UNIVERSAL CHARACTERS FROM THE MACDONALD IDENTITIES

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ABSTRACT. The main result of [Kos] shows that a sequence of complexes associated with the MacDonald identities gives a sequence of universal characters which extends and generalises some representations of the exceptional series given in [Del96]. This proves the conjectures in [MP03].

In this paper we use a twisted version of this definition to define a sequence of complexes for the first row of the Freudenthal magic square.

We also calculate the dimensions of these characters for the special linear groups and for the symplectic groups. This gives two new explicit expressions for the D'Arcais polynomials.

1. INTRODUCTION

The motivation for this paper is to define a universal sequence of characters. Here universal is used in the sense of [LM] to mean that we have a sequence of characters for the universal Lie algebra introduced in [Vog99]. The first four terms of this sequence were defined in [Del96, (C)] for the exceptional series of simple complex Lie algebras. These characters were further studied in [MP03] where the authors made a number of conjectures. These papers considered for each exceptional simple Lie algebra $\mathfrak{g}$ and for sufficiently small $k \geq 1$ the k-th eigenspace of the quadratic Casimir acting on the k-th exterior power of the adjoint representation. One of the mysterious features of these papers is that there are low values of k such that this definition gives the character of this representation for groups of high rank but when this character is extended to the groups of low rank it gives a virtual character.

In this paper we adopt a different definition of these characters given in [Kos]. This paper gives, for each simple Lie algebra $\mathfrak{g}$ and each $k \geq 1$ a bounded complex of $\mathfrak{g}$-modules; and one of the main results is that if $k \leq \check{h}$ (the dual Coxeter number of $\mathfrak{g}$) then the character of this complex is the character of the k-eigenspace. This definition explains the appearance of virtual characters and all the conjectures in [MP03] are immediate consequences.

Since the tables of [CdM96] contain several virtual characters this suggests that the category which is conjectured to exist in [Del96, (G)] should have objects which are graded. The fact that the characters in

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this paper are the characters of complexes suggests that maybe this category should be a triangulated category.

Rather surprisingly the definition of these complexes arises from the proof in [GL76] of the Weyl-Kac denominator formula for the affine Lie algebra of $\mathfrak{g}$.

2. Symmetric functions

We introduce a sequence of generating functions by giving the generating function.

Definition 2.1. *Define symmetric functions $X_{r,s}$ for $r, s \geq 0$ by*

$$\sum_{r,s\geq 0} X_{r,s} q^r t^s = \prod_{k\geq 1} \sum_{r\geq 0} q^{kr} t^r e_r = \prod_{k\geq 1} \prod_{i\geq 1} (1 + q^k t x_i) \tag{1}$$

These are polynomials which are symmetric in the indeterminates $\{x_i\}$ and e_r is the r-th elementary symmetric polynomial.

These can be interpreted as the symmetric functions associated to a sequence of polynomial functors. Let V be a finite dimensional vector space and let $qV[q]$ be the graded vector space whose component in degree n is isomorphic to V for $n \geq 1$ and which is zero for $n \leq 0$. Then take the exterior algebra of this graded vector space $\bigwedge(qV[q])$ as a bigraded vector space. Define $X_{r,s}(V)$ to be the component of $\bigwedge(qV[q])$ of degree (r, s). Elements of this subspace are given by taking $v_1 \wedge \cdots \wedge v_s$ where $v_i \in t^{n_i}V$ and $n_1 + \cdots n_s = r$. The vector spaces $X_{r,s}(V)$ for $1 \leq r, s \leq 5$ are given by putting $\mathfrak{g}_k = V$ for $k \geq 1$ in the table (8).

This gives a sequence of graded symmetric functions or graded polynomial functors by taking

$$X_r = \sum_{s=0}^{r} X_{r,s}$$

The first observation is that this is a divided power sequence of symmetric functions in the sense of [Loe90]. In terms of the polynomial functors this means that, for all $r \geq 0$, and all pairs of vector spaces (U, V) there are natural isomorphisms

$$X_r(U \oplus V) \cong \oplus_{k=0}^{r} X_k(U) \otimes X_{r-k}(V) \tag{2}$$

This follows both from the definition (1) and also from the interpretation as a polynomial functor.

