A WEIL-PETERSSON TYPE METRIC ON SPACES OF METRIC GRAPHS

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0. The introduction

Given a compact topological surface V with negative Euler characteristic, the Teichmüller space $\mathrm{Teich}(V)$ describes the marked Riemann metrics (with constant curvature $\kappa = -1$) which it supports. More precisely, $\mathrm{Teich}(V)$ is the set of equivalence classes (V_g, ϕ), where V_g is a hyperbolic surface with Riemann metric g and $\phi : V \to V_g$ is a homeomorphism, with (V_{g_1}, ϕ_1) equivalent to (V_{g_2}, ϕ_2) if there is an isometry $\psi : V_{g_1} \to V_{g_2}$ such that $\psi \circ \phi_1$ is isotopic to ϕ_2. The moduli space $\mathrm{Mod}(V)$ describes the unmarked Riemann metrics on V and is obtained by quotienting $\mathrm{Teich}(V)$ by the Mapping Class Group of V.

There are several different metrics which can naturally be defined on $\mathrm{Teich}(V)$, for example, the Teichmüller metric and the Weil-Petersson metric, both of which are invariant under the Mapping Class Group and descend to $\mathrm{Mod}(V)$. There is a particularly illuminating formulation of the Weil-Petersson metric, due to Wolpert, in terms of the second derivative of the length of a typical geodesic on V [Wo]. Recently, a more dynamical characterization of this was proposed by Curt McMullen, who thereby extended the notion of the Weil-Petersson metric to a variety of settings (e.g., from Fuchsian to Quasi-Fuchsian groups) [Mc]. In this note we will introduce an analogue of the Weil-Petersson metric for families of metric graphs, and explore its properties through some simple examples.

To formulate an analogous definition for families of metric graphs we can replace the surface V by a finite (undirected) graph $\mathcal{G}$ with edge set $\mathcal{E}$. We can replace the Riemann metrics by edge weightings $l : \mathcal{E} \to \mathbb{R}^+$.

Definition. Let $\mathcal{M}_{\mathcal{G}}$ denote the space of all edge weightings $l : \mathcal{E} \to \mathbb{R}^+$ on $\mathcal{G}$.

Of course, the constant curvature -1 condition gives a natural normalization to metrics in $\mathrm{Teich}(V)$ or $\mathrm{Mod}(V)$ and it is natural to introduce a constraint on edge weightings on $\mathcal{G}$. One normalization (corresponding to curvature -1 metrics giving constant area $-2\pi\chi(V)$) would be that the sum of the edge lengths is equal to one, i.e., $\sum_{e \in \mathcal{E}} l(e) = 1$. However, the following, more dynamical approach, is useful.

Let us first consider a surface V. For any (not necessarily constant) negative curvature Riemannian metric g on on the surface V there are a countable infinity of closed geodesics γ with least period $l(\gamma)$. We can use these to define the *entropy* of the metric (i.e., the topological entropy of the associated geodesic flow)

$$h(g) = \lim_{T \to +\infty} \frac{1}{T} \log \mathrm{Card}\{\gamma : l(\gamma) \leq T\}. \tag{0.1}$$

Of course, for Riemann surfaces of constant curvature $\kappa < 0$, we have that $h(g) = \sqrt{|\kappa|}$ and therefore, if we normalize the surfaces to have $\kappa = -1$, then we have that $h = 1$. By analogy with (0.1), we may normalize edge weightings on a graph $\mathcal{G}$ by their *entropy* characterized as, say, the rate of growth of closed loops, i.e.,

$$h(l) = \lim_{T \to +\infty} \frac{1}{T} \log \mathrm{Card}\{\gamma : l(\gamma) \leq T\}, \tag{0.2}$$

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where $\gamma = (e_0, e_1, \cdots, e_n = e_0)$ is a closed cycle of edges in $\mathcal{G}$ (without backtracking) and $l(\gamma) = \sum_{i=0}^{n-1} l(e_i)$.

Remark. In fact, this is equivalent to the asymptotic volume which shows that $h(l)$ depends only on edge lengths of the metric tree $\mathcal{T}$ which is the universal cover of the graph $\mathcal{G}$, i.e., if $\Gamma \cong \pi_1(\mathcal{G})$ is the covering group and $v \in \mathcal{T}$ is any vertex then

$$h(l) = \lim_{T \to +\infty} \frac{1}{T} \log \mathrm{Card}\{g \in \Gamma : d(gv, v) \leq T\}.$$

The entropy leads to the following dynamical normalization of edge weightings on graphs.

Definition. Let

$$\mathcal{M}^1_{\mathcal{G}} = \{l : \mathcal{E} \to \mathbb{R}^+ : h(l) = 1\},$$

denote the space of all edge weightings with entropy $h(l) = 1$.

