

Name: .....

Supervisor: .....

Class Teacher: .....

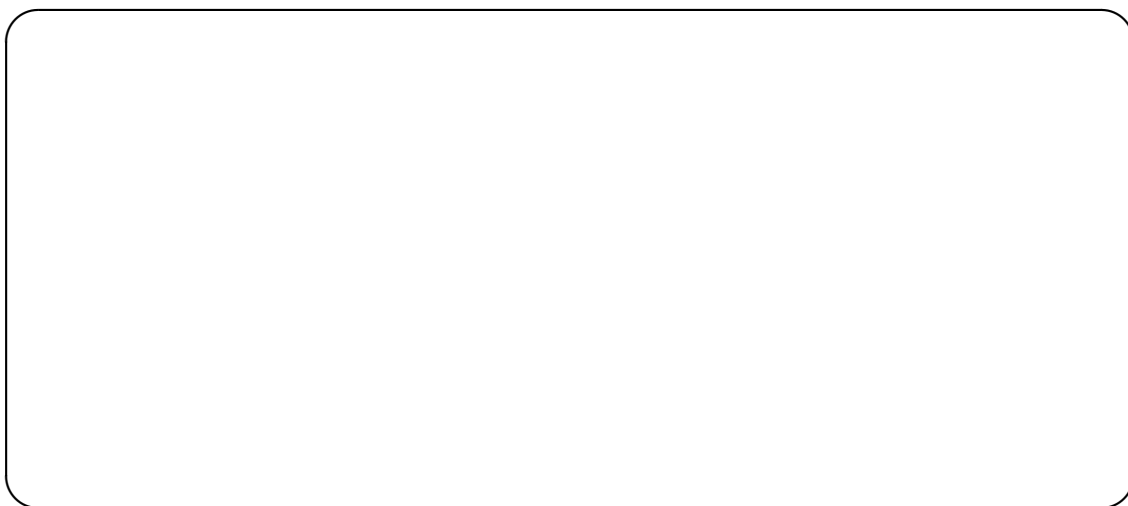
---

MA131 - Analysis 1  
Workbook 7 Assignments

**Due in 17th Nov**

---

**Assignment 1** Look again at the series  $\sum_{n=1}^{\infty} \frac{1}{2^n}$ . Plot on two small separate graphs both the sequences  $(a_n) = (\frac{1}{2^n})$  and  $(s_n) = (\sum_{k=1}^n \frac{1}{2^k})$ .



**Assignment 2** Find the sum of the series  $\sum_{n=1}^{\infty} (\frac{1}{10^n})$ .



**Assignment 3**

Reread your answer to exercise 4 and then write out a full proof that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots = 1$$

**Assignment 4**

Show that the series  $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \dots$  diverges to  $+\infty$ . [Hint: Calculate the partial sums  $s_1, s_3, s_6, s_{10}, \dots$ ]

**Assignment 5**

Prove the theorem [Hint: Use the GP formula to get a formula for  $s_n$ ].

**Assignment 6**

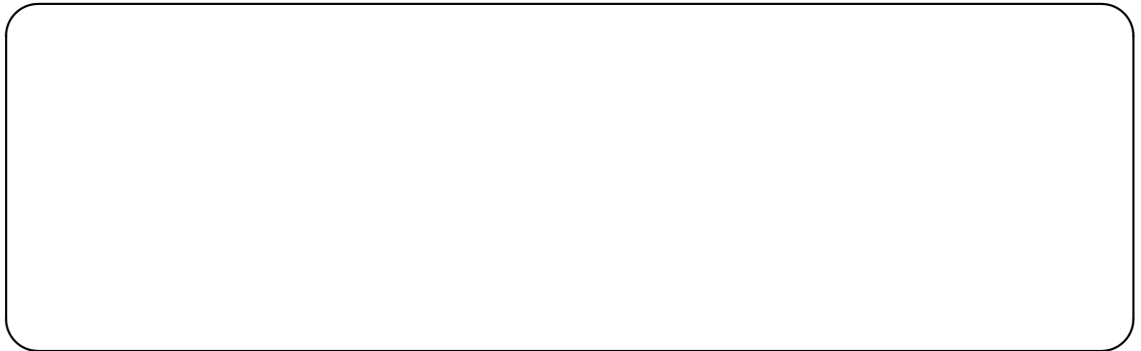
Prove that the Harmonic Series diverges. Structure your proof as follows:

