies \pi_j \sim \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \text{tr} \pi_j = \text{rk} \pi_j = \dim S_j^G
\]

Hence \( \text{LHS} = p_{SG}(t) \). □

Corollary 10.3. \( |G| = d_1 \cdots d_n \).

Corollary 10.4. \( d_1 + \ldots + d_n = n + \frac{1}{2}\text{number of reflections} \).

Proof. Multiply 10.2 by \((1-t)^n\):

\[
\frac{1}{|G|} \sum_{x \in G} \chi_x(t)(1-t)^n = \prod_{j=1}^{n} \frac{1 - t}{1 - t^{d_j}}
\]

\[
\frac{1}{|G|} \sum_{x \in G} \frac{(1-t)^n}{\det(1-tg)} = \prod_{j=1}^{n} \frac{1}{1 + t + t^2 + \ldots + t^{d_j-1}} = \alpha(t)
\]

\[
= \frac{1}{|G|} \left( \sum_{g \in G} \frac{1}{1 + \frac{2|\Phi|}{1+t}} + \frac{h(t) \cdot (1-t)^2}{(1+t+\ldots+t^{d_j-1})^2} \right)
\]

\([h(1) \text{ is defined}]

Plug in \( t = 1 \), then \( \frac{1}{|G|} = \prod_{j=1}^{n} \frac{1}{d_j} \implies |G| = d_1 \cdots d_n
\]

Apply \( \frac{\partial}{\partial t} \):

\[
\frac{|\Phi|}{2|G|} \frac{-2}{(1+t)^2} + \tilde{h}(t)(1-t) = \sum_{j=1}^{n} \alpha(t) (1 + t + \ldots + t^{d_j-1}) \left( \frac{-1 + 2t + 3t^2 + \ldots + (d_j-1)t^{d_j-2}}{(1 + t + \ldots + t^{d_j-1})^2} \right)
\]

55
Plug in $t = 1$:

$$-\frac{|\Phi|}{4|G|} = \sum_{j=1}^{n} \frac{d_j}{d_1 \cdots d_n} \left( -\frac{(d_j - 1)d_j}{2d_j^2} \right)$$

As $|G| = d_1 \cdots d_n$

$$-\frac{|\Phi|}{4} = \sum_{j=1}^{n} \frac{d_j - 1}{2} = \frac{1}{2} \left( \sum_{j=1}^{n} d_j - n \right)$$

Example:

- $I_2(m)$ has fundamental degrees $2, m$.
  
  $$|G| = 2m, \ |\Phi| = 2((m + 2) - 2) = 2m$$

- $A_n$ has fundamental degrees $2, 3, \ldots, n + 1$
  
  $$|G| = (n+1)!, \ |\Phi| = 2((2+3+\ldots+n+1) - n) = 2 \left( \frac{n(n+3)}{2} - n \right) = n^2 + 3n - 2n = n^2 + n$$

56
11 Coxeter Elements

Let \((G, V)\) be a finite reflection group, \(\Phi \subset V\) a root system.
A Coxeter element is \(w = S_{\alpha_1} \cdots S_{\alpha_n}\) for some simple system \(\Pi = \{\alpha_1, \ldots, \alpha_n\}\).

There are two choices involved:

- simple system
- order on \(\Pi\)

In \(S_n\) the choices will lead to

\[
S_{\alpha_1} = (a_1 a_2), \quad S_{\alpha_2} = (a_2 a_3) \cdots S_{\alpha_{n-1}} = (a_{n-1} a_n)
\]

where \(\{a_1, \ldots, a_n\} = \{1, \ldots, n\}\). Hence \(w = (a_1 \ldots a_n)\). In \(S_n\), Coxeter elements are cycles of order \(n\).

