Name: .... Supervisor: ....

Class Teacher: ....

MA131 - Analysis 1
Workbook 7 Assignments

Due in 17th Nov

Assignment 1 Look again at the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Plot on two small separate graphs both the sequences $(a_n) = \left(\frac{1}{2^n}\right)$ and $(s_n) = \left(\sum_{k=1}^{n} \frac{1}{2^k}\right)$.

Assignment 2 Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{1}{10^n}\right)$.

Assignment 3 Reread your answer to exercise 4 and then write out a full proof that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots = 1$$

Assignment 4 Show that the series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \dots$ diverges to $+\infty$. [Hint: Calculate the partial sums $s_1, s_3, s_6, s_{10}, \dots$.]

Assignment 5 Prove the theorem [Hint: Use the GP formula to get a formula for s_n].

Assignment 6 Prove that the Harmonic Series diverges. Structure your proof as follows:

1. Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$ be the partial sum. Show that $s_{2n} \geq s_n + \frac{1}{2}$ for all n. (Use the idea in the cunning grouping above).
2. Show by induction that $s_{2^n} \geq 1 + \frac{n}{2}$ for all n.
3. Conclude that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Assignment 7 Give, with reasons, a value of N for which $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \geq 10$.

Assignment 8 Prove the shift rule.

Assignment 9 Prove this result. Your proof must use the axiom of completeness or one of its consequences - make sure you indicate where this occurs.

Assignment 10

1. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then the sequence (a_n) tends to zero.
   (Hint: Notice that $a_{n+1} = s_{n+1} - s_n$ and use the Shift Rule for sequences.)
2. Is the converse true: If $(a_n) \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges?

Assignment 11 Prove the Comparison Test [Hint: Consider the partial sums of both $\sum b_n$ and $\sum a_n$ and show that the latter is increasing and bounded].

Assignment 12 Use the Comparison Test to determine whether each of the following series converges or diverges. In each case you will have to think of a suitable series with which to compare it.

$$\text{(i)} \quad \sum \frac{2n^2+15n}{n^3+7} \qquad \text{(ii)} \quad \sum \frac{\sin^2 nx}{n^2} \qquad \text{(iii)} \quad \sum \frac{3^n+7^n}{3^n+8^n}$$

Assignment 13 Consider the series $\sum_{n=0}^{\infty} \frac{1}{n!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots$ and its partial sums $s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots+\frac{1}{n!}$.

1. Show that the sequence (s_n) is increasing.
2. Prove by induction that $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ for $n > 0$.
3. Use the comparison test to conclude that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Assignment 14 Use the Binomial Theorem to show that

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{(k-1)}{n}\right) \le \sum_{k=0}^{n}\frac{1}{k!}$$

Conclude that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le e$.

Assignment 15 Consider the inequality in equation (1). Fix m and let $n \to \infty$ to conclude that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \ge \sum_{k=0}^{m}\frac{1}{k!}$. Then let $m \to \infty$ and show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \ge e$.

Assignment 16

1. Show that $\left(1 - \frac{1}{n+1}\right) = \frac{1}{(1+1/n)}$ and hence find $\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n$.
2. Use the shift rule to find $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$.

Assignment 17

Prove this result by contradiction. Structure your proof as follows:

1. Suppose $e = \frac{p}{q}$ and show that $e - \sum_{i=1}^{q+1} \frac{1}{(i-1)!} = \frac{p}{q} - \sum_{i=1}^{q+1} \frac{1}{(i-1)!} = \frac{k}{q!}$ for some positive integer k.
2. Show that $e - \sum_{i=1}^{q+1} \frac{1}{(i-1)!} = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \cdots < \frac{1}{q!}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)$ and derive a contradiction to part 1.