1. Let  $s_n = \sum_{k=1}^n \frac{1}{k}$  be the partial sum. Show that  $s_{2n} \geq s_n + \frac{1}{2}$  for all  $n$ . (Use the idea in the cunning grouping above).
2. Show by induction that  $s_{2^n} \geq 1 + \frac{n}{2}$  for all  $n$ .
3. Conclude that  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

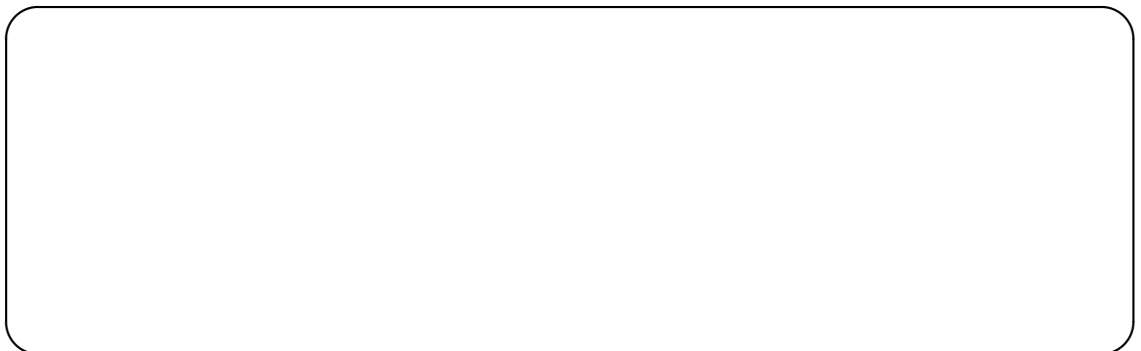
**Assignment 7** Give, with reasons, a value of  $N$  for which  $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \geq 10$ .



**Assignment 8** Prove the shift rule.

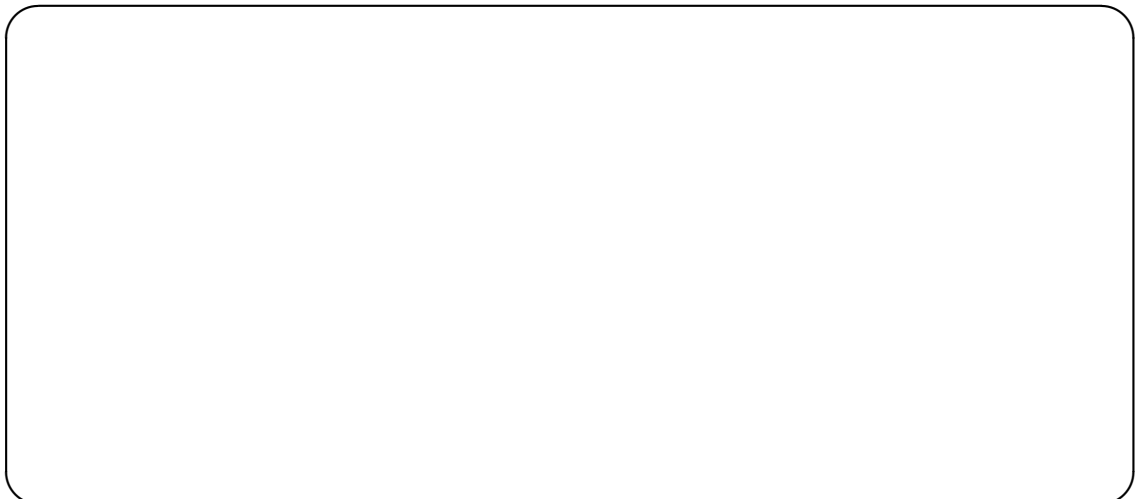


**Assignment 9** Prove this result. Your proof must use the axiom of completeness or one of its consequences - make sure you indicate where this occurs.



