0. Introduction

Given a connected finite $d$-regular graph $\mathcal{G}$ (for $d \geq 3$) one can associate the Ihara zeta function $\zeta_{\mathcal{G}}(z)$, a complex function defined in terms of closed cycles in the graph. It transpires that $\zeta_{\mathcal{G}}(z)$ is the reciprocal of a polynomial, whose explicit form was given by Ihara [Ihara] (cf. also Bass [Bass] and [KS] for generalisations). For an interesting example of such a graph, consider the group $\text{PGL}(2, \mathbb{Q}_p)$ which has a natural action on a $(p+1)$-regular tree $T$. If $\Gamma < \text{PGL}(2, \mathbb{Q}_p)$ is a torsion free discrete subgroup (necessarily isomorphic to a free group) then the quotient $\mathcal{G} = T/\Gamma$ is a $(p+1)$-regular tree.

Consider next an infinite graph $\mathcal{G}_\infty$, which is either the Cayley graph of an infinite group or associated to a sequence of $d$-regular graphs $\{\mathcal{G}_n\}$ which converge (in a suitable sense) to the infinite graph. Following Grigorchuk and Zuk [GZ] we can then associate to $\mathcal{G}_\infty$ the logarithm of its zeta function $\log \zeta_{\mathcal{G}_\infty}(z)$ either by using an integral with respect to the spectral measure or by taking suitable limits of the functions $\log \zeta_{\mathcal{G}}(z)$. In particular, $\log \zeta_{\mathcal{G}_\infty}(z)$ has a formal power series which converges to an analytic function in the region $|z| < \frac{1}{d-1}$. The approach for infinite graphs was further studied by Guido-Isola-Lapidus, [GIL] and Clair-Mokhtan-Sharghi [CMS] using Fudge-Kadison determinants and traces on von Neumann algebras.

The purpose of this note is to consider new expressions for the zeta functions which naturally lead to the analytic extensions of $\log \zeta_{\mathcal{G}_\infty}(z)$ as new explicit convergent series, which is particularly useful in numerical evaluation of the zeta function in the region of its analytic extension. In the case that the associated spectra measure is a Gibbs measure, there is an approach using classical Fredholm-Grothendieck determinants for transfer operators. In some other cases, the transfer operator can be applied directly. In either case, this viewpoint has significant advantages. Firstly, it gives an explicit convergent expression for $\log \zeta_{\mathcal{G}_\infty}(z)$ within its domain of analyticity. Secondly, it leads to a simple approach to numerical computation of $\log \zeta_{\mathcal{G}_\infty}(z)$ within the domain of analyticity.

1. Zeta Functions for Graphs

In this section we begin by recalling the properties of the Ihara zeta function for finite $d$-regular graphs and their extension to suitable infinite graphs.

1.1 Ihara zeta function. Let $d \geq 3$. Given a finite $d$-regular graph $\mathcal{G}$ (with vertices $V$) we first define the zeta function.
**Definition.** We can define the Ihara zeta function as

$$\zeta_G(z) = \exp \left( -\sum_{k=1}^{\infty} \frac{c_k z^k}{k} \right)$$

where $c_n = \text{Card} \{ \text{closed loops of length } n \}$, which converges for $|z| < \frac{1}{d-1}$.

It is possible to say quite a lot about the analytic extension of the zeta function. We first denote by $M$ the the $|V| \times |V|$ adjacency matrix defined by

$$M(v,v') = \begin{cases} 1 & \text{if } v, v \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

The zeta function $\zeta_G(z)$ has the following properties.

**Theorem (Ihara).** For a $d$-regular graph $G$:

1. the Ihara zeta function has a closed form given by

$$\frac{1}{\zeta_G(z)} = (1 - z^2)^{|V|/2} \det((1 + (d-1)z^2)I - zM);$$

2. the zeros of $1/\zeta_G(z)$ lie on the following union of a circle and two intervals:

$$X = \left\{ z \in \mathbb{C} : |z| = \frac{1}{(d-1)^{1/2}} \right\} \cup \left[-1, -\frac{1}{(d-1)}\right] \cup \left[\frac{1}{(d-1)}, 1\right].$$