Since we do not fix a marking on $\mathcal{G}$ (given by a homotopy equivalence to the graph with one vertex and $r = \mathrm{rank}(\pi_1(\mathcal{G}))$ edges, say), $\mathcal{M}^1_{\mathcal{G}}$ is an analogue of $\mathrm{Mod}(V)$, rather than $\mathrm{Teich}(V)$. (A more precise analogue of $\mathrm{Mod}(V)$ is the complex obtained by attaching spaces $\mathcal{M}^1_{\mathcal{G}}$ for graphs $\mathcal{G}$ with a given fundamental group.)

In §1 we recall the definition of Wolpert and McMullen of the Weil-Petersson metric. In §2 we give a definition of an analogue of the Weil-Petersson metric for graphs. In §3 we describe a number of its properties, which help to illustrate the usefulness of the definition. In §4 we illustrate the definition for a variety of examples of graphs whose fundamental group is the free group on 2 generators. This draws an interesting connection with the Culler-Vogtmann space [Vo], also known as the *outer space*, in rank 2.

1. The Weil-Petersson Metric on Moduli Space

We begin by reviewing some of the results of Wolpert and McMullen for Riemann surfaces which are relevant to our analysis.

Let $\mathrm{Mod}(V)$ be the moduli space of Riemann metrics for a compact surface. We can consider a C^1 family of metrics $g_\lambda \in \mathrm{Mod}(V)$, $0 \leq \lambda \leq 1$. Let SV be the unit tangent bundle of V with respect to the metric g_{λ_0}, say. Let μ_{λ_0} be the corresponding Haar measure on SV. We denote by $\phi_t^{(\lambda_0)} : SV \to SV$ the geodesic flow. Since the geodesic flow for g_t is volume preserving then

$$\int \dot{g}_{\lambda_0}(v, v)\, d\mu_{\lambda_0}(v) = 0, \tag{1.1}$$

(cf. [Besse]), where we consider the expansion

$$g_\lambda = g_{\lambda_0} + \dot{g}_{\lambda_0}(\lambda - \lambda_0) + O((\lambda - \lambda_0)^2).$$

Definition. If we write $F(v) := \dot{g}_{\lambda_0}(v, v)$ we can define the *variance* by the following equivalent formulae

$$\sigma^2 := \lim_{T \to +\infty} \frac{1}{T} \int \left(\int_0^T F(\phi_t v)\, dt \right)^2 d\mu(v) = \int_{-\infty}^{+\infty} \left(\int_{SV} F(\phi_t v) F(v)\, d\mu(v) \right) dt$$

(cf. [KS]).

We are now able to formulate the characterization of the Weil-Petersson metric by McMullen [Mc].

Proposition 1.1 (McMullen). *The Weil-Petersson metric* $\|\dot{g}\|^2_{WP}$ *is proportional to* σ^2.

Proposition 1.1 comes from a reinterpretation of Wolpert's formula for the Weil-Petersson metric in terms of the second derivative of lengths of generic geodesics. A simple interpretation for σ^2 is via the Central Limit Theorem, which we can express in terms of closed orbits. Let γ be a free homotopy class and let $l_g(\gamma)$ denote its length with respect to the metric g_λ.

Proposition 1.2 (Central Limit Theorem for surfaces). *For $a < b$ we have*

$$\lim_{T\to+\infty} Te^{-T}\mathrm{Card}\left\{\gamma \,:\, l_{g_{\lambda_0}}(\gamma) \le T \text{ and } \frac{1}{\sqrt{T}} \left.\frac{\partial \log l_{g_t}(\gamma)}{\partial \lambda}\right|_{\lambda=\lambda_0} \in [a,b]\right\}$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\int_a^b e^{-y^2/(2\sigma^2)}\, dy.$$

Proof. This follows from [Ra] once we also observe that

$$\left.\frac{\partial \log l_{g_t}(\gamma)}{\partial \lambda}\right|_{\lambda=\lambda_0} = \frac{1}{l_{g_{\lambda_0}}(\gamma)}\left.\frac{\partial l_{g_\lambda}(\gamma)}{\partial \lambda}\right|_{\lambda=\lambda_0} = \frac{1}{l_{g_{\lambda_0}}(\gamma)}\int_0^{l_{g_{\lambda_0}}(\gamma)} \dot{g}_{\lambda_0}(\phi_t^{(\lambda_0)} v_\gamma)\, dt$$

where $v_\gamma \in T_\gamma V$. □

Remark. There are several alternative equivalent definitions of σ. It can be interpreted as the value at zero of the spectral density. It can also be written in terms of an asymptotic quantity for lengths of weighted closed geodesics, and thus in terms of variants on zeta functions.