Now take V to be a Lie algebra $\mathfrak{g}$. Then $t\mathfrak{g}[t] \cong t\mathbb{C}[t] \otimes \mathfrak{g}$ is a graded Lie algebra with bracket given by

$$[t^r \otimes x, t^s \otimes y] = t^{r+s} \otimes [x, y]$$

Then we consider $\bigwedge(t\mathfrak{g}[t])$ as the Koszul complex of this graded Lie algebra. The differential has degree $(-1, 0)$; so for each $r \geq 0$ $X_r(\mathfrak{g})$ is a bounded complex.

3. D'Arcais polynomials

Let V be a vector space. The dimension of the graded vector space $X_r(V)$ is, by definition, $\sum_{s=0}^{r}(-1)^s \dim X_{r,s}(V)$. Let $\varphi(q)$ be the Euler product

$$\varphi(q) = \prod_{k\geq 1}(1-q^k)$$

Putting $x_i = 1$ for $1 \leq i \leq \dim(V)$ and $x_i = 0$ for $i > \dim(V)$ and $t = -1$ in the definition (2.1) we see that

$$\sum_{r\geq 0} \dim X_r(V) q^r = \varphi(q)^{\dim(V)}$$

Definition 3.1. *Define the sequence of polynomials* $\{p_r : r \geq 0\}$ *by the generating function*

$$\sum_{r\geq 0} p_r(D) q^r = \varphi(q)^D = \exp\left(-D \sum_{k\geq 1} \frac{\sigma(k)}{k} q^k\right) \tag{3}$$

Then we have $\dim X_r(V) = p_r(\dim(V))$.

This definition of these polynomials is given in [D'A13] together with a calculation of $p_r(D)$ for $r \leq 5$. These polynomials are also given in [Com74, page 159] where they are called the D'Arcais polynomials. The polynomials $p_r(D)$ for $r \leq 9$ are given in [MP03]. If we modify the definition and take $(-1)^r r! p_r(-D)$ then we get polynomials whose coefficients are non-negative integers. These polynomials are in the On-line Encyclopaedia of Integer Sequences as sequence A00829.

It follows from (2) that this is a sequence of divided powers. This means that

$$p_n(D_1 + D_2) = \sum_{r+s=n} p_r(D_1) p_s(D_2)$$

It is clear that although $p_r(D)$ is a polynomial with rational coefficients evaluating $p_r(D)$ for D a positive integer gives an integer. Polynomials with these properties are known as numerical polynomials. Other examples of numerical polynomials are the polynomials $\binom{D}{m}$ and it is known that these are an integral basis for numerical polynomials. This shows that we can write

$$p_n(D) = \sum_{m=0}^{n} c(n,m) \binom{D}{m} \tag{4}$$

where the coefficients $c(n,m)$ are integers. These coefficients are given in [Loe90, Corollary 1.3].

4. MacDonald identities

The connection with the MacDonald identities can be seen by applying the polynomial functors X_r to a simple Lie algebra $\mathfrak{g}$.

By the definition (1), the generating function for the character of $X_r(\mathfrak{g})$ is

$$\prod_{k\geq 1}\left[(1-q^k)^\ell \prod_{\alpha\in\Phi}(1-q^k e(\alpha))\right] \tag{5}$$

where ℓ is the rank of $\mathfrak{g}$ and Φ is the set of non-zero roots of $\mathfrak{g}$.

Then the MacDonald identities express this character as a linear combination of characters of simple $\mathfrak{g}$-modules; see [Mac72] and [Kac85, (12.1.4) and Exercise 12.6].

Let $\mathfrak{g}$ be a Lie algebra with an automorphism σ of order m. Let $\omega = \exp(\frac{2\pi i}{m})$ and for $j \in \mathbb{Z}$ let $\mathfrak{g}_j$ be the eigenspace of σ with eigenvalue ω^j. In particular, $\mathfrak{g}_0$ is the invariant subalgebra.

