

Name: .....

Supervisor: .....

Class Teacher: .....

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MA131 - Analysis 1  
Workbook 6 Assignments  
**Due in 10th Nov**

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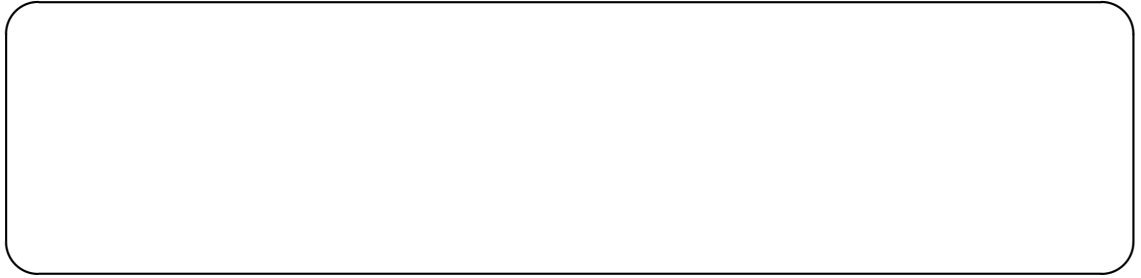
**Assignment 1**

Consider the sequence  $a_n = \left(1 + \frac{1}{n}\right)^n$ . Show that  $\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n+1}\right) \left(1 - \frac{1}{(n+1)^2}\right)^n$  and then use Bernoulli's inequality to show that  $a_{n+1} \geq a_n$ . Show that  $\left(1 + \frac{1}{2n}\right)^n = \frac{1}{\left(1 - \frac{1}{2n+1}\right)^n}$  and then use Bernoulli's inequality to show that  $\left(1 + \frac{1}{2n}\right)^n \leq 2$ . Hence show that  $(a_{2n})$  is bounded. Using the fact that  $(a_n)$  is increasing, show that it is bounded and hence convergent.


**Assignment 2**

Show that  $\left(1 - \frac{1}{n}\right) = \frac{1}{\left(1 + \frac{1}{n-1}\right)}$  and hence that  $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n$  exists.

**Assignment 3** Prove the Bolzano-Weierstrass Theorem. [Hint: It is easy using the results from workbook 3.]



**Assignment 4** Explain why  $(a_n)$  and  $(b_n)$  converge to a limit  $L$ . Explain why it is possible to find a subsequence  $(x_{n_i})$  so that  $x_{n_k} \in [a_k, b_k]$  and show that this subsequence is convergent.



**Assignment 5** Cleverclog's Test says that a sequence converges if and only if  $a_{n+1} - a_n \rightarrow 0$ . Give an example to show that Cleverclog's test is completely false (alas).



**Assignment 6** Suppose  $(a_n) \rightarrow a$ . Show that  $|a_n - a_m| \leq |a_n - a| + |a - a_m|$ . Use this fact to prove that  $(a_n)$  is Cauchy.