2. The Weil-Petersson Type Metric on Spaces of Metric Graphs.

We can associate to the graph $\mathcal{G}$ a subshift of finite type whose states are oriented edges of $\mathcal{G}$. More precisely:

1. Each edge $e \in \mathcal{E}$ corresponds to two oriented edges which, abusing notation, we shall denote by e and $\overline{e}$. We shall write $\mathcal{E}^o$ for the set of oriented edges; and
2. We say that $e' \in \mathcal{E}^o$ follows $e \in \mathcal{E}^o$ if e' begins at the terminal endpoint of e. We then define a $|\mathcal{E}^o| \times |\mathcal{E}^o|$ matrix A, with rows and columns indexed by $\mathcal{E}^o$, by

$$A(e,e') = \begin{cases} 1 \text{ if } e' \text{ follows } e \text{ and } e' \neq \overline{e} \\ 0 \text{ otherwise.}\end{cases}$$

The shift space

$$\Sigma_A = \{\underline{e} = (e_n)_{n\in\mathbb{Z}} \in (\mathcal{E}^o)^{\mathbb{Z}} \,:\, A(e_n, e_{n+1}) = 1 \ \forall n \in \mathbb{Z}\}$$

can be naturally identified with the space of all (two-sided) infinite paths (with a distinguished zeroth edge) in the graph $\mathcal{G}$. Then Σ_A is a compact zero dimensional space with respect to the Tychanoff product topology. We define the shift map $\sigma : \Sigma_A \to \Sigma_A$ by $(\sigma\underline{e})_n = e_{n+1}$, $n \in \mathbb{Z}$. Clearly, this is a homeomorphism.

Given any continuous function $f : \Sigma_A \to \mathbb{R}$ we can define the *pressure function* $P : C(\Sigma_A, \mathbb{R}) \to \mathbb{R}$ defined by

$$P(f) = \limsup_{n\to+\infty} \frac{1}{n}\log\left(\sum_{\sigma^n x = x} e^{f(x)+f(\sigma x)+\cdots+f(\sigma^{n-1}x)}\right).$$

We can associate to the directed graphs the weightings of the edges coming from those for the undirected graph. This leads to a locally constant function $l : \Sigma_A \to \Sigma_A$ defined by $l((e_n)_{n=-\infty}^{\infty}) = l(e_0)$. This function satisfies $l \circ i = l$ under the involution $i : \Sigma_A \to \Sigma_A$ given by $i((e_n)_{n=-\infty}^{\infty}) = (e_{-n})_{n=-\infty}^{\infty}$.

The following results on the function l are easily seen,

Lemma 2.1.

1. *The entropy h is characterized by $P(-h(l)l) = 0$.*
2. *The entropy $\mathcal{M}_{\mathcal{G}} \ni l \mapsto h(l) \in \mathbb{R}^+$ varies analytically for $l > 0$.*

Proof. The first result is standard and follows from the variational principle

$$P(-tl) = \sup\left\{h(m) - t\int l\,dm \,:\, m \text{ is a } \sigma\text{-invariant probability measure}\right\},$$

for $t \in \mathbb{R}$ [PP]. The second result follows by the analyticity of the pressure function, as seen from its characterization in terms of the maximal eigenvalue of a positive matrix, and the implicit function theorem. □

We can define an analogue of the tangent space (at $l \in \mathcal{M}_{\mathcal{G}}$) to be a subspace of the locally constant functions (depending only on the zeroth coordinate e_0). More precisely, we first define the Markov measure μ associated to the matrix $Q(e,e') = A(e,e')e^{-l(e)}$. Let $pQ = p$ be the left eigenvector.

Definition. We define the *tangent space* to $\mathcal{M}_{\mathcal{G}}$ at $l \in \mathcal{M}_{\mathcal{G}}$ by

$$T_l\mathcal{M}_{\mathcal{G}} = \left\{f(\underline{e}) = f(e_0) \,:\, \sum_{e\in\mathcal{E}^o} f(e)p_e = 0\right\},$$

i.e., the signed edge weightings whose appropriately weighted sum is zero.

This is a finite dimensional space (indeed the dimension is precisely $|\mathcal{E}^o| - 1$). Alternatively, we can associate to l the measure $\mu = \mu_l$ on Σ corresponding to the equilibrium state for $-h(l)l$, i.e., the unique σ-invariant probability measure μ on Σ_A such that

$$P(-h(l)l) = h(\mu) - \int l\,d\mu.$$

Then we could write $T\mathcal{M}_{\mathcal{G}} = \{f : \Sigma_A \to \mathbb{R} \,:\, \mu(f) = 0 \text{ and } f(\underline{e}) = f(e_0)\}$.

Definition. Given $f \in T_l\mathcal{M}_{\mathcal{G}}$ we define the *variance* by

$$\sigma^2(f) = \lim_{n\to+\infty} \frac{1}{n}\int (f^n(\underline{e}))^2\,d\mu(\underline{e}).$$

Since we are dealing with functions that depend only on one coordinate the formula for the variance is particularly simple.