Now let $L(\mathfrak{g})$ be the graded Lie algebra $\mathbb{C}[t,t^{-1}]\otimes\mathfrak{g}$ with bracket determined by

$$[t^r\otimes x, t^s\otimes y] = t^{r+s}\otimes[x,y]$$

Then extend the action of σ on $\mathfrak{g}$ to an action on $L(\mathfrak{g})$ by

$$\sigma(t^r\otimes x) = \omega^{-r}t^r\otimes\sigma(x)$$

Now enlarge $L(\mathfrak{g})$ by taking a central extension with centre of degree zero and adjoin outer derivations also of degree zero. This gives a graded Lie algebra $\widehat{L}(\mathfrak{g})$ whose component in degree zero is $\mathfrak{g}_0\oplus\mathfrak{h}$. Extend the action of σ on $L(\mathfrak{g})$ to an action on $\widehat{L}(\mathfrak{g})$ by taking the action on $\mathfrak{h}$ to be trivial. Define $\widehat{L}(\mathfrak{g},\sigma)$ to be the invariant subspace of $\widehat{L}(\mathfrak{g})$. This is a graded Lie algebra whose component in degree zero is $\mathfrak{g}_0\oplus\mathfrak{h}$ and if $r\neq 0$ then the component in degree r is $t^r\otimes\mathfrak{g}_r$. Take $\mathfrak{u}_+$ to be sum of the components in positive degrees and $\mathfrak{u}_-$ to be the sum of the components in negative degrees; so we have a decomposition

$$\widehat{L}(\mathfrak{g},\sigma) = \mathfrak{u}_-\oplus(\mathfrak{g}_0\oplus\mathfrak{h})\oplus\mathfrak{u}_+ \tag{6}$$

The Lie algebra $\mathfrak{g}_0$ acts on the graded Lie algebra $\mathfrak{u}_-$ and so the exterior algebra $\bigwedge(\mathfrak{u}_-)$ is a bigraded representation of $\mathfrak{g}_0$. The Koszul differential makes $\bigwedge(\mathfrak{u}_-)$ into a complex of graded representations of $\mathfrak{g}_0$. The homology groups of this complex are the homology groups of $\mathfrak{u}_-$. By the Euler-Poincaré principle $\bigwedge(\mathfrak{u}_-)$ and $H(\mathfrak{u}_-)$ are equal as bigraded characters of $\mathfrak{g}_0$.

Now take $\mathfrak{g}$ to be a simple algebra and σ to be an automorphism induced by an automorphism of the Dynkin diagram of $\mathfrak{g}$. Then there is a one dimensional central extension of $L(\mathfrak{g})$ such that $\widehat{L}(\mathfrak{g},\sigma)$ is an affine Kac-Moody algebra. Furthermore the subalgebra $\mathfrak{u}_-\oplus\mathfrak{g}_0\oplus\mathfrak{z}$ in the decomposition (6) is the standard maximal parabolic subalgebra. The Levi subalgebra is $\mathfrak{g}_0\oplus\mathfrak{z}$ and the nilradical is the graded Lie algebra $\mathfrak{u}_-$. The homology groups $H_n(\mathfrak{u}_-)_k$ are computed in [GL76].

The generalisation of (1) to the twisted case is:

$$\sum_{r,s\geq 0} X_{r,s}(\mathfrak{g},\sigma)q^r t^s = \prod_{k\geq 1}\sum_{r\geq 0} q^{kr}t^r \bigwedge^r(\mathfrak{g}_k) \tag{7}$$

The following table gives these $\mathfrak{g}_0$-modules for $r, s \leq 5$. Each column corresponds to a fixed value of s and each row to a fixed value of r so that the graded module $X_r(\mathfrak{g},\sigma)$ is given by a row of this table.

