Name: ....   Supervisor: ....

Class Teacher: ....

MA131 - Analysis 1
Workbook 6 Assignments

Due in 10th Nov

Assignment 1 Consider the sequence $a_n = \left(1+\frac{1}{n}\right)^n$. Show that $\frac{a_{n+1}}{a_n} = \left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{(n+1)^2}\right)^n$ and then use Bernoulli's inequality to show that $a_{n+1} \geq a_n$.
Show that $\left(1+\frac{1}{2n}\right)^n = \frac{1}{\left(1-\frac{1}{2n+1}\right)^n}$ and then use Bernoulli's inequality to show that $\left(1+\frac{1}{2n}\right)^n \leq 2$. Hence show that (a_{2n}) is bounded. Using the fact that (a_n) is increasing, show that it is bounded and hence convergent.

Assignment 2 Show that $\left(1-\frac{1}{n}\right) = \frac{1}{\left(1+\frac{1}{n-1}\right)}$ and hence that $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$ exists.

Assignment 3 Prove the Bolzano-Weierstrass Theorem. [Hint: It is easy using the results from workbook 3.]

Assignment 4 Explain why (a_n) and (b_n) converge to a limit L. Explain why it is possible to find a subsequence (x_{n_i}) so that $x_{n_k} \in [a_k, b_k]$ and show that this subsequence is convergent.

Assignment 5 Cleverclog's Test says that a sequence converges if and only if $a_{n+1} - a_n \to 0$. Give an example to show that Cleverclog's test is completely false (alas).

Assignment 6 Suppose $(a_n) \to a$. Show that $|a_n - a_m| \leq |a_n - a| + |a - a_m|$. Use this fact to prove that (a_n) is Cauchy.

Assignment 7 Let (a_n) be a Cauchy sequence. By putting $\varepsilon = 1$ in the Cauchy criterion prove that every Cauchy sequence is bounded. Now use the Bolzano-Weierstrass Theorem together with the identity

$$|a_n - a| \leq |a_n - a_{n_i}| + |a_{n_i} - a|$$

to prove that every Cauchy sequence is convergent.

Assignment 8 Define a sequence by $a_0 = 1$ and $a_{n+1} = \cos(a_n/2)$. Use the inequality $|\cos(x) - \cos(y)| \leq |x - y|$ (which you may assume) to show that (a_n) is strictly contracting with contracting factor $l = 1/2$.

Assignment 9 The aim of this question is to show that a strictly contracting sequence (a_n) is Cauchy. Show by induction on n that $|a_{n+1} - a_n| \leq |a_1 - a_0| l^n$. Then suppose that $n > m$ and use the triangle inequality in the form:

$$|a_n - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots |a_{m+1} - a_m|$$

to show that (a_n) is Cauchy.

Assignment 10 Using the sequence and inequality given in Assignment **??** show that $\cos(a_n/2) \to \cos(a/2)$. Hence show that the sequence (a_n) converges to the unique solution of $x = \cos(x/2)$.

Assignment 11 How do the definitions for the decimal digits change for a negative real number x?

Assignment 12 Check that the sequence of sums is monotonic and bounded. Use the bounded increasing sequence version of the Completeness Axiom to show that the infinite decimal represents a real number.

Assignment 13 Prove that $0.999\dot{9} = 1$.

Assignment 14 Suppose $x = p/q$ for integers p, q where the only prime factors of q are 2's and 5's. Show that x has a terminating decimal representation. [Hint: show that $x = p'/10^n$ for some integer p' and some $n \geq 0$.]

Assignment 15 Show that if x has a terminating decimal expansion then $x = p/q$ for integers p, q where the only prime factors of q are $2's$ and $5's$.

Assignment 16 Express the recurring decimal $1.23\dot{4}5\dot{6}$ as a fraction.

Assignment 17 To show this, suppose that x has a decimal representation that has recurring blocks of length k. Explain why $10^k x - x$ must have a terminating decimal representation. Now use the characterisation of terminating decimals to show that $x = \frac{p}{q(10^k-1)}$ for some integers p, q where q has no prime factors except 2's and 5's.

Assignment 18 Explain in words why the above result is true, without giving a detailed proof. Include in your explanation the reason, when finding the repeating decimal for the number p/q, that the length of the repeating block will be at most $q - 1$.