Lemma 2.2. *Given $f \in T_l\mathcal{M}_{\mathcal{G}}$ we can write*

$$\sigma^2(f) = \sum_{e\in\mathcal{E}^o} p_e f(e)^2.$$

Proof. We can write

$$\begin{aligned}
&\frac{1}{n}\int\left(\sum_{i=0}^{n-1} f(e_i)\right)^2 d\mu(\underline{e}) = \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int f(e_i)f(e_j)\,d\mu(\underline{e}) \\
&= \frac{1}{n}\sum_{i=0}^{n-1}(n-i)\int f(e_0)f(e_i)\,d\mu(\underline{e}) \\
&= \int |f(e_0)|^2\,d\mu(\underline{e}) + \frac{1}{n}\sum_{k=1}^{n}(n-k)\sum_{e_0}\sum_{i_1\cdots i_{k-1}}\sum_{e_k}\mu[x_0,i_1,\cdots,i_{k-1},e_k]f(e_0)f(e_k) \\
&= \int |f(e_0)|^2\,d\mu(\underline{e}) + \frac{1}{n}\sum_{e_0}p_{e_0}f(e_0)\sum_{k=1}^{n}(n-k)\sum_{i_1\cdots i_{k-1}}\sum_{e_k}P(e_0,i_1)\cdots P(i_{k-1},e_k)f(e_k) \\
&= \int |f(e_0)|^2\,d\mu(\underline{e}) + \frac{1}{n}\sum_{e_0}p_{e_0}f(e_0)\sum_{k=1}^{n}(n-k)\sum_{e_k}P^n(e_0,e_k)f(e_k) \\
&= \int |f(e_0)|^2\,d\mu(\underline{e}) + \frac{1}{n}\left(\sum_{e_0}p_{e_0}f(e_0)\right)\sum_{k=1}^{n}(n-k)\left(\sum_{e_k\in\mathcal{E}}p_{e_k}f(e_k)+O(\theta^n)\right) \\
&= \int |f(e_0)|^2\,d\mu(\underline{e}) + o(1),
\end{aligned}$$

for some $0<\theta<1$, since $\sum_{e\in\mathcal{E}^o}p(e)f(e)=0$. □

Finally, we are in a position to define the analogue of the Weil-Petersson metric in the context of weighted graphs.

Definition (Weil-Petersson metric for graphs). We can define a norm on the tangent space $T_l\mathcal{M}_\mathcal{G}$ by

$$\|f\|_{WP}^2 = \sigma^2(f),$$

where $f\in T_1\mathcal{M}_\mathcal{G}$. We can then define the length of any continuously differentiable curve $\gamma:[0,1]\to\mathcal{M}_\mathcal{G}$ by

$$L(\gamma) = \int_0^1 \|\dot\gamma\|_{WP}\,dt$$

and thus define a path space metric on $\mathcal{M}_\mathcal{G}$ by $d(l_1,l_2)=\inf_\gamma\{L(\gamma)\}$, where the infimum is taken over all continuously differentiable curves with $\gamma(0)=l_1$ and $\gamma(1)=l_2$.

3. Properties of the Weil-Petersson Metric for Graphs

In this section we want to establish some of the basic properties of the Weil-Petersson metric for graphs. We begin with the following analyticity result.

Theorem 1 (Analyticity of metric). *The metric $\|\cdot\|_{WP}$ is real analytic on $\mathcal{M}_\mathcal{G}$.*

This follows immediately from Lemma 2.1.

It is known that the Weil-Petersson metric on Teichmüller space is incomplete, but has strictly negative curvature. It is natural to ask if there are analogous results in the case of spaces of graphs. In this context we have the following:

Theorem 2 (Properties of the metric).
- (i) *There exist examples of graphs for which the metric $\|\cdot\|_{WP}$ is not complete*
- (ii) *There exist examples of graphs for which the curvature of the metric $\|\cdot\|_{WP}$ takes both negative and positive values.*

We present examples of the properties in (i) and (ii) in later sections.

The following can be compared with Wolpert's Theorem [Wolpert] gives, perhaps, the most intuitive definition in the present setting. Consider a family of closed paths $\{\gamma_n\}$ which become evenly distributed with respect to μ_{λ_0} (i.e., γ corresponds to $\sigma^n \underline{e} = \underline{e}$ with $l_\lambda(\gamma) = \sum_{j=0}^{n-1} l_\lambda(e_j)$ and the measure $\frac{1}{n}\sum_{j=0}^{n-1} \delta_{\sigma^j \underline{e}}$ converges in the weak star topology to μ_{λ_0} on Σ).