**Lemma 11.1.** Let \(G\) a group, \(X\) a finite forest (graph without cycles). Suppose \(f : \text{Ver}(X) \rightarrow G\) is a function s.t. if two vertices \(a, b \in \text{Ver}(X)\) are not connected, then

\[
f(a)f(b) = f(b)f(a)
\]

Then for any choice of a total order on the vertices of \(X\), say \(\text{Ver}(X) = \{a_1, \ldots, a_n\}\), the conjugacy class \(Cl_G(g)\) of \(g = f(a_1) \cdots f(a_n)\) does not depend on the order chosen.

**Proof.** Induction on \(n = |\text{Ver}(X)|\).

\(n=1\): \(g = f(a_1)\), nothing to prove

\(n<k\) proved

\(n=k\) Easy case: \(X\) has no edges, then all \(f(a)f(b) = f(b)f(a)\), so \(g = f(a_1) \cdots f(a_n)\) is independent of the choice of the order.

If \(X\) has an edge, there is a vertex \(b\) connected with exactly one other vertex \(a\). For each function \(\psi : \{1, \ldots, n\} \rightarrow \text{Ver}(X)\), let \(g_{\psi} = f(\psi(1)) \cdots f(\psi(n))\).

Pick a particular \(\pi : \{1, \ldots, n\} \rightarrow \text{Ver}(X)\) s.t. \(\pi(n) = b, \pi(n-1) = a\). It suffices to find a \(s \in G\) s.t. \(g_\pi = s g_{\psi} s^{-1}\).

**Case 1** \(\psi(n) = n, \psi(n-1) = a\).

Let \(Y = X \setminus \{b\}\) and \(f' : Y \rightarrow G\) be \(f'(c) = \begin{cases} f(a)f(b) & c = a \\ f(c) & c \neq a \end{cases}\). Then

\[
g_{\psi} = f(\psi(1)) \cdots f(\psi(n-2)) f(a) f(b) = f'(\psi(1)) \cdots f'(\psi(n-1))
\]

By induction assumption it is conjugate to \(f'(\pi(1)) \cdots f'(\pi(n-1)) = g_\pi\).
Case 2 \( \psi(n) = b, \psi(j) = a, \ j < n - 1. \)
Let \( X_0 = \{ \psi(k) : k < j \}, \ X_1 = \{ \psi(k) : j < k < n \} \). Then define
\[
\psi_i : \psi^{-1}(X_i) \rightarrow X_i, \ \psi_i(c) = \psi(c)
\]
Then
\[
g_\psi = g_{\psi_0}f(a)g_{\psi_1}f(b) = g_{\psi_0}f(a)f(b)g_{\psi_1}^{-1} = g_{\psi_1}g_{\psi_0}f(a)f(b)
\]
This is conjugate to \( g_\pi \) by Case 1.

Case 3 \( \psi(j) = b, \ j < n, \ X_0, X_1, \psi_0, \psi_1 \) as above. Then
\[
g_\psi = g_{\psi_0}f(b)g_{\psi_1} \sim g_{\psi_1}g_{\psi_0}f(b)
\]
done by Case 2.

\[\square\]

**Proposition 11.2.** Coxeter elements form a conjugancy class in \( G \).

**Proof.** Let \( C = \{ \text{Coxeter elements} \} \). Then
\[
xS_{\alpha_1} \cdots S_{\alpha_n}x^{-1} = (xS_{\alpha_1}x^{-1}) \cdots (xS_{\alpha_n}x^{-1}) = S_{x(\alpha_1)} \cdots S_{x(\alpha_n)} \in C
\]
Hence \( xCx^{-1} = C \), so \( C \) is the union of some conjugacy classes. Let \( w_1 = S_{\alpha_1} \cdots S_{\alpha_n}, \ w_2 = S_{\beta_1} \cdots S_{\beta_n} \in C \). Then there is a \( g \in G \) s.t. \( \{ g(\alpha_1), \ldots, g(\alpha_n) \} = \{ \beta_1, \ldots, \beta_n \} \). Hence \( gw_1g^{-1} = S_{g(\alpha_1)} \cdots S_{g(\alpha_n)} \) is conjugate to \( w_2 \) by Lemma 11.1. \( \square \)