(8)

	5	4	3	2	1
1					$\mathfrak{g}_1$
2				$\wedge^2(\mathfrak{g}_1)$	$\mathfrak{g}_2$
3			$\wedge^3(\mathfrak{g}_1)$	$\mathfrak{g}_1.\mathfrak{g}_2$	$\mathfrak{g}_3$
4		$\wedge^4(\mathfrak{g}_1)$	$\wedge^2(\mathfrak{g}_1).\mathfrak{g}_2$	$\wedge^2(\mathfrak{g}_2)+\mathfrak{g}_1.\mathfrak{g}_3$	$\mathfrak{g}_4$
5	$\wedge^5(\mathfrak{g}_1)$	$\wedge^3(\mathfrak{g}_1).\mathfrak{g}_2$	$\wedge^2(\mathfrak{g}_2).\mathfrak{g}_1+\wedge^2(\mathfrak{g}_1).\mathfrak{g}_3$	$\mathfrak{g}_2.\mathfrak{g}_3+\mathfrak{g}_1.\mathfrak{g}_4$	$\mathfrak{g}_5$

5. Twisted case

In this section we consider the first row of the Freudenthal magic square as a series of Lie algebras. Our interest in this row is that there are two results which apply to the other three rows but which break down for this row. The first is that the method of [LM02] for finding dimension formulae breaks down; the other is that it is noted in [Wes03] that all rows except the first have uniform R-matrices. Here we apply the construction of the previous section to twisted affine Kac-Moody algebras. This gives an sequence of complexes for the first row of the Freudenthal magic square.

Here we recall the Tits construction of the first two rows of the magic square. This construction was originally given in [Tit66] and the following discussion is based on the account in [BS03]. Let J be a Jordan algebra with identity 1 and with an inner product $\langle -,-\rangle$ which satisfies

$$\langle x, y.z\rangle = \langle x.y, z\rangle$$

for all $x, y, z \in J$. Denote the subspace orthogonal to the identity by $\mathrm{Im}(J)$. Denote the Lie algebra of derivations of J by $\mathfrak{Der}(J)$. Then there is a product on $\mathfrak{Der}(J)\oplus\mathrm{Im}(J)$ which gives a semisimple Lie algebra. For $x \in J$ let L_x be the endomorphism of J given by left multiplication by x. Then the product is given by the product on $\mathfrak{Der}(J)$, the action on $\mathrm{Im}(J)$ and the map

$$\wedge^2(J) \to \mathfrak{Der}(J) \qquad x\wedge y \mapsto [L_x, L_y]$$

By construction the linear involution σ which acts by 1 on $\mathfrak{Der}(J)$ and -1 on $\mathrm{Im}(J)$ is a Lie algebra involution. The first two rows of the magic square are obtained by taking a real division algebra $\mathbb{A}$ and then taking J to be the Jordan algebra of Hermitian 3×3 matrices with entries in $\mathbb{A}$. The Jordan product is given by the anticommutator of the matrix product, the unit is the unit of matrix multiplication and the inner product of two matrices is given by taking one half of the trace of the matrix product.

If $\mathbb{A}$ is associative then we can also construct these Lie algebras with involution by taking $\mathfrak{sl}_3(\mathbb{A})$ with involution given by $A \mapsto -A^*$.

The involution σ is induced by an involution of the Dynkin diagram of $\mathfrak{Der}(J) \oplus \mathrm{Im}(J)$ and so gives a twisted affine Kac-Moody algebra. The involution σ induces an involution on the set of isomorphism classes of finite dimensional representations; this involution maps a representation to the dual representation.

The following table gives the types of the semisimple Lie algebras and the twisted affine Lie algebra (in Kac's notation) for each real division algebra.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathfrak{g}_0$	A_1	A_2	C_3	F_4
$\mathfrak{g}$	A_2	$2A_2$	A_5	E_6
$\widehat{L}(\mathfrak{g}, \sigma)$	$A_2^{(2)}$	$A_2^{(1)}$	$A_5^{(2)}$	$E_6^{(2)}$

Then evaluating the characters in (8) gives the following characters for the first row of the magic square.