Theorem 3 (Random Geodesic Theorem). *Let $l_\lambda \in \mathcal{M}_\mathcal{G}$ be a family of length functions for $0 \leq \lambda \leq 1$. Then for $0 < \lambda_0 < 1$,*

$$\lim_{n\to+\infty} -\frac{d^2}{d\lambda^2} \log\left(l_\lambda(\gamma_n)\right)\Big|_{\lambda=\lambda_0} = \|v\|^2_{WP}$$

where $v = l'(\lambda_0) \in T\mathcal{M}$ is the tangent vector at $\lambda = \lambda_0$.

Proof. At the symbolic level we can characterize $h(l)$ and μ symbolically in terms of the pressure $P(\cdot)$. In particular, differentiating the expression $P(-l_\lambda) = 0$ on $\mathcal{M}_\mathcal{G}$ we see that $\int \dot{l}_{\lambda_0}\, d\mu_{l_\lambda} = 0$. Differentiating again we see that $\mathrm{var}(\dot{l}_{\lambda_0}) + \int \ddot{l}_{\lambda_0}\, d\mu_{l_{\lambda_0}} = 0$. We can write that

$$\frac{d^2}{d\lambda^2}\log\left(l_\lambda(\gamma)\right)\Big|_{\lambda=\lambda_0} = \frac{d}{d\lambda}\left(\frac{\dot{l}_\lambda(\gamma)}{l_\lambda(\gamma)}\right)\Big|_{\lambda=\lambda_0} = \frac{\ddot{l}_{\lambda_0}(\gamma)}{l_{\lambda_0}(\gamma)} - \left(\frac{\dot{l}_{\lambda_0}(\gamma)}{l_{\lambda_0}(\gamma)}\right)^2.$$

For typical geodesics $\frac{\dot{l}_{\lambda_0}(\gamma_n)}{l_{\lambda_0}(\gamma_n)} \to 0$ and $\frac{\ddot{l}_{\lambda_0}(\gamma_n)}{l_{\lambda_0}(\gamma_n)} \to \int \ddot{l}_{\lambda_0}\, d\mu_{l_{\lambda_0}}$. □

4. Some Examples

In this section we will consider some simple examples. In particular, we consider the graphs whose fundamental group in F_2, the free group of rank 2. In subsequent sections we will use them to illustrate properties of the metric.

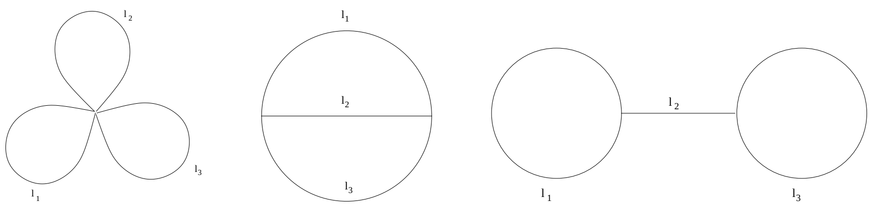

Figure 1. The three examples: (I) A rose (with $n = 2$); (II) A belt buckle; and (III) A dumbell

4.1. Example I (A rose).

A particularly simple example of an undirected graph is the rose with n-petals ($n \geq 2$).

Lemma 4.1. *The entropy h of the rose example is characterized by*

$$\sum_{i=1}^{k} e^{-hl_i} = 1$$

Proof. This is immediate from the definitions. □

For the figure eight graph (corresponding to $n = 2$) with edge lengths $l_1, l_2 > 0$ the entropy h has to satisfy $1 = e^{-hl_1} + e^{-hl_2}$. Thus for any $l_1 > 0$ we can find a $l_2 = l_2(l_1) > 0$ for which $h(l_1, l_2) = 1$.

4.2. Example II (Belt buckles). We can consider a graph with two vertices, each connected by three edges. By shrinking any of the edges to a point we get the homotopy equivalent figure eight graph (i.e., a 2-rose, as above).

In the associated tree the valency is three and the three edges meeting at a vertex and have the three lengths $l_1, l_2, l_3 > 0$.

Lemma 4.2. *The entropy $h = h(l) > 0$ is characterized by*

$$2 = e^{h(l_1+l_2+l_3)} - e^{hl_1} - e^{hl_2} - e^{hl_3}$$

In particular, we see that for $l = (l_1, l_2, l_3) \in \mathcal{M}_{\mathcal{G}}^1$ we require that $l_3 > 0$ satisfies

$$e^{-l_3} = \frac{e^{l_1+l_2} - 1}{2 + e^{l_1} + e^{l_2}}. \tag{4.1}$$

Proof. Let us define a matrix

$$M_l(h) = \begin{pmatrix} 0 & e^{-hl_1} & e^{-hl_1} \\ e^{-hl_2} & 0 & e^{-hl_2} \\ e^{-hl_2} & e^{-hl_2} & 0 \end{pmatrix}.$$

Then the entropy is characterized as the value for which the maximal eigenvalue λ_l of M_l is equal to 1 . Since we can compute:

$$\det(I - M_l) = e^{-l_1-l_2-l_3}(-2 + e^{h(l_1+l_2+l_3)} - e^{hl_1} - e^{hl_2} - e^{hl_3})$$

the identity in the statement follows. □

Remark. Unlike the previous example, we see that there are values $l_1, l_2 > 0$ for which we can find no corresponding value of l_3 for a graph with entropy 1.