Let \( W \) be a finite Coxeter group, \( w = \pi_1 \cdots \pi_n \) a Coxeter element, \( \{ a_1, \ldots, a_n \} \) the vertices of the Coxeter graph. Then \( h = |w| \) is called the Coxeter number of \( W \).
Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( w|_{V(W)} \). Since \( w^h = 1, \ \lambda_k^h = 1 \implies \lambda_k = e^{\frac{2\pi i m_k}{h}} \) for a unique \( 0 \leq m_k < h \).
We call \( m_1, \ldots, m_n \) the exponents of \( W \).

Finite Coxeter graphs are bipartite \( (X = X_1 \cup X_2) \) **Example:**

- \( D_6 \):
• $E_8$:

For $a, b \in X_i \implies \pi b = \overline{b\pi}$. Hence $\omega_j = \prod_{a \in X_j} \pi$ does not depend on the order of $X_j$. Moreover, $\omega_j^2 = \prod \pi^2 = 1$.

$\omega = \omega_1 \omega_2$ is called canonical Coxeter element.

$G = \langle \omega_1, \omega_2 \rangle$ is a dihedral group of order $2h$.

**Proposition 11.3.** Let us order the exponents $0 \leq m_1 \leq \ldots \leq m_n < h$. Then

1. $m_1 \geq 1$
2. $\forall j: h - m_j$ is an exponent
3. $\sum_{j=1}^{n} m_j = \frac{h}{2}$

**Proof.**

1. if $m_1 = 0$ then $e^0 = 1$ is a eigenvalue of $\omega$. Hence $\exists x \in V(W)$ s.t. $x \neq 0, \omega x = x$.

\[
\omega_2 x = \omega_1 \omega_1 \omega_2 x = \omega_1 \omega_1 x = \omega_1 x
\]

Let $a \in X_j: S_a(c) = \overline{a} \cdot x = x - 2 \langle e_a, x \rangle e_a$. Hence

\[
\omega_1 x = x + \sum_{a \in X_1} \alpha_a e_a = \omega_2 x = x + \sum_{b \in X_2} \alpha_b e_b
\]

\[
\implies \sum_{a \in X_1} \alpha_a e_a = \sum_{b \in X_2} \alpha_b e_b
\]

and therefore all $\alpha_a, \alpha_b = 0$. So $\omega_1 x = \omega_2 x = 0$. Consider

\[
\omega_1 |_{V(W)}^{a_j \in X_1} = S_{a_1} \cdots S_{a_k}:
\begin{align*}
V^1_1 \ni y & \mapsto y \\
V_1 \ni y & \mapsto -y
\end{align*}
\]

where $V_i = \bigoplus_{a \in X_j} \mathbb{R} e_a$.

Hence $x \in V^1_1 \cap V^1_2$ and $V^1_1 \cap V^1_2 = 0$ because $V = V_1 \oplus V_2 = V^1_1 \oplus V^1_2$ since the form is non-degenerate. But this is a contradiction ($x = 0$) proving that $m_1 > 0$. 

59
2. $G$ acts on $V(W)$. Let $f(z)$ be the characteristic polynomial of $\omega|_{V(W)}$. Then

$$f(z) = f_1(z) \cdots f_s(z)$$

where $f_j(z) \in \mathbb{R}[z]$ and monic irreducible in $\mathbb{R}[z]$.

**Case 1** $f_j(z) = z - \lambda$, $\lambda \in \mathbb{R}$.

Since $\omega^h = 1 \implies \lambda^h = 1 \implies \lambda = \pm 1$. Note that 1 is not an eigenvalue of $\omega$ by part (1), so $\lambda = -1$, $h$ is even and $\lambda = e^{\frac{2\pi i}{h}}$, $\frac{h}{2}$ is the exponent.