**1.2 The Laplacian.** Given any finite $d$-regular graph $G = (V, E)$ we can associate the Laplacian $\Delta_G : L^2(V, \mathbb{R}) \to L^2(V, \mathbb{R})$ defined by

$$\Delta_G h(v) = \frac{1}{d} \sum_{v' \sim v} h(v')$$

where $v \in V$ and $v \sim v'$ denotes a neighbouring vertex. The operator is self adjoint and thus the spectrum lies in the the interval $[-1, 1]$. As for any self-adjoint operator, one can write $\Delta_G = \int_{-1}^{1} \lambda dE(\lambda)$, where $E$ is a spectral measure taking values in projections on $L^2(V, \mathbb{R})$. For any $v \in V$ we can associate the spectral measure $\mu(B) = \langle E(B) \delta_v, \delta_v \rangle$ on Borel sets $B$ (which is independent of the choice of vertex $v$). Equivalently, consider the measure

$$\mu = \frac{1}{|\text{spec}(\Delta_G)|} \sum_{\lambda \in \text{spec}(\Delta_G)} \delta_\lambda$$

supported on the finite set of eigenvalues in the spectrum of $\Delta_G$.\(^1\) We can then write

$$\log \zeta_G(z) = -\left( \frac{d-2}{2} \right) \log(1 - z^2) - \int_{-1}^{1} \log(1 - zd + (d-1)z^2) d\mu(x)$$

where $\mu$ is the spectral measure associated to $G$.

\(^1\)We could have defined $\sum_{v' \sim v}(h(v) - h(v'))$, giving a closer analogy with the laplacian on manifolds. However, the present normalization is more convenient for taking limits of measures.
1.3 Infinite graphs. For definiteness, consider the particular case of taking limits of graphs and associating a zeta function. Let $G_n$ be a finite family of graphs and let $\mu_n$ be the associated measures. In this context, there are two natural ways to approach this, which can be shown to be equivalent using the following result.

**Proposition 1.2 (Serre ([Serre], cf. [GZ, Theorem 4.2]).** The convergence of $\mu_n \to \mu_\infty$ in the weak star topology is equivalent to the formal power series

$$\log \zeta_G(z) = - \sum_{k=1}^{\infty} c_k^{(n)} \frac{z^k}{k}$$

converging termwise to a limit, i.e., $c_k = \lim_{n \to +\infty} c_k^{(n)}$ exists for each $k \geq 1$.

When convergence in either of these equivalent senses holds then we can define the zeta function for the limiting graph $G$ by:

$$\log \zeta_G(z) := \lim_{n \to +\infty} \frac{1}{|V_n|} \log \zeta_{G_n}(z).$$

The uniqueness of the definition in the connected component $V_0$ of $\mathbb{C} - X$ follows from the uniqueness of the analytic extension. Furthermore, given the explicit form of the Ihara zeta function for finite graphs (in Part (i) of Ihara’s theorem) we have the following:

**Proposition 1.3 (Grigorchuk-Zuk) [GZ, Theorem 5.1].** When the limit of $\log \zeta_{G_n}(z)$ exists it takes the form

$$\log \zeta_{G_\infty}(z) := - \frac{d-2}{2} \log(1 - z^2) - \int_{-1}^{1} \log(1 - zd\xi + (d-1)z^2) d\mu_\infty(\xi) \quad (1.2)$$

where $\mu_\infty$ is the weak star limit of the associated spectral measures $\mu_n$.

In particular, the expression for $\log \zeta_{G_\infty}(z)$ converges to an analytic function on $|z| < \frac{1}{d-1}$ and has an analytic extension to $V_0$.

More generally, for the Cayley graph associated to an infinite graph we can associate the logarithm of the zeta function $\log \zeta_G(z)$ by using the definition (1.2) where $\mu_\infty$ is taken to be the Kesten spectral measure for the random walk.

2. Gibbs measures and integrals

2.1 Gibbs measures. In order to calculate the analytic extension of $\log \zeta_{G_\infty}(z)$ using (1.2) we need to address the same problem for the integral. We can consider a more general problem.

Assume that we have a piecewise real analytic expanding map $T : X \to X$ for a cube $X \subset \mathbb{R}^d$, say, and a real analytic function $g : X \to \mathbb{R}$ (which has an analytic extension to a the neighbourhood $U$, say).

**Definition.** The $T$-invariant Gibbs measure $\mu$ associated to $g$ is the unique $T$-invariant measure on $[-1, 1]$ which realises the supremum in

$$P(g) = \sup\{h(m) + \int g dm : m = T\text{-invariant probability}\}.$$

Consider the pressure function $t \mapsto P(t \log(f(z, \cdot)) + g(\cdot))$ where $z \in V_0$. The following lemma is now standard.
Lemma 2.1 (Ruelle). Assume that \( P(g) = 0 \). The map \( t \mapsto P(t \log(f(z, \cdot)) + g(\cdot)) \) is analytic in a neighbourhood of \( t = 0 \). Moreover, we can write
\[
\int_X \log(f(z, x)) d\mu(x) = \frac{\partial}{\partial t} P(t \log(f(z, \cdot)) + g(\cdot))|_{t=0}. \tag{2.1}
\]

