VAN DER WAERDEN'S THEOREM ON ARITHMETIC PROGRESSIONS

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Abstract. This is a short exposition of the dynamical approach to the proof of van der Waerden's theorem on arithmetic progressions.

0. Introduction

Let $\mathbb{Z} = \cup_{i=1}^N C_i$ be a finite partition of the integers (i.e., $C_i \cap C_j$ for $i \neq j$). Of course, at least one of sets must have infinitely many integers, but the following theorem shows there is a far stronger result.

Theorem (van der Waerden, 1927 [6]). *There exists $1 \leq i \leq N$ such that C_i contains arithmetic progressions of arbitrary length, (i.e., $\forall k \geq 1$, $c \in \mathbb{Z}$, and $d \in \mathbb{N}$ such that $c + jd \in C_i$ for $0 \leq j \leq k-1$)*

Here k is called the *length* of the arithmetic progression.

Examples.

(1) We could let C_1 be the odd numbers and let C_2 be the even numbers, then both sets have arithmetic progressions of all lengths;
(2) We could let C_1 be $\pm$prime numbers and C_2 the compliment.

The original proof of van der Waerden is combinatiorial, and was one of the "Three pearls of number theory" in Khintchine's famous book [5]. There is a dynamical approach to this theorem due to Furstenberg and Weiss [4]. We can associate to the partition a sequence $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma = \{1, \cdots, N\}^{\mathbb{Z}}$ defined by

$$x_n = i \text{ if } n \in C_i.$$

If Σ has the usual (Tychanoff product) topology, we let $Y = \overline{\cup_{n\in\mathbb{Z}} \sigma^n x}$ be the closure of the orbit of x. We define the continuous *shift map* $\sigma : Y \to Y$, by $\sigma((w_n)_{n\in\mathbb{Z}}) = (w_{n+1})_{n\in\mathbb{Z}}$. A basic key observation is the following:

Lemma 1 (Dynamical approach to arithmetic progressions). *Assume that for some $1 \leq i \leq N$ and $[i] = \{w = (w_n)_{n\in\mathbb{Z}} : w_0 = i\}$ we have that*

$$Y \cap [i] \cap \sigma^{-d}[i] \cap \sigma^{-2d}[i] \cap \cdots \cap \sigma^{-(k-1)d}[i] \neq \emptyset,$$

for some $d \geq 1$ and $k \geq 1$ C_i contains an arithmetic progression of length k.

Proof. Since the intersection is an open set in Y, it follows that it contains $\sigma^n x$, for some $n \in \mathbb{Z}$. However, this means $x_{n+jd} = i$ for $0 \leq j \leq d-1$. □

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Minimal subsets

It helps to consider the restriction of σ to a smaller set. We say that a σ-invariant subset $X \subset Y$ is *minimal* if the restriction $\sigma : X \to X$ is minimal (i.e., for every $w \in X$ we have $X = \overline{\cup_{n\in\mathbb{Z}}\sigma^n w}$).

Lemma 2 (Existence of Minimal sets).

(1) *There exists a minimal subset $X \subset Y$.*
(2) *For any open set $V \subset X$ there exists $M > 0$ such that