	A_1	A_2	C_3	F_4
1	$[4]$	$[1,1]$	$[0,1,0]$	$[0,0,0,1]$
2	$[6]$	$[0,3]+[3,0]$	$[1,0,1]$	$[0,0,1,0]$
3	0	0	$[0,0,2]$	$[0,1,0,0]$
4	0	$-[1,4]-[4,1]$	$-[3,0,1]$	0
5	$[10]$	$-[3,3]$	$-[2,0,2]-[4,1,0]$	$-[1,1,0,0]$

6. $\mathfrak{b}$-normal ideals

In this section we compute the dimension of $X_r(\mathfrak{g})$ for $\mathfrak{g}$ a classical simple Lie algebra. The corresponding calculations for the exceptional simple Lie algebras are given in [MP03]. The basis for these calculations is another definition of $X_r(\mathfrak{g})$ which is given in [Kos65].

Let $\mathfrak{g}$ be a simple Lie algebra and let $\bigwedge \mathfrak{g}$ be the exterior algebra of the adjoint representation. Then the maximal eigenvalue of the quadratic Casimir acting on $\bigwedge^k \mathfrak{g}$ is at most k. Here the quadratic casimir has been normalised so that it acts as the identity on the adjoint representation. Furthermore there is an integer p such that k occurs as an eigenvalue if and only if $1 \leq k \leq p$.

The values of p were evaluated in [Mal45]. These are given for the classical groups by

A_ℓ	$B_\ell, \ell \geq 4$	C_ℓ	$D_\ell, \ell \geq 4$
$[(\ell+1)^2/4]$	$\ell(\ell-1)/2+1$	$\ell(\ell+1)/2$	$\ell(\ell-1)/2$

and for the remaining simple groups by

G_2	B_3	F_4	E_6	E_7	E_8
3	5	9	16	27	36

For $1 \le k \le p$, define $M_k(\mathfrak{g}) \subset \bigwedge^k \mathfrak{g}$ to be the eigenspace of the quadratic Casimir with eigenvalue k.

Choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and put $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$. Then an ideal $\mathfrak{a} \subset \mathfrak{b}$ is called $\mathfrak{b}$-normal if $[\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{a}$ and is nilpotent if $\mathfrak{a} \subset \mathfrak{u}$. There are only finitely many $\mathfrak{b}$-normal nilpotent ideals.

Remark 6.1. *The number of these ideals is given in terms of the Coxeter number, h, of $\mathfrak{g}$ and the exponents $\{e_i : i \in I\}$ by*

$$\prod_{i \in I} \frac{(h + e_i + 1)}{(e_i + 1)} \tag{9}$$

Define a partial order on the positive roots by $\alpha \ge \beta$ if $\alpha - \beta$ when written as a linear combination of simple roots has all non-zero coefficients positive. Then there is a natural bijection between the $\mathfrak{b}$-normal nilpotent ideals and the order ideals in this poset. The ideal, $\mathfrak{a}(J)$, corresponding to an order ideal, J, is the subspace it spans. Let J be an order ideal and put $w(J) = \wedge_{\alpha \in J} \alpha$. Then trivially $\mathfrak{a}(J)$ has dimension $|J|$ iff $w(J) \ne 0$ moreover this occurs iff $\mathfrak{a}(J)$ is abelian. If J is an order ideal such that $\mathfrak{a}(J)$ is abelian then put $\lambda(J) = \sum_{\alpha \in J} \alpha$; this is a dominant integral weight. Then the main result of [Kos65] is the following explicit description of the representations $M_k(\mathfrak{g})$;

$$M_k(\mathfrak{g}) = \oplus V(\lambda(J)) \tag{10}$$

where the sum is over all order ideals, J, of size k such that $\mathfrak{a}(J)$ is abelian. In particular, if $k > p$ then there are no $\mathfrak{b}$-normal abelian ideals of dimension k.

However if we are computing characters or dimensions then we can replace the sum over order ideals, J, such that $\mathfrak{a}(J)$ is abelian by the sum over all order ideals, J, since any J such that $\mathfrak{a}(J)$ is non-abelian will contribute zero to the sum. One advantage of this is that it is easier to parametrise the order ideals than the abelian ideals. The other advantage is that if we take any of the three sequences of Lie algebras $\mathfrak{sl}(N)$, $\mathfrak{so}(N)$, $\mathfrak{sp}(2N)$ then the parametrisation of the ideals of dimension k for fixed k is independent of N provided N is sufficiently large. Then we show below that, for each series, the dimension of $V(\lambda)$ (for N sufficiently large) is a polynomial function in N. This gives for each series and each k an explicit expression for $\dim M_k$ as a polynomial in N.