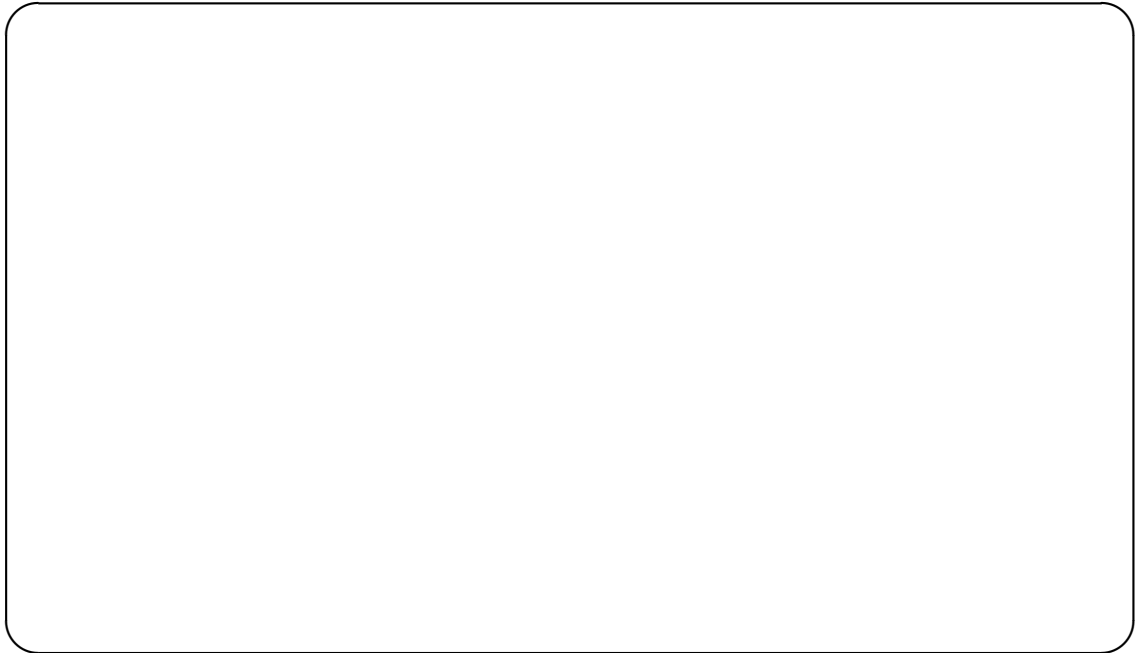


**Assignment 7**

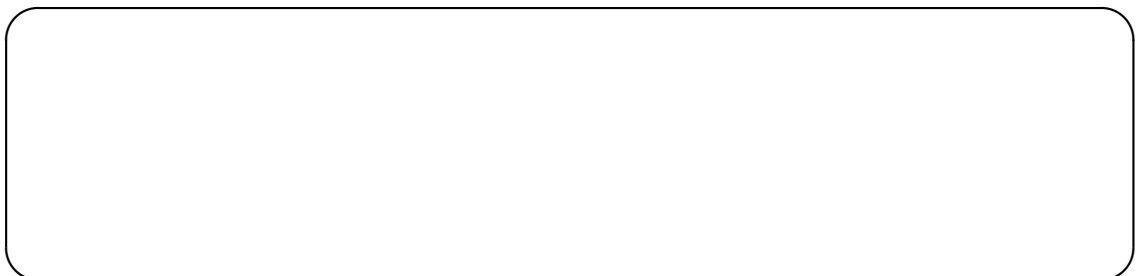
Let  $(a_n)$  be a Cauchy sequence. By putting  $\varepsilon = 1$  in the Cauchy criterion prove that every Cauchy sequence is bounded. Now use the Bolzano-Weierstrass Theorem together with the identity

$$|a_n - a| \leq |a_n - a_{n_i}| + |a_{n_i} - a|$$

to prove that every Cauchy sequence is convergent.

**Assignment 8**

Define a sequence by  $a_0 = 1$  and  $a_{n+1} = \cos(a_n/2)$ . Use the inequality  $|\cos(x) - \cos(y)| \leq |x - y|$  (which you may assume) to show that  $(a_n)$  is strictly contracting with contracting factor  $l = 1/2$ .