4.3. Example III (Dumbells). We can consider a graph with two vertices, each being the start and end of an edge, and joined by a third edge.

Lemma 4.3. *The entropy h of the dumbell example is characterized by*

$$4 = e^{-2l_2h} + (e^{-(2l_2+l_3)h} - e^{-(2l_1+l_2)h}) + e^{-(l_1+2l_2+l_3)h}.$$

Proof. This follows from considering the rate of growth of closed loops. Let us define a matrix

$$Q_l = \begin{pmatrix} e^{-l_1} & 0 & 0 & 0 & e^{-l_1} & 0 \\ e^{-l_2} & 0 & 0 & e^{-l_2} & 0 & 0 \\ 0 & e^{-l_3} & e^{-l_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-l_1} & e^{-l_1} & 0 \\ 0 & 0 & 0 & e^{-l_2} & 0 & e^{-l_2} \\ 0 & e^{-l_3} & 0 & 0 & 0 & e^{-l_3} \end{pmatrix}.$$

The maximal eigenvalue $\lambda = \lambda(l_1, l_2, l_3)$ of the natural matrix satisfies

$$\det(zI - Q_l) = -4 + e^{2l_2}\lambda^2 + (e^{2l_2+l_3} - e^{2l_1+l_2})\lambda^3 + e^{l_1+2l_2+l_3}\lambda^4 = 0.$$

This completes the proof of the lemma. □

This example also allows us to see that the entropy may be unbounded on the space of all graphs.

Corollary 4.3.1. *In this example we see that if $l_2 \to 0$ and $l_3 \to 0$ then the entropy tends to infinity (whenever the length l_1 is fixed).*

By shrinking the edge joining the two vertices, we get the homotopy equivalent figure eight graph (i.e., a 2-rose, as in Example I above).

4.4. Culler-Vogtmann space in rank 2. The three examples above occur very naturally in the context of the study of outer automorphisms of graphs. We recall that the Culler-Vogtmann space (or outer space) $CV(F_2)$ corresponds to metrics (i.e., length functions) on the three types of graph above in the same free homotopy class [Vo].

The graphs in Examples II and III are parameterized by the three lengths $l_1, l_2, l_3 > 0$ of the edges and so, subject to the normalization that $h(l_1, l_2, l_3) = 1$, the corresponding moduli space is two dimensional However, these are joined by one dimensional curves corresponding to the graphs in Example II. Different simplicies corresponding to Example II are joined together along three edges. This is also where one edge of the simplicies corresponding to Example III, which appear like "fins".

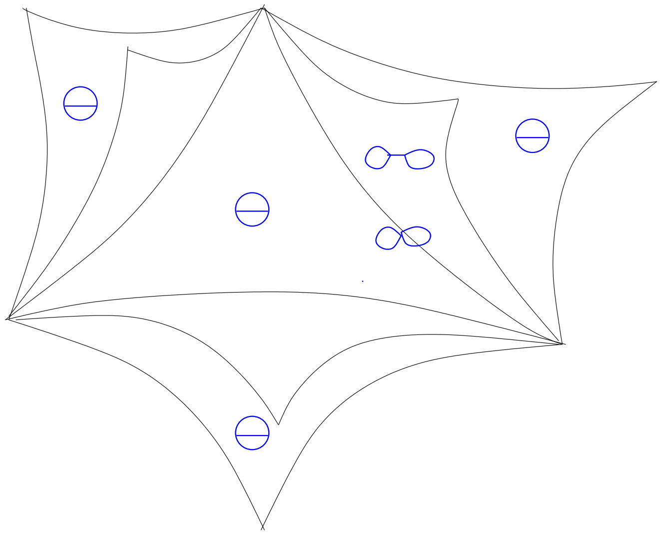

Figure 2. The three types of examples of graphs corresponding to $CV(F_2)$

5. Proof of Theorem 2

We can use the examples introduced in the previous section to provide those explicit examples needed for Theorem 2.