2.2. Gibbs measures and transfer operators. We can consider the contracting local inverses \( T_i : X \to X \) \((i = 1, \ldots, k)\) and the linear operator \( \mathcal{L} : B \to B \) defined on the Banach space \( B \) of bounded analytic functions \( w : U \to \mathbb{C} \) (with the supremum norm \( ||\cdot||_\infty \)) by
\[
\mathcal{L}w(x) = \sum_{i=1}^k e^{g(T_i x)} w(T_i x).
\]

We then have the following useful theorem on the spectrum of the operator [Ruelle].

Lemma 2.2 (Ruelle). The operator \( \mathcal{L} : B \to B \) is compact. In particular, for any \( 0 < \theta < 1 \) on can associate:

1. a positive eigenfunction \( h \in B \) and eigenmeasure \( \nu \in B^* \); and
2. a finite rank operator \( Q : B \to B \) with \( \rho(Q) < \theta \),

such that
\[
\mathcal{L}^n w = e^{nP(g)} \left( \int w d\nu \right) h + Q^n(g) + O(\theta^n).
\]

for \( w \in B \).

In the particular case that \( \mathcal{L}1 = 1 \) we see that \( h = 1 \) and \( \nu = \mu \) is the Gibbs measure associated to \( g \).

2.3. Gibbs measures and determinants. We can use the following result to give another expression for (2.1)

Lemma 2.3.

1. The function
\[
d(w, z, t) = \exp \left( -\sum_{n=1}^\infty \frac{w^n}{n} Z_n(t \log f(z, \cdot) + g(\cdot)) \right) \tag{2.2}
\]
where, for \( n \geq 1 \), we write
\[
Z_n(t \log f(z, \cdot) + g(\cdot)) = \sum_{T^n x = x} \frac{\exp((t \log(f(z, \cdot)) + g(\cdot))^n(x))}{\det(I - D(T^n)(x)^{-1})}, \tag{2.3}
\]
is analytic \( w \in \mathbb{C}, z \in V_0 \) and \( t \) is sufficiently small. Furthermore, we can write
\[
d(w, z, t) = 1 + \sum_{n=1}^\infty a_n(z, t) w^n
\]

where \( |a_n| = O(\theta^{n^2}) \) for some \( 0 < \theta < 1 \);

2. Given \( w, t \) the value \( e^{-P(t \log f(z, \cdot) + g(\cdot))} \) is a simple zero for \( d(z, w, t) \);

3. We can write
\[
\int_{-1}^1 \log(f(z, x)) d\mu(x) = \frac{\partial d(e^{-P(g)}, t)}{\partial t} |_{t=0} - \frac{\partial d(z, t)}{\partial z} |_{z = e^{-P(g)}}. \tag{2.4}
\]
Proof of Lemma 2.3. Part (1) and Part (2) are a direct application of the work of Ruelle, after Grothendieck. For Part (3) we observe that \( d(z, \exp(-P(t \log f(z, \cdot) + g(\cdot)), t) = 0 \), thus we can use Lemma 2.1 and the Implicit Function Theorem to derive (2.4). \( \square \)

Remark. The operator \( \mathcal{L} \) in §2.2 is actually of trace class and the complex function in (2.2) corresponds to the determinant of \( \mathcal{L} \).

3. Two Theorems

We can now state two theorems which can often be used to provide analytic extensions of the logarithm of the zeta function. They are both rather technical, but in the next section we shall apply them to some interesting example.

3.1. The first theorem. We begin with the following hypothesis which applies in many examples.

Hypothesis I. Assume that there is a Gibbs measure \( \mu \), such that

\[
\log \zeta_{G_\infty}(z) = -\frac{d-2}{2} \log(1 - z^2) - \int_X f(z, x) d\mu(z)
\]

where \( f(x, \cdot) : [-1, 1] \to \mathbb{R} \) are real analytic (with an analytic extension to a common open neighbourhood \( U \supset [-1, 1] \)).

This brings us to our main result:

Theorem 1. Under the above hypothesis, for any \( z \in V_0 \) we can write

\[
\log \zeta_{G_\infty}(z) = -\frac{d-2}{2} \log(1 - z^2) - \sum_{k=1}^\infty d_k(z) \tag{3.1}
\]

where:

(1) \( d_k(z) \) are explicit polynomials in \( \{ \log f(z, x) : T^l x = x, 1 \leq l \leq k \} \); and

(2) \( |d_k(z)| = O(\theta^{n^2}) \).