6.1. **Special linear groups.** The $\mathfrak{b}$-normal nilpotent ideals, or order ideals, of $\mathfrak{sl}(n)$ have been studied extensively. They are parametrised by partitions, λ, which satisfy $\lambda_i \le n - i$ for $1 \le i \le n$. In particular the number of order ideals is the Catalan number.

Definition 6.1. *A partition with k parts is a decreasing sequence of k positive integers. The notation is $\lambda = (\lambda_1, \lambda_2, \dots \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$.*

The conjugate partition is denoted by λ'.

The Frobenius notation for a partition λ is

$$\lambda = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

where $a_i = \lambda_i - i$ and $b_i = \lambda'_i - i$ for $1 \leq i \leq r$. Here r is determined by the conditions $a_i \geq 0$ and $b_i \geq 0$ for $1 \leq i \leq r$ and $a_i < 0$ and $b_i < 0$ for $i > r$.

We associate a dominant weight $\omega(\lambda)$ to a partition with k parts by

$$\omega(\lambda) = \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1})\omega_i + \lambda_k \omega_k \tag{11}$$

Let $*$ be the Dynkin diagram automorphism which reverses the interval. Then this induces an involution on weights. This involution is given on the fundamental weights by $\omega_i^* = \omega_{N-i}$. Then we associate a dominant weight $\omega(\lambda, \mu)$ to a pair of partitions by

$$\omega(\lambda, \mu) = \omega(\lambda) + \omega^*(\mu) \tag{12}$$

The decomposition of the tensor powers of the adjoint representation of $\mathfrak{sl}(N)$ is given in [BD02]. Here we do not specify N but simply assume that it is sufficiently large. Then the irreducible components of $\otimes^k ad$ are indexed by pairs of partitions (μ, λ) with $|\mu| = |\lambda| \leq k$. This means that the decomposition of the tensor powers of the adjoint representation of $\mathfrak{gl}(N)$ or $\mathfrak{sl}(N)$ is given by

$$\otimes^k \mathfrak{g} = \sum_{(\lambda,\mu)} W_k(\lambda, \mu) \otimes V(\lambda, \mu)$$

where $V(\lambda, \mu)$ is the representation with heighest weight $\omega(\lambda, \mu)$. The sum is over all pairs of partitions such that $|\lambda| = |\mu| \leq k$. The dimension of the vector space $W_k(\lambda, \mu)$ is the multiplicity of $V(\lambda, \mu)$ in $\otimes^k \mathfrak{g}$ and these multiplicities are given in [BD02, Theorem 2.2].

Let ϵ_i be an orthonormal basis of a Euclidean space. Then the roots of $\mathfrak{sl}(N)$ can be taken to be the set

$$\{(\epsilon_i - \epsilon_j) : i \neq j\}$$

Then we associate to a partition λ the order

$$S(\lambda) = \{(\epsilon_i - \epsilon_j) : 1 \leq i \leq \lambda_{N-j}\}$$

For N sufficiently large, this is a bijection between orders of size k and partitions of k. This is essentially the observation that Young's lattice is also the finitary distributive lattice $J_j(\mathbb{N}^2)$, see [Sta97, Exercise 3.63].

The following lemma shows that the highest weight associated to the order $S(\lambda)$ is the highest weight $\omega(\lambda, \lambda')$ defined by (11) and (12).

Lemma 6.1. *The weight associated to* $S(\lambda)$ *is* $\omega(\lambda) + \omega^*(\lambda')$.