**Assignment 10**

1. Prove that if  $\sum_{n=1}^{\infty} a_n$  converges then the sequence  $(a_n)$  tends to zero.  
(Hint: Notice that  $a_{n+1} = s_{n+1} - s_n$  and use the Shift Rule for sequences.)
2. Is the converse true: If  $(a_n) \rightarrow 0$  then  $\sum_{n=1}^{\infty} a_n$  converges?



**Assignment 11**

Prove the Comparison Test [Hint: Consider the partial sums of both  $\sum b_n$  and  $\sum a_n$  and show that the latter is increasing and bounded].

**Assignment 12**

Use the Comparison Test to determine whether each of the following series converges or diverges. In each case you will have to think of a suitable series with which to compare it.

$$(i) \sum \frac{2n^2 + 15n}{n^3 + 7} \quad (ii) \sum \frac{\sin^2 nx}{n^2} \quad (iii) \sum \frac{3^n + 7^n}{3^n + 8^n}$$

**Assignment 13**

Consider the series  $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$  and its partial sums  $s_n = \sum_{k=0}^n \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!}$ .

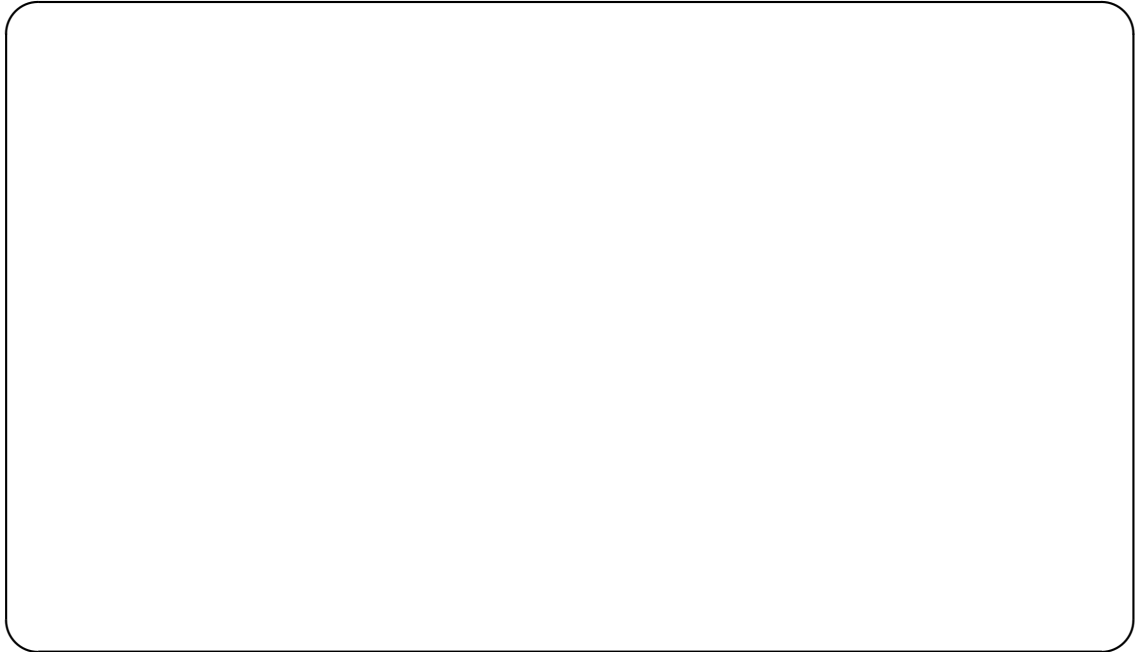
1. Show that the sequence  $(s_n)$  is increasing.
2. Prove by induction that  $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$  for  $n > 0$ .
3. Use the comparison test to conclude that  $\sum_{n=0}^{\infty} \frac{1}{n!}$  converges.