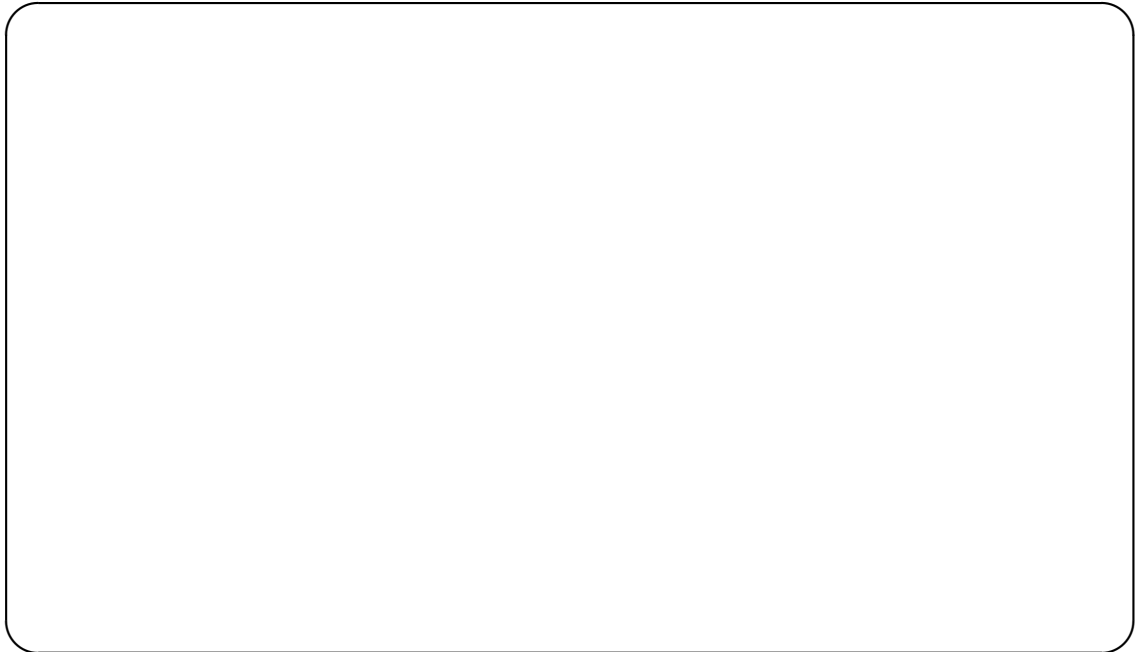


**Assignment 9**


The aim of this question is to show that a strictly contracting sequence  $(a_n)$  is Cauchy. Show by induction on  $n$  that  $|a_{n+1} - a_n| \leq |a_1 - a_0|l^n$ . Then suppose that  $n > m$  and use the triangle inequality in the form:

$$|a_n - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots + |a_{m+1} - a_m|$$

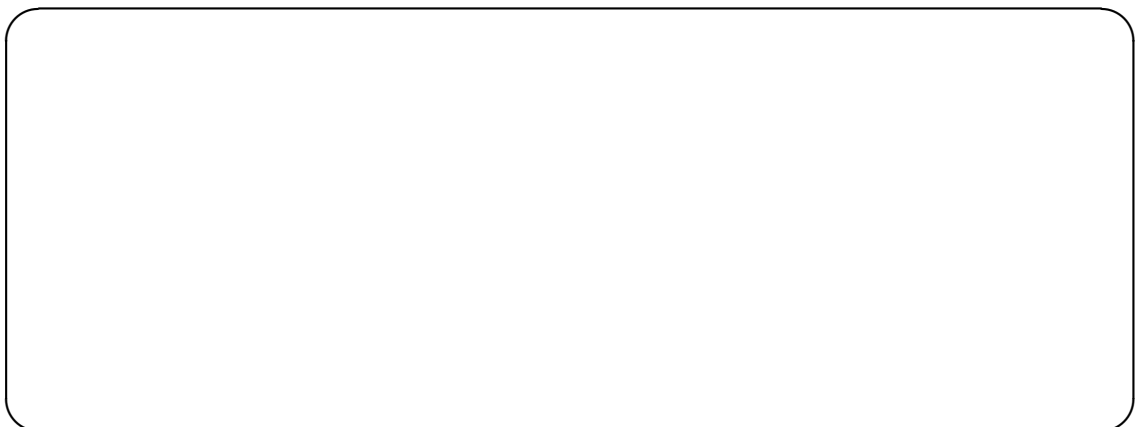
to show that  $(a_n)$  is Cauchy.

**Assignment 10**

Using the sequence and inequality given in Assignment ?? show that  $\cos(a_n/2) \rightarrow \cos(a/2)$ . Hence show that the sequence  $(a_n)$  converges to the unique solution of  $x = \cos(x/2)$ .