Proof. From the definitions the weight associated to $S(\lambda)$ is

$$\sum_{\alpha \in S(\lambda)} \alpha = \sum_i \left(\lambda_i \epsilon_i - \sum_{k=1}^{\lambda_i} \epsilon_{n-k+1} \right)$$

The dominant weight we have associated to the partition is

$$\left(\sum_i (\lambda_i - \lambda_{i+1}) \omega_i \right) + \left(\sum_j (\lambda'_j - \lambda'_{j+1}) \omega_{n-j} \right)$$

The fundamental weights are expressed in terms of the basis ϵ_i by

$$\omega_k = \sum_{i=1}^{k} \epsilon_i - \frac{k}{N} \sum_{j=1}^{N} \epsilon_j$$

Substituting gives

$$\sum_i (\lambda_i - \lambda_{i+1}) \omega_i = \sum_i \lambda_i \epsilon_i - \frac{|\lambda|}{N} \sum_{j=1}^{N} \epsilon_j$$

$$\sum_j (\lambda'_j - \lambda'_{j+1}) \omega_{n-j} = \sum_{k=1}^{\lambda_i} \epsilon_{n-k+1}$$

□

Take a partition λ, draw the Ferrer's diagram and mark in each square (i, j), the hooklength $h(i, j)$. These hooklengths are given by

$$h(i, j) = \lambda_i + \lambda'_j - i - j + 1$$

Proposition 6.1. *For each partition* λ *and each sufficiently large* N, *the dimension of the representation* $V(\lambda, \lambda')$ *of* $\mathfrak{sl}(N)$ *is given by evaluating the following polynomial at* $D = N^2 - 1$.

$$\dim V(\lambda, \lambda') = \prod_{(i,j)} \left(\frac{D - h(i,j)^2 + 1}{h(i,j)^2} \right) \tag{13}$$

Note that $\dim V(\lambda, \lambda') = \dim V(\lambda', \lambda)$. Using this observation the following example gives $\dim V(\lambda, \lambda')$ for all λ with $|\lambda| \leq 4$.

Example 6.1. *Here are some examples of these polynomials.*

$$\begin{aligned}
\dim V(2; 1, 1) &= D(D - 3)/4 \\
\dim V(3; 1, 1, 1) &= D(D - 3)(D - 8)/36 \\
\dim V(2, 1; 2, 1) &= D^2(D - 8)/9 \\
\dim V(4; 1, 1, 1, 1) &= D(D - 3)(D - 8)(D - 15)/576 \\
\dim V(3, 1; 2, 1, 1) &= D^2(D - 3)(D - 15)/64 \\
\dim V(2, 2; 2, 2) &= D(D - 3)^2(D - 8)/144
\end{aligned}$$

Note that these are not numerical polynomials although they do become numerical if we substitute $D = N^2 - 1$ where here N is an indeterminate.

Proof. Let λ be a partition with r parts. Then the Frobenius notation for the usual partition associated to $\omega(\lambda, \lambda')$ is

$$a_i = \lambda_i - i + r \qquad b_i = N - \lambda_{r-i+1} - i$$

for $1 \leq i \leq r$. This can be seen by drawing the Ferrers diagram.

Then the dimension of this representation is given by

$$\left(\prod_{i=1}^{r} \frac{1}{[(\lambda_i - i + r)!]^2} \frac{(N + \lambda_i - i + r)!}{(N - \lambda_i + i - r - 1)!} \right) \left| \frac{1}{N + \lambda_i - \lambda_{r-j+1} - i - j + r + 1} \right|$$

The determinant can be evaluated by the methods of [Rob61] and gives

$$\left(\prod_{1 \leq i < j \leq r} (\lambda_i - \lambda_j + j - i)^2 \right) \left(\prod_{1 \leq i,j \leq r} \frac{1}{(N + \lambda_i - \lambda_j + j - i)} \right)$$