5.1. Part (a): Incompleteness of the metric. We consider the particular case of Example I with $n = 2$, i.e., where there are two undirected loops. The associated directed graph has a transition matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

and an associated weighted matrix (with $h = 1$)

$$Q_l = \begin{pmatrix} e^{-l_1} & e^{-l_1} & 0 & e^{-l_1} \\ e^{-l_2} & e^{-l_2} & e^{-l_2} & 0 \\ 0 & e^{-l_1} & e^{-l_1} & e^{-l_1} \\ e^{-l_2} & 0 & e^{-l_2} & e^{-l_2} \end{pmatrix}.$$

We can consider the right eigenvalue $Qv = v$ and define a stochastic matrix $P(i,j) = Q(i,j)v_i/v_j$. The maximal eigenvalue in the case of two loops is checked to be

$$\lambda(l_1, l_2) = \frac{1}{2}\left(e^{-l_1} + e^{-l_2} + \sqrt{e^{-2l_1} + e^{-2l_2} + 14e^{-(l_1+l_2)}}\right).$$

Since $\mathcal{M}^1_{\mathcal{G}}$ is one dimensional, we see that the lengths l_1 and l_2 to be related by $\lambda(l_1, l_2) = 1$. In particular, we can solve

$$e^{-l_1} = \frac{1 - e^{-l_2}}{1 + 3e^{-l_2}}. \tag{5.1}$$

The stochastic matrix P then has an asymptotic form

$$P(x) = \begin{pmatrix} -3+4x & 2-2x, & 0 & 2-2x \\ x/2 & 1-x & x/2 & 0 \\ 0 & 2-2x & -3+4x & 2-2x \\ x/2 & 0 & x/2 & 1-x \end{pmatrix} + O(1-x)$$

where $x = e^{-l_2}$. The right eigenvector $p(x)$ associated to the eigenvalue 1 is then easily seen to be

$$p = \left(\frac{x}{4}, 1-x, \frac{x}{4}, 1-x\right).$$

Using (5.1) we can consider a curve

$$c : l_2 \mapsto (l_1, l_2) = \left(-\log\left((1 - e^{-l_2})/(1 + 3e^{-l_2})\right), l_2\right) \in \mathcal{M}$$

parameterized by l_2. The tangent vector is then

$$c'(l_2) := \left(\frac{4e^{l_2}}{e^{2l_2} + 2e^{l_2} - 3}, 1\right) \in \mathcal{M}^1_{\mathcal{G}}.$$

However, we have to scale the tangent vector according to the variation to have length

$$\sqrt{\left(\frac{4e^{l_2}}{e^{2l_2} + 2e^{l_2} - 3}\right)^2 \left(\frac{1 - e^{-l_2}}{1 - 3e^{-l_2}/4}\right) + \left(\frac{e^{-l_2}}{4(1 - 3e^{-l_2}/4)}\right)}.$$
$$\sim \sqrt{\frac{4}{3l_2} + 1} \sim \frac{2}{\sqrt{3l_2}} \text{ as } l_2 \to 0.$$

In particular, since $\int_0^1 x^{-1/2}dx$ is convergent we see that the metric is incomplete, i.e., we arrive at $l_2 = 0$ in finite time with respect to the metric.

5.2. Part (b): Curvature of the metric. We consider formulae for the Gaussian curvature of the metric for Example II, with two vertices and 3 edges, with lengths $l_1, l_2, l_3 > 0$.

To compute the variance associated to the natural Markov measure, we have to associate to the matrix $M_{abc}(h)$, with this choice of c. The associated stochastic matrix becomes

$$P_l = \begin{pmatrix} 0 & \frac{1+e^{-l_1}}{1+e^{l_2}} & \frac{-e^{-l_1}+e^{l_2}}{1+e^{l_2}} \\ \frac{1+e^{-l_2}}{1+e^{l_1}} & 0 & \frac{-e^{-l_2}+e^{l_1}}{1+e^{l_1}} \\ \frac{1+e^{l_2}}{2+e^{l_1}+e^{l_2}} & \frac{1+e^{l_1}}{2+e^{l_1}+e^{l_2}} & 0 \end{pmatrix}$$

The right eigenvalue for this is

$$p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \text{ where } \begin{aligned} p_1 &= \frac{e^{l_1}(1+e^{l_2})^2}{2(-1+3e^{l_1+l_2}+e^{2l_1+l_2}+e^{l_1+2l_2})} \\ p_2 &= \frac{e^{l_2}(1+e^{l_1})^2}{2(-1+3e^{l_1+l_2}+e^{2l_1+l_2}+e^{l_1+2l_2})} \\ p_3 &= \frac{-2-e^{l_1}-e^{l_2}+2e^{l_1+l_2}+e^{2l_1+l_2}+e^{l_1+2l_2}}{2(-1+3e^{l_1+l_2}+e^{2l_1+l_2}+e^{l_1+2l_2})} \end{aligned}$$