**Assignment 14**

Use the Binomial Theorem to show that

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{(k-1)}{n}\right) \leq \sum_{k=0}^n \frac{1}{k!}$$

Conclude that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \leq e$ .

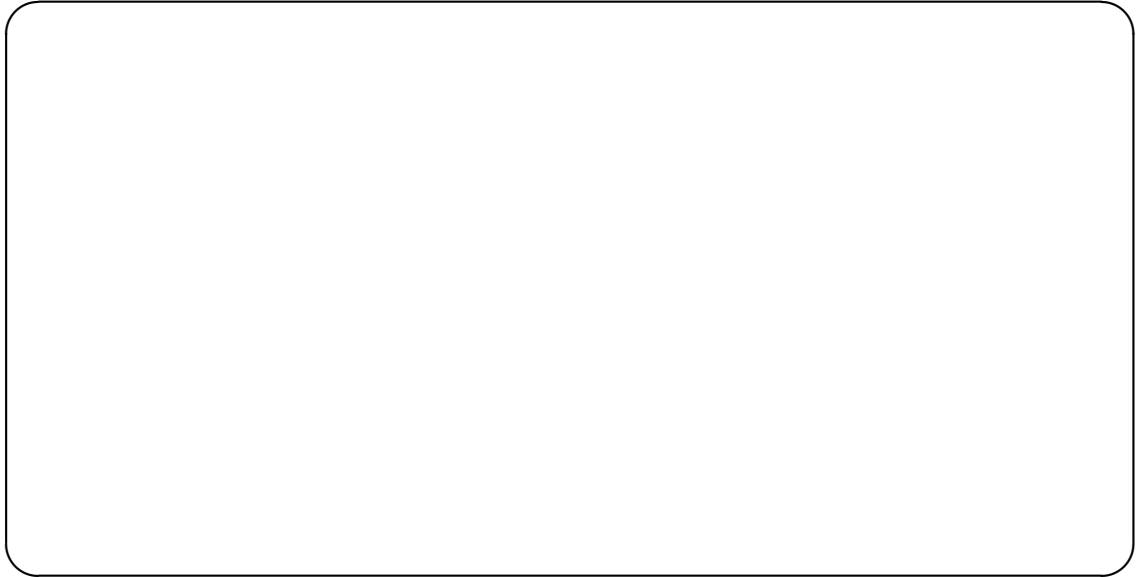
**Assignment 15**

Consider the inequality in equation (1). Fix  $m$  and let  $n \rightarrow \infty$  to conclude that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \geq \sum_{k=0}^m \frac{1}{k!}$ . Then let  $m \rightarrow \infty$  and show that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \geq e$ .



**Assignment 16**

1. Show that  $\left(1 - \frac{1}{n+1}\right) = \frac{1}{(1+1/n)}$  and hence find  $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right)^n$ .
2. Use the shift rule to find  $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n$ .

**Assignment 17**

Prove this result by contradiction. Structure your proof as follows:

1. Suppose  $e = \frac{p}{q}$  and show that  $e - \sum_{i=1}^{q+1} \frac{1}{(i-1)!} = \frac{p}{q} - \sum_{i=1}^{q+1} \frac{1}{(i-1)!} = \frac{k}{q!}$  for some positive integer  $k$ .
2. Show that  $e - \sum_{i=1}^{q+1} \frac{1}{(i-1)!} = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \dots < \frac{1}{q!} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)$  and derive a contradiction to part 1.