□

Proposition 6.2. *For $k \geq 0$, the polynomial $p_k(D)$ is given by*

$$p_k(D) = \sum_{|\lambda|=k} \dim V(\lambda, \lambda') \tag{14}$$

where $\dim V(\lambda, \lambda')$ *is given by (13).*

Proof. Denote the polynomial defined by (6.2) and (13) by $p'_k(D)$. Then choose k and take N sufficiently large. Then $p'_k(N^2 - 1) = \dim M_k(\mathfrak{sl}(N))$ and $p_k(N^2 - 1) = \dim X_k(\mathfrak{sl}(N))$. However, by [Kos], we have $\dim M_k(\mathfrak{sl}(N)) = \dim X_k(\mathfrak{sl}(N))$ for N sufficiently large. This shows that $p'_k(N^2 - 1) = p_k(N^2 - 1)$ for N sufficiently large and so $p'_k(D) = p_k(D)$. □

If we adopted (14) as the definition of the polynomial $p_k(D)$ then it is simple to check; that the coefficient of D^k is $1/k!$; that $p_k(-1)$ is the number of partitions of k; and that, for $k > 0$, the constant term is zero.

Here we put $D = p^2 - 1$ for some positive integer p and attempt to evaluate the generating function. The first observation is that the sum over partitions is in fact a sum over p-cores. A p-core is a partition with no border strip or equivalently, no hook, of length p. The equivalence of these two definitions is given in ([Sta99, Exercise 7.59 (f)]). This gives the following:

Corollary 6.1. *The abelian subalgebras of $SL(p)$ of are parametrised by partitions λ such that*

(1) $|\lambda| \leq p^2/4$
(2) λ *is a* p*-core*
(3) $\lambda_i \leq p - i$ *for* $1 \leq i \leq p$

Moreover the dimension of the ideal is $|\lambda|$.

6.2. **Symplectic groups.** In this section we give an algorithm for computing the analogous polynomials for the symplectic groups. For our purposes, the symplectic group $\mathfrak{sp}(2N)$ can be regarded as the orthogonal group $\mathfrak{so}(-2N)$. Therefore we do not have to treat the orthogonal groups separately.

Take the same map from partitions to dominant weights as before. Then the fundamental representation, V, corresponds to the partition (1) and the adjoint representation to the partition (2). Then the representations which occur in $\otimes^k V$ are indexed by partitions of k' where $k' \leq k$ and $k - k'$ is even. Since the adjoint representation is a direct summand of $V \otimes V$, the representation $\otimes^k \mathfrak{g}$ is a direct summand of $\otimes^{2k} V$. This shows that the components of $\otimes^k \mathfrak{g}$ can be labelled by partitions of $2k'$ for some k' with $k' \leq k$.

Next we describe an algorithm which for each k produces a finite set of partitions of $2k$. The corresponding representations are the components of X_k.

Definition 6.2. *Let $O \subset \mathbb{N}^2$ be the subposet given by*

$$O = \{(i,j) \in \mathbb{N}^2 : j \geq i\}$$

Then if $S \in J_f(O)$ is an order ideal in this poset define $\lambda(S)$ by

$$\lambda(S) = \sum_{(i,j)\in S} (e_i + e_j)$$

Then $\lambda(S)$ is a partition and $|\lambda(S)| = 2|S|$.

There is a bijection between ideals with k elements and strict partitions of k . Given an ideal S take λ_i to be the number of elements in the set $\{j : (i,j) \in S\}$. Conversely given a strict partition λ define the ideal S by

$$S = \{(i,j) : i \leq j \leq i + \lambda_i - 1\}$$

The number of strict partitions of k is also the number of partitions of k into odd parts, see [HW79, Theorem 344].

Now given a partition λ the dimension of the associated representation of $\mathfrak{sp}(2N)$ is given in [ESK79, (3.29)]. The formula for $\dim V(\lambda)$ is

$$\frac{\prod_{(i>j)}(N + \lambda_i + \lambda_j - i - j + 2) \prod_{(i\leq j)}(N - \lambda'_i - \lambda'_j + i + j)}{\prod_{(i,j)}(\lambda_i + \lambda'_j - i - j + 1)}$$

Proposition 6.3. *For $k \geq 0$,*

$$p_k(N(N+1)/2) = \sum_{S:|S|=k} \dim V(\lambda(S))$$

The proof is essentially the same as for proposition (6.2). However this result is less interesting than proposition (6.2) since it is not clear from the definition that this gives a polynomial in $N(N+1)/2$.

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