In order to associate the metric we want to define an inner product on the tangent space to $\mathcal{M}^1_{\mathcal{G}}$. We can consider the parameterization $\psi : (l_1, l_2) \mapsto (l_1, l_2, l_3) \in \mathcal{M}^1_{\mathcal{G}}$, where $l_3 = l_3(l_1, l_2)$ is given by (4.1). We then define

$$D_1\psi(l_1, l_2, l_3) := \left(1, 0, \frac{\partial \psi}{\partial l_1}\right) \text{ and } D_2\psi(l_1, l_2, l_3) := \left(0, 1, \frac{\partial \psi}{\partial l_2}\right).$$

We can then define the metric (in terms of its first fundmental form)

$$ds^2 = E(l_1, l_2)dl_1^2 + 2F(l_1, l_2)dl_1dl_2 + G(l_1, l_2)dl_2^2,$$

where

$$\begin{aligned}
E(l_1, l_2) &= \mathrm{var}(D_1\psi, D_1\psi) = \sum_{i=1}^{3} p_i(l_1, l_2)(D_1\psi_i(l_1, l_2, l_3))^2 \\
&= 1 + p_3(l_1, l_2)(D_1\psi_i(l_1, l_2, l_3))^2,
\end{aligned}$$

$$\begin{aligned}
F(l_1, l_2) &= \mathrm{var}(D_1\psi, D_2\psi) = \sum_{i=1}^{3} p_i(l_1, l_2)(D_1\psi_i(l_1, l_2, l_3))(D_2\psi(l_1, l_2, l_3)) \\
&= 1 + p_3(l_1, l_2)(D_1\psi_i(l_1, l_2, l_3))(D_2\psi_i(l_1, l_2, l_3)),
\end{aligned}$$

$$G(l_1, l_2) = \mathrm{var}(D_1\psi, D_1\psi) = \sum_{i=1}^{3} p_i(l_1, l_2)(D_2\psi_i(l_1, l_2, l_3))^2.$$

The explicit formulae are easy to compute, but rather lengthy to write down. We can write the Gaussian curvature of $\mathcal{M}^1_{\mathcal{G}}$ at a point (l_1, l_2, l_3) using the following standard formula [Gr].

Lemma 5.1 (Brioschi formula). *If a metric has local coordinates*

$$ds^2 = E(u, v)du^2 + F(u, v)dudv + G(u, v)dv^2$$

then the curvature is given by

$$\begin{aligned}
\kappa(l_1, l_2, l_3) &= \frac{1}{(EG - F^2)^2} \\
&\times \left(\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - -\frac{1}{2}G_{vv} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right).
\end{aligned}$$

The plot of the curvature for $0 < a, b < 2.5$ is presented in the Figure 2

To complete the proof of Theorem 2 (b) it only suffices to show that the curvature takes both positive values and negative values. We can evaluate numerically curvature at $(l_1, l_2) = (1, 10)$ to give $\kappa(1, 10) = 0 \cdot 0457317 \cdots$. We can evaluate the curvature at $(a, b) = (1, 1)$ to give $\kappa(1, 1) = -0 \cdot 507639$. □

Remark. We can also consider the asymptotic curvature as $l_1, l_2 \to 0$. The value of the curvature at the origin is $\kappa(0, 0) = -\frac{25}{16}$ and

$$\begin{aligned}
\kappa(a, b) = &-\frac{25}{16} + \frac{63}{64}a + \frac{63}{64}b - \frac{635}{1024}a^2 - \frac{779}{512}ab - \frac{635}{1024}b^2 \\
&+ O\left(a^3, a^2b, ab^2, b^3\right).
\end{aligned}$$

This is in contrast to the situation for the usual Weil-Petersson metric, for which the curvature tends to minus infinity in the cusp.

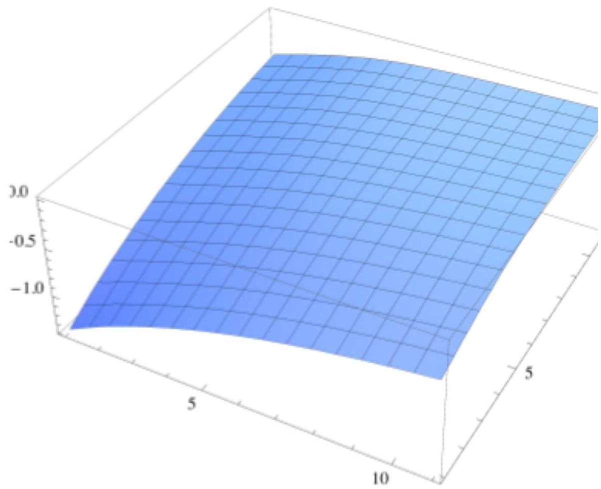

FIGURE 3. The curvature of a portion of $\mathcal{M}_{\mathcal{G}}^1$.

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