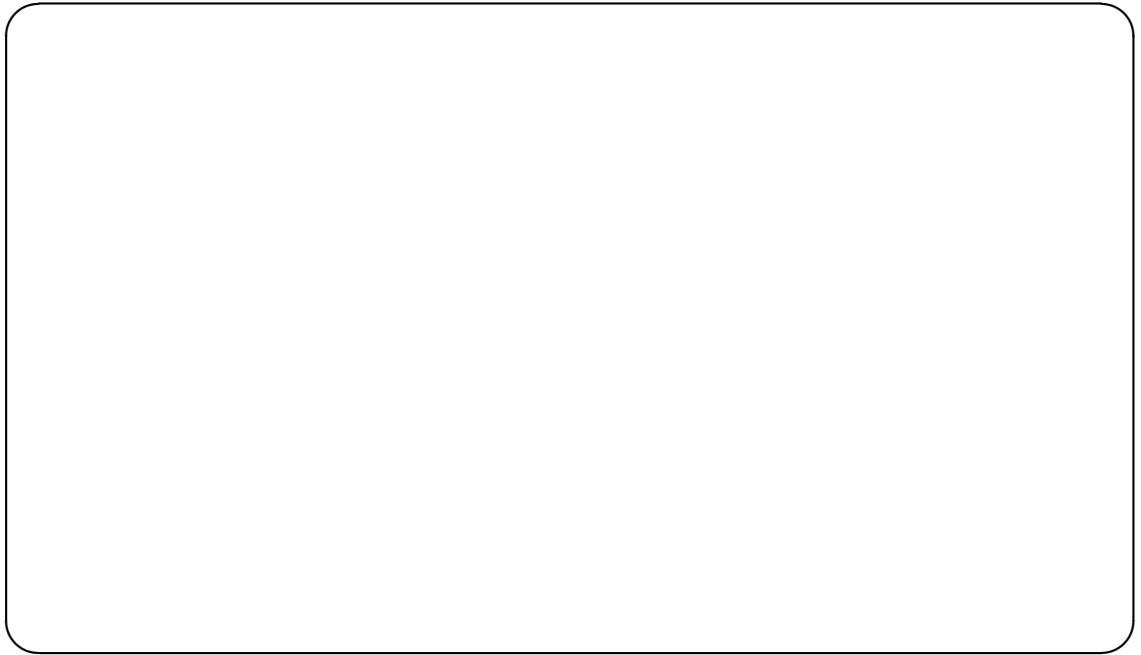
**Assignment 11**

How do the definitions for the decimal digits change for a negative real number  $x$ ?



**Assignment 12**

Check that the sequence of sums is monotonic and bounded. Use the bounded increasing sequence version of the Completeness Axiom to show that the infinite decimal represents a real number.

**Assignment 13**

Prove that  $0.999\dot{9} = 1$ .

**Assignment 14**

Suppose  $x = p/q$  for integers  $p, q$  where the only prime factors of  $q$  are 2's and 5's. Show that  $x$  has a terminating decimal representation. [Hint: show that  $x = p'/10^n$  for some integer  $p'$  and some  $n \geq 0$ .]



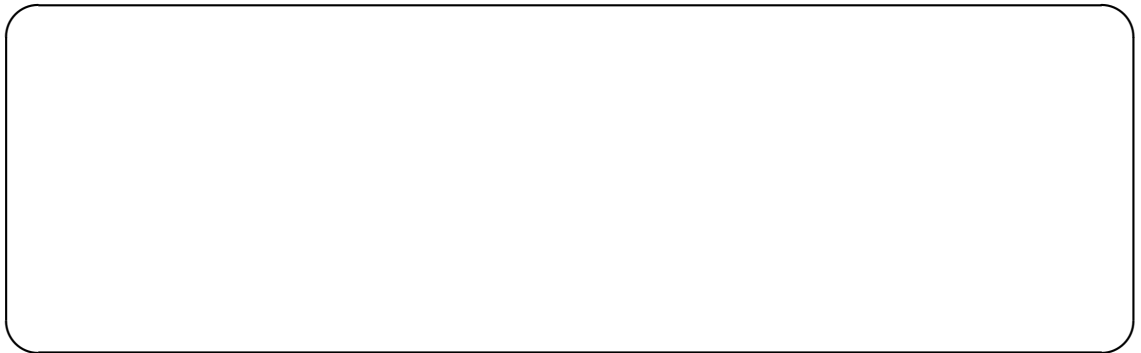
**Assignment 15** Show that if  $x$  has a terminating decimal expansion then  $x = p/q$  for integers  $p, q$  where the only prime factors of  $q$  are 2's and 5's.



**Assignment 16** Express the recurring decimal  $1.2345\bar{6}$  as a fraction.



**Assignment 17** To show this, suppose that  $x$  has a decimal representation that has recurring blocks of length  $k$ . Explain why  $10^k x - x$  must have a terminating decimal representation. Now use the characterisation of terminating decimals to show that  $x = \frac{p}{q(10^k - 1)}$  for some integers  $p, q$  where  $q$  has no prime factors except 2's and 5's.



**Assignment 18**

Explain in words why the above result is true, without giving a detailed proof. Include in your explanation the reason, when finding the repeating decimal for the number  $p/q$ , that the length of the repeating block will be at most  $q - 1$ .