Proof. Using (2.4) we can write

\[
\int_{-1}^1 \log(f(z, x))d\mu(x) = \left. \frac{\partial \exp(-P(t \log f(z, \cdot) + g(\cdot)))}{\partial t} \right|_{t=0} = \sum_{k=1}^\infty \left. \frac{\partial a_k(z, t)}{\partial t} \right|_{t=0} e^{-kP} \tag{3.2}
\]

where

\[
\left. \frac{\partial d(z, e^{-P}, t)}{\partial t} \right|_{t=0} = \sum_{k=1}^\infty \left. \frac{\partial a_k(z, t)}{\partial t} \right|_{t=0} e^{-kP} \quad \text{and}
\]

\[
\left. \frac{\partial d(z, w, 0)}{\partial w} \right|_{w=e^{-P}} = \sum_{k=1}^\infty k a_k(z, 0) e^{-(k-1)P}.
\]

For \( |w| \) sufficiently small we can also expand

\[
d(w, z, t) = 1 + \sum_{k=1}^\infty \frac{1}{k!} \left( -\sum_{n=1}^\infty \frac{w^n}{n} Z_n(t \log f(z, \cdot) + g(\cdot)) \right)^k
\]

\[= 1 + \sum_{N=1}^\infty w^N \left( \sum_{n_1 + \cdots + n_k = N} \frac{1}{k!} Z_{n_1} \cdots Z_{n_k} \right) \tag{3.3}
\]
One sees from the definition of $Z_n$ that it depends only on the values of the functions at $\{ x : T^nx = x \}$, and thus by (3.3) that $a_n(z,t)$ depends only on the values of the functions at $\cup_{n=1}^\infty \{ x : T^nx = x \}$. On the other hand, when $t = 0$ then $d(w,z,0)$, and its derivative $\frac{\partial d(z,w,0)}{\partial w}|_{e^{-r}}$ is clearly independent of $\log f(z,\cdot)$. Thus the expression (3.0) follows with

$$d_k(z) = \frac{\partial^k a_k(z,t)}{\partial t^k} |_{t=0} e^{-r}.$$ and part (1) follows from (3.1).

It is easy (using Cauchy’s theorem) that since $|a_n|O(e^{-ck^2})$, for suitable $c > 0$, we can estimate $|\frac{\partial a_k}{\partial t}(z,t)|_{t=0} = O(e^{-c'k^2})$ and $|ka_k(z,0)| = O(e^{-c'k^2})$, for any $c > c' > 0$. This proves part (2).

A simple special case. In several examples $\mu$ can be taken merely to be the normalised Lebesgue measure on $[-1,1]$ then we can let $T : [-1,1] \to [-1,1]$ be the doubling map

$$T(x) = \begin{cases} 
2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1 \\
2x & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\
2x + 1 & \text{if } -1 \leq x \leq -\frac{1}{2}
\end{cases}$$

and $g = -\log 2$. Then we can write

$$Z_n = \frac{1}{2^n - 1} \sum_{T^n x = x} \exp \left( t \sum_{j=0}^{n-1} \log f(z, T^j x) \right).$$

We can then construct $d(w,z,t)$ from the definition and the integral $\frac{1}{2} \int_{-1}^{1} \log(f(z,x))dx$ by (3.1).

3.2. The second theorem. We now consider a different hypothesis which applies in some other examples.

Hypothesis II. Assume that we can find a suitable choice of transfer operator and write

$$\log \zeta_G(\infty)(z) = -\log(1 - z^2) - \sum_{n=0}^\infty \mathcal{L}^n f(z,\cdot)(x_0)$$

(3.4)

where $f(x,\cdot) : [-1,1] \to \mathbb{R}$ are real analytic (with an analytic extension to a common open neighbourhood $U \supset [-1,1]$) and $x_0 \in [-1,1]$.

This brings us to our second theorem:

**Theorem 2.** Under the above hypothesis, for any $z \in V_0$ we can write the analytic extension in the form

$$\log \zeta_G(\infty)(z) = -\log(1 - z^2) - (I - \mathcal{L})^{-1} f(z,\cdot)(x_0)$$

**Proof.** This is a direct application of the spectral properties of $\mathcal{L}$ in Lemma 2.2.

**Remark.** Theorem 3.1 is appears slightly more complicated to implement, but has the advantage that it gives a method for explicitly constructing the analytic extension whenever it exists. Theorem 3.2 is easier to use, when applicable, but the size of the extension depends on the size of the second eigenvalue of the operator $\mathcal{L}$. 
4. Examples

