MA251 - Algebra I

Assignment 3 **November 2009**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your university ID number in the top right-hand corner. Use the problems at the beginning as well as exercises in the lecture notes for a warm up. Solutions to the **FOUR TEST** *problems must be handed in by* **15.00** *on* **MONDAY 16 NOVEMBER** *(Monday of the seventh week of term), or they will not be marked. There will be an award of 5 extra marks for clarity, so do a good job.*

These are practice problems for you to sharpen your teeth on.

P1. Calculate the rank and signature of the quadratic forms corresponding to the matrices

$$\text{(i)} \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1 \\ 1/2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

P2. Call two quadratic forms on the n-dimensional vector space V over field K *equivalent* if one can be obtained from the other by a change of coordinates. How many equivalence classes are there when (i) $K = \mathbb{C}$ and $n = 4$; and (ii) $K = \mathbb{R}$ and $n = 3$.

P3. What is the answer to parts (i) and (ii) of Question P2 for general n?

P4. Classify the following curves and surfaces (ellipse, parabola, etc.):
(i) $x^2 - y^2 + 2xy - 1 = 0$ (2 dimensions);
(ii) $x^2 - y^2 + 2xy - 1 = 0$ (3 dimensions);
(iii) $x^2 + 2xy + y^2 + x + 1 = 0$ (2 dimensions);
(iv) $x^2 + y^2 - 2z^2 - x - y - 4z = 0$ (3 dimensions);
(v) $x^2 + y^2 - z^2 + 6x - 4y - 2z + 12 = 0$ (3 dimensions);
(vi) $x^2 + y^2 - z^2 + 2xy - 2xz - 2yz - y = 0$ (3 dimensions).

P5. (i) Show that any 2×2 real orthogonal matrix is equal to

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

for some θ. (Hence the matrix represents a rotation about the origin or a reflection about a line through the origin in the 2-dimensional plane.)

(ii) Show that a 3×3 real orthogonal matrix A represents either a rotation about a line through the origin, or a reflection about a plane through the origin followed by a rotation (*Hint*: First show that A has an eigenvector $\mathbf{v}$, and change basis to include $\mathbf{v}$.)

P6. Let A be any invertible $n \times n$ matrix over $\mathbb{R}$.
(i) Show that AA^{T} is symmetric and positive definite.
(ii) Show that there is a symmetric positive definite matrix S with $S^2 = AA^{\mathrm{T}}$.
(iii) Show that there is a symmetric positive definite matrix S and an orthogonal matrix R such that $A = SR$. (*Hint*: same S as in (ii).)

P7. Prove that the eigenvalues of a complex Hermitian matrix are all real.

P8. Find a 2×2 complex matrix which is both Hermitian and unitary and whose entries are not all real numbers.

The following problems are test problems for you to submit for marking. Write concise but complete solutions only to the questions asked. Additional 5 marks are awarded for clarity.

1. Prove that the 2-dimensional quadratic from $q(x,y) = \alpha x^2 + \beta xy + \gamma y^2$ is positive definite if and only if $\alpha > 0$ and $4\alpha\gamma - \beta^2 > 0$. [2 marks]

2. Find orthogonal matrices P such that $P^{\mathrm{T}}AP$ is diagonal for

$$\text{(i) } A = \begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix} \qquad \text{(ii) } \begin{pmatrix} 4 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 3 \end{pmatrix}.$$

[2,3 marks]

3. For the following Euclidean vector spaces (V, ω) and a basis $f_1, \ldots f_n$, run Gram-Schmidt orthonormalisation process to arrive at an orthonormal basis.

(i) $V = \mathbb{R}^3$ with the the standard dot-product $\omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T\mathbf{w}$; $f_1 = (1,0,0)^T$, $f_2 = (1,1,1)^T$ and $f_3 = (1,-1,1)^T$. [2 marks]

(ii) $V = \mathbb{R}[X]_{\leq 4}$ is the space of real polynomials of degree at most 4,

$$\omega(f(X), g(X)) = \int_{-1}^{1} f(X)g(X)dX \ ;$$

$f_i = X^{i-1}$, $i = 1,2,3,4,5$. [2 marks]

(iii) $V = \mathbb{R}[X]_{\leq 4}$ is the space of real polynomials of degree at most 4,

$$\omega(f(X), g(X)) = \int_{-\infty}^{+\infty} e^{-X^2} f(X)g(X)dX \ ;$$

$f_i = X^{i-1}$, $i = 1,2,3,4,5$. (*Hint:* You will need the Gaussian integral $\int_{-\infty}^{+\infty} X^{2k}e^{-X^2}dX = 2^{-k}(2k-1)!!\sqrt{\pi}$ where $(2k-1)!!$ is the double factorial defined by $n!! = n \cdot (n-2)!!$ and $(-1)!! = 1$.) [2 marks]

4. Let $V, <, >$ be an Euclidean space. The Gram matrix of vectors $\mathbf{v}_1, \ldots \mathbf{v}_k \in V$ is

$$G(\mathbf{v}_1, \ldots \mathbf{v}_k) = \begin{pmatrix} <\mathbf{v}_1, \mathbf{v}_1> & <\mathbf{v}_1, \mathbf{v}_2> & \cdots & <\mathbf{v}_1, \mathbf{v}_k> \\ \vdots & \vdots & & \vdots \\ <\mathbf{v}_k, \mathbf{v}_1> & <\mathbf{v}_k, \mathbf{v}_2> & \cdots & <\mathbf{v}_k, \mathbf{v}_k> \end{pmatrix}$$

(i) Prove that if $\mathbf{v}_1, \ldots \mathbf{v}_k$ are linearly dependent then $\det(G(\mathbf{v}_1, \ldots \mathbf{v}_k)) = 0$. [1 mark]

(ii) Prove that if $\mathbf{v}_1, \ldots \mathbf{v}_k$ are linearly independent then $\det(G(\mathbf{v}_1, \ldots \mathbf{v}_k)) > 0$. (*Hint:* Gram matrix is the matrix of $<, >$ restricted to the span of $\mathbf{v}_i$-s.) [2 marks]

(iii) The Gram inequality $\det(G(\mathbf{v}_1, \ldots \mathbf{v}_k)) \geq 0$, which you have just proved, has a very important special case of $k = 2$. In this special case, it is called the Schwarz inequality. Write the Schwarz inequality explicitly (by using a formula for a 2×2-determinant) and explain how the Schwarz inequality can be used to define an angle between two vectors $x, y \in V$. [1 mark]

(iv) We define a Euclidean distance on V by $d(x,y) = \sqrt{<x-y, x-y>}$ Using the Schwarz inequality, prove the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$. [1 mark]

(v) Prove that a linear map $T : V \to V$ is orthogonal if and only if it is distance preserving, i.e. $d(a,b) = d(T(a), T(b))$ for all $a, b \in V$. [2 marks]