**Case 2** $f_j(z) = (z - \lambda)(z - \lambda^*)$, $\lambda \notin \mathbb{R}$.

Then if $\lambda = e^{\frac{2\pi i}{h}m}$ then $\lambda^* = e^{-\frac{2\pi i}{h}m} = e^{\frac{2\pi i}{h}(h - m)}$.

3. This also gives part (3) because the average of exponents in each $f_j(z)$ is $\frac{h}{2}$.

\[\square\]

**Lemma 11.4.** Let $W$ be finite and connected. Then there are $z_1, z_2 \in V$ s.t.

1. $z_1, z_2$ are linearly independent
2. $P = \mathbb{R}z_1 \oplus \mathbb{R}z_2$ is $G = D_{2h}$-linear
3. $\omega|_P = S_{z_1}^\perp$
4. $z_i \in \overline{C}$, the closure of the fundamental chamber
5. $P \cap C = \mathbb{R}_{>0}z_1 + \mathbb{R}_{>0}z_2$

**Proof.** $\omega_a$ is the dual basis to $e_a$. That means $\langle \omega_a, e_a \rangle = \delta_{ij}$.

Easy claims:

- $\overline{C} = \langle x : \langle x, e_a \rangle \geq 0 \rangle = \sum_a \mathbb{R}_{\geq 0} \omega_a$
- $V_j^\perp = \bigoplus_{a \notin X_j} \mathbb{R} \omega_a$
- angle between $\omega_a$ and $\omega_b = \pi$ -- angle between $e_a$ and $e_b$
- The angles between $\omega_a$ and $\omega_b$ are acute and $\langle \omega_a, \omega_b \rangle \geq 0$ and $\langle \omega_a, \omega_b \rangle = 0 \iff \langle e_a, e_b \rangle = 0 \iff \omega_a$ and $\omega_b$ commute.

$Q = (\langle e_a, e_b \rangle)_{a,b}$ is the matrix of

- symmetric positive definite bilinear form $\langle , \rangle$ in the basis $e_a$,
- linear map $q : V \rightarrow V, q(\omega_a) = e_a$ in the basis $\omega_a$.

In particular, its eigenvalues are positive and real. Let $\lambda$ be the largest eigenvalue of $Q$. Consider the new form $[ , ]$ given by $Q - \lambda I$ in the basis $e_a$.

Note: $[x, x] \leq 0$ and $[x, x] = 0 \iff x \in \ker[ , ] = K$. $K \neq 0$, $K$ is eigenspace of $q$.

Pick $z \neq 0$, $z = \sum_a \alpha_a \omega_a \in K$ and let $\tilde{z} = \sum |\alpha_a| \omega_a$.

Now, the idea is to use that $z_j = \sum_{a \in X_j} |\alpha_a| \omega_a$, but we have no time to finish the proof. \[\square\]
Theorem 11.5. If $W$ is finite connected of rank $n$ (i.e. the number of vertices in the Coxeter graph is $n$), then

1. $m_1 = 1$, $m_n = h - 1$
2. $|\Phi| = h n$

Proof. (Idea)
$\alpha \in \Phi \longrightarrow H_\alpha \cap P$ is $G$-conjugate to $Rz_1$ or $Rz_2$.

- if $h$ is odd:

There are $\frac{n}{2}$ hyperplanes $H_\alpha$ s.t. $H_\alpha \cap P$ is one of these $h$ lines

- if $h$ is even:

There are $\frac{h}{2}$ lines with one intersection pattern and $\frac{h}{2}$ lines with another intersection pattern

\[ \square \]

Theorem 11.6. Let $W$ be finite and connected of rank $n$, $1 \leq m_1 \leq \ldots \leq m_n = h - 1$ the fundamental exponents, $2 \leq d_1 \leq \ldots \leq d_n$ the fundamental degrees. Then:

\[ \forall j : d_j = m_j + 1 \]

Proof. (idea)
Compute $Jac = \det \left( \frac{\partial f_i}{\partial z_k} \right)$ in two different ways.