Example 1 (Cayley graph of $\mathbb{Z}^2$). The Cayley graphs $G$ of $\mathbb{Z}$ (or the free groups $F_m$) satisfy $\zeta_G(z) = 1$ [GZ, §6]. For $k = 2$ the Cayley graph of $\mathbb{Z}^k$ the zeta function is given by

$$
\log \zeta_G(z) = -\log(1 - z^2) - \int_{-1}^{1} \int_{-1}^{1} \log(f(z, x_1 + x_2)) \frac{dx_1 dx_2}{\pi^2 \sqrt{1 - x_1^2} \sqrt{1 - x_2^2}}
$$

(4.1)

where $f(z, x) = 1 - 4zx + 3z^2$. However, we can simply change variables in (4.1) using $x_1 = \sin \left(\frac{\pi y_1}{2}\right)$ and $x_2 = \sin \left(\frac{\pi y_2}{2}\right)$ to write the integral in (4.1) as

$$
\frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \left(1 - 4z \left(\sin \left(\frac{\pi y_1}{2}\right) + \sin \left(\frac{\pi y_2}{2}\right)\right) + 3z^2\right) dy_1 dy_2.
$$

Since integral is with respect to normalised Lebesgue measure on $[-1, 1]$ which is a special case of a Gibbs measure, we can apply Theorem 3.1

![Figure 1. The absolute value of the integral term of $\log \zeta_{G_\infty}(z)$](image.png)

Example 2 (Fabikowski-Gupta). Following [GZ], Corollary 10.2, we can write the zeta function for for the Fabikowski-Gupta (described in [BG]) takes the form:

$$
\log \zeta_G(z) = -\log(1 - z^2) - \frac{1}{3} \log(1 - z - 3z^2)

- \sum_{n=1}^{\infty} \log \left(1 - z \left(1 \pm \sqrt{6 \pm \sqrt{\cdots \pm \sqrt{6}}}\right)\right) \frac{1}{3^{n+1}}
$$
In particular, the last two terms can be written as \(((I - \mathcal{L})^{-1}f(z, \cdot)) \cdot (1)\) where \(f(z, x) = \log(1 - zx + 3z^2)\) and

\[
\mathcal{L}w(x) = \frac{1}{3} w(T_1 z) + \frac{1}{3} w(T_2 z)
\]

where \(T_1(x) = 1 - \sqrt{x + 5}\) and \(T_2(x) = 1 + \sqrt{x + 5}\).

**Figure 2.** The absolute value of the integral term of \(\log \zeta_{G_\infty}(z)\) in Example 2.

**Example 3 (cf. [GZ], Proposition 9.4).** Grigorchuk and Zuk consider an example of a graph \(G_\infty\) with \(d = 8\) associated to an automatic group for which the zeta function satisfies

\[
\log \zeta_{G_\infty}(z) = -3 \log(1 - z^2) - \frac{1}{2} \int_{-1}^{1} \log(F(z, x)) \frac{dx}{\pi \sqrt{1 - x^2}}
\]

where \(F(z, x) = 1 - 4z - 2z^2 - 28z^2 + 49z^4 + 16z^2x\). We can write \(x = \sin(\frac{\pi y}{2})\) then \(\frac{dx}{dy} = \frac{\pi}{2} \cos(\frac{\pi y}{2})\). This converges for \(|z| < \frac{1}{7}\) and has an analytic extension to \(V_0\). Thus we want to evaluate

\[
\int_{-1}^{1} \log(F(z, \sin(\frac{\pi y}{2}))) dy
\]

We can let \(z = \frac{1}{10}\) and we have the following approximations \(I_n\) to the value of the integral:
Figure 3. The absolute value of the integral term of $\log \zeta_{G_\infty}(z)$ in Example 3.

Table 1. Values of $p(c)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$I_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.2</td>
</tr>
<tr>
<td>2</td>
<td>-0.27</td>
</tr>
<tr>
<td>3</td>
<td>-0.292107</td>
</tr>
<tr>
<td>4</td>
<td>-0.302979</td>
</tr>
<tr>
<td>5</td>
<td>-0.310813</td>
</tr>
<tr>
<td>6</td>
<td>-0.315728</td>
</tr>
<tr>
<td>7</td>
<td>-0.31867</td>
</tr>
<tr>
<td>8</td>
<td>-0.3204</td>
</tr>
<tr>
<td>9</td>
<td>-0.321421</td>
</tr>
<tr>
<td>10</td>
<td>-0.322035</td>
</tr>
</tbody>
</table>

References


**Mark Pollicott**, Department of Mathematics, University of Warwick, Coventry CV4 7AL

*E-mail address: M.Pollicott@warwick.ac.uk*