PHD PROPOSAL: MCKAY CORRESPONDENCE FOR DERIVED CATEGORIES

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ABSTRACT. The McKay correspondence relates the representation theory of a finite subgroup $G$ of $SL(2, \mathbb{R})$ and the homology or cohomology of a crepant resolution $Y \to \mathbb{C}^2/G$. A generalisation of this replaces $\mathbb{C}^2$ by an algebraic manifold $M$ of dimension $n$ and $G$ by a finite group of automorphisms of $M$. My proposed PhD project will study this generalisation in the context of derived categories of coherent sheaves, with the aim of attacking some important problems and conjectures raised by Reid.

INTRODUCTION AND MOTIVATION

The classical McKay correspondence is the correspondence between the representation theory of a finite subgroup $G \leq SL(2, \mathbb{C})$ and the cohomology of a minimal resolution of $\mathbb{C}^2/G$. We can generalise this by replacing $\mathbb{C}^2$ by a smooth quasi-projective variety (algebraic manifold) $M$, of dimension $n$ and $G$ by a finite subgroup of the group of automorphisms of $M$. The quotient $M/G$, which we shall denote by $X$, is quasi-projective and singular. To resolve the singularity, we consider a crepant resolution

$$Y \to X,$$

that is, a resolution which does not change the canonical class of $X$. In general, these resolutions are hard to come by, especially in higher dimensions. Miles Reid, my proposed advisor, founded the theory of the derived Mckay Correspondence. It is the relation between the geometry of $Y$ and the $G$-equivariant geometry of $M$. I would like to phrase this relation in terms of the derived categories of coherent sheaves on $X$ and $Y$, so I would like to consider the relation between $D^b(Y)$ and $D^b_G(M)$, the derived category of the abelian category of $G$-equivariant sheaves on $M$. The central conjecture in this area is the following.

**Conjecture 1** (Reid). Suppose $Y \to M/G$ is a crepant resolution. Then

$$D^b_G(M) \simeq D^b(Y).$$

This is unproved as far as I know. However, a very important paper in this area is [BKR01], where this conjecture is shown to be true for some special cases. Namely, suppose that $Y$ is an object called a $G$-Hilbert scheme. If the dimension of the fibre product $Y \times_X Y$ is at most $n$, then $Y$ is crepant.
and we get the desired equivalence. This equivalence is given by a very remarkable functor called the Fourier-Mukai transform. These are defined by Mukai in [Muk84] and are studied extensively by Bridgeland [Bri98], Bondal and Orlov [BO95], Maciocia [Mac96] and many others. I have done a final year dissertation on Fourier-Mukai transforms under the supervision of Maciocia, who was the teacher of Bridgeland (now FRS) and who brought together the three authors of the prestigious paper [BKR01].

The quotient $X$ may in fact have several non-isomorphic crepant resolutions, say $Y, Y'$ are two of those. If the conjecture is true, then $Y$ and $Y'$ are D-equivalent. It is a well-known fact that crepant resolutions of the same variety are $K$-equivalent, for example, see [Huy06]. This fits well with the conjecture that D-equivalence follows from $K$-equivalence for varieties. This theory connects to various parts of mathematics, most notably to mathematical physics via the homological mirror symmetry.

Objectives and project outline

My PhD project will consider this conjecture due to Reid and I will try to, at the very least, to resolve some further special cases of it. A good candidate for a crepant resolution of $M$ seems to be the $G$-Hilbert scheme $G$-Hilb $M$, which is a mysterious object in itself. For example, it is not known whether it is irreducible or connected in general, but it is for some special cases, as shown by Nakamura in [Nak01]. This already leads to a number of interesting research problems.

**Problem 1.** Let $A$ be a $G$-Hilbert scheme. What are the requirements for $A$ to be irreducible? Can these requirements be made minimal in some appropriate sense, in order to include as many cases of $G$-Hilbert schemes as possible?

Even if we do not know whether $A$ is irreducible, it should be possible to proceed by picking an irreducible component $Y$ of $G$-Hilb $M$, having certain properties. This leads to further worthwhile problems.

**Problem 2.** Is $Y \rightarrow X$ a resolution? That is, is $Y$ smooth? When is $Y$ crepant?

Unfortunately, $G$-Hilb $M$ seems to be crepant in a very small number of cases. So, we can consider another problem.

**Problem 3.** Change the functor $G$-Hilb $M$ to get other cases of crepant resolution.

A very important result which relates to the conjecture is Orlov's theorem, which says that any equivalence of derived categories of two smooth projective varieties is a Fourier-Mukai transform. This result is highly nontrivial and is proved in [Orl03]. A big advantage of considering $G$-Hilbert schemes is that there is a good candidate for the Fourier-Mukai kernel. For example,
this is used to prove the main claim of [BKR01]. The main problem is the following.

**Problem 4.** Is there a desired derived equivalence as outlined in Conjecture 1?

If a crepant resolution exists, a possible angle of attacking the problem is to exhibit the resolution as representing a functor, for example, as a moduli space of some kind of quiver representations and stability conditions, which gives a Fourier-Mukai kernel. After that, using the results of Bondal and Orlov, Maciocia and Bridgeland, and Reid, it will be possible to prove the conjecture by standard formal arguments.

**Conclusion**

The theory of derived McKay correspondence, and, in a broader context, the approach to algebraic geometry through derived categories, has seen quite a lot of interest in the last 20 years, which resulted in many remarkable discoveries, e.g., the notion of stability conditions on a triangulated category due to Bridgeland. The researchers in this field have also gained prestigious mathematical awards, for example Reid’s LMS Pólya prize (2014) or Bridgeland’s election to the Royal Society (2014). Warwick is an ideal place for this project because of the various activities around the proposed topic, most notably the current EPSRC Symposium on derived categories. Another great advantage of Warwick is the presence of other mathematicians who work in nearby fields, like Rumynin, or Böhning, who is due to join the faculty in 2015.

**References**


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