\[ \square \]

Corollary 11.7. $|W| = \prod_j (m_j + 1)$, $|\Phi| = 2 \sum_j m_j$. 61
Corollary 11.8. Let \( W \) be connected.
\[
-1 = \begin{pmatrix}
  -1 \\
  . \\
  . \\
  1
\end{pmatrix} \in \rho(W) \iff \text{all } m_i \text{ are odd}
\]

Proof.
"\( \Rightarrow \)" \((-1) \circ I_j = I_j \in S^W \). Therefore \((-1) \cdot x_k = x_k \Rightarrow (-1)I_j = (-1)^d_j I_j \Rightarrow (-1)^d_j = 1.
So all \( d_j \) are even and therefore all \( m_j \) are odd.

"\( \Leftarrow \)" \( m_1 = 1, m_n = h - 1 \) are both odd \( \Rightarrow \) \( h \) is even
\[
(\omega|_{V(W)})^h \sim \begin{pmatrix}
  e^{2\pi i m_1} \\
  . \\
  . \\
  e^{2\pi i m_n}
\end{pmatrix} = \begin{pmatrix}
  e^{\pi i m_1} \\
  . \\
  . \\
  e^{\pi i m_n}
\end{pmatrix} = \begin{pmatrix}
  -1 \\
  . \\
  . \\
  -1
\end{pmatrix}
\]

\( \Box \)

Proposition 11.9. Let \( W \) be finite, connected, crystallographic. Then \( \forall \ m \ s.t. \ 1 \leq m \leq h - 1, \ \gcd(m,h) = 1 \ then \ m \ is \ an \ exponent. \)

Proof. Cyclotomic polynomial
\[
\Phi_h(z) = \prod_{1 \leq m \leq h-1 \atop \gcd(m,h)=1} (z - e^{\frac{2\pi i}{h} m})
\]
is irreducible in \( \mathbb{Z}[z] \).
The characteristic polynomial \( f_\omega(z) \) of \( \omega|_{V(W)} \) is in \( \mathbb{Z}[z] \).
For \( m = 1 \): \( (z - e^{\frac{2\pi i}{h}})|f_\omega(z) \) in \( \mathbb{C}[z] \). Therefore
\[
\Phi_h(z)|f_\omega(z) \text{ in } \mathbb{Z}[z]
\]
and therefore all \( e^{\frac{2\pi i}{h} m} \) are eigenvalues.

Application: \( E_8 \) has rank 8 and 240 roots. Hence \( h = \frac{\left|\Phi\right|}{n} = \frac{240}{8} = 30 \). The number of coprimes of 30 is
\[
\varphi(30) = \varphi(2)\varphi(5)\varphi(3) = 1 \cdot 4 \cdot 2 = 8
\]
Hence the exponents are 1, 7, 11, 13, 17, 19, 23, 29 and so
\[
\left|W\right| = 2 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24 \cdot 30
\]

|   | \( \left|\Phi\right| \) | \( h = \frac{\left|\Phi\right|}{n} \) | "easy exponents" | missing exponents | \( \left|W\right| \) |
|---|---|---|---|---|---|
| \( E_6 \) | 72 | 12 | 1, 5, 7, 11 | 4, 8 |   |
| \( E_7 \) | 126 | 18 | 1, 5, 7, 11, 13, 17 | 9 |   |
| \( E_8 \) | 240 | 30 | 1, 7, 11, 13, 17, 19, 23, 29 | none |   |
| \( F_4 \) | 48 | 12 | 1, 5, 7, 11 | none | \( 2 \cdot 6 \cdot 8 \cdot 12 = 2^7 \cdot 3^2 \) |
| \( H_3 \) | 30 | 10 | 1, 9 | 5 | \( 2 \cdot 6 \cdot 10 = 120 \) |
| \( H_4 \) | 120 | 30 | 1, 29 | 11, 19 | \( 2 \cdot 12 \cdot 20 \cdot 30 = (120)^2 \) |
Calculations:
Since exponents come in pairs \((m, h-m)\) the only stand alone exponent is \(\frac{h}{2}\). This gives missing \(E_7\) and \(H_3\) components.

\(E_6\): two exponents are missing. We know that \(z^{12} - 1\) is satisfied by Coxeter transformations.

\[
z^{12} - 1 = \phi_1\phi_6\phi_4\phi_3\phi_2\phi_1
\]

if \(\lambda = e^{\frac{2\pi m}{12}}\) where \(m\) is a missing component, there are 4 possible cases:

- \(\phi_2(\lambda) = 0 \rightarrow 6,6\)
- \(\phi_4(\lambda) = 0 \rightarrow 3,9\)
- \(\phi_3(\lambda) = 0 \rightarrow 4,8\)
- \(\phi_6(\lambda) = 0 \rightarrow 2,10\)

One needs to write the Coxeter transformations itself to see that \(\phi_3(\lambda) = 0\).

\(H_4\): can be realised by matrices with coefficients in \(\mathbb{Q}(\sqrt{5})\), \(z^{30} - 1 = \phi_{30} \cdots\). Over \(\mathbb{Q}(\sqrt{5})\), \(\phi_{30} = \phi_{30}^{(1)}\phi_{30}^{(0)}\), the product of 2 irreducibles of degree 4. Exponents correspond to the factor \(\phi_{30}^{(0)}\) and we know that \(\phi_{30}^{(0)}(e^{\frac{2\pi i}{30}}) = 0\). The roots of \(\phi_{30}^{(0)}\) are:

\(e^{\frac{2\pi i}{30}}, e^{\frac{2\pi i}{30} - 29}, e^{\frac{2\pi i}{30} + 11}, e^{\frac{2\pi i}{30} + 19}\)

\(H_3\): \(H_3 = \text{Sym(Icosahedron)} = \text{Sym(Dodecahedron)}\).

Every reflection of the Dodecahedron fixes two opposite edges. Hence

\[|\Phi| = 2|\text{reflections}| = 2 \frac{|\text{edges}|}{2} = 30\]

and

\[|W| = \frac{20}{|\text{vertices}|} \cdot 6 = 120\]

63
\( F_4, H_4: \) \( W(F_4) = \text{Sym}(24 - \text{cells}), \ W(H_4) = \text{Sym}(120 - \text{cells}) = \text{Sym}(600 - \text{cells}). \) We want to use a trick here:

Let \( x, y \in \mathbb{H} \), then \( S_x(y) = -x\bar{y}x. \) Hence if \( G \leq \mathbb{H}^* \), \( |G| < \infty \), \( |G| \) even \( \implies G \) is a root system. Let \( \{q : \|q\| = 1\} = U \subset \mathbb{H}^* \) and consider

\[
\psi : U \xrightarrow{\text{"SU}_2(\mathbb{C})"} SO(3)(\mathbb{R}) = SO(\mathbb{H}_{im}) : \ q \mapsto (x \mapsto q\bar{x}q)
\]

which is 2:1. Then \( \Phi_{H_4} = \psi^{-1}(\text{Rot Symm}(\text{Dodec})) \implies |\Phi_{H_4}| = 2 \cdot 60 = 120. \)
\( \Phi_{F_4} = \psi^{-1}(\text{Rot Symm}(\text{Cube})) \implies |\Phi_{F_4}| = 2 \cdot 24 = 48. \)

\( E_8, E_7: \) There exists a 8-dimensional algebra \( \Phi \) (octonions). If \( G \leq \Phi^* \), \( |G| < \infty \), \( |G| \) is even \( \implies G \) is a root system.

\begin{itemize}
  \item \( \Phi_{E_8} \) is the "group" of invertible octavian integers
  \item \( \Phi_{E_7} \) are the elements of order 4 in it.
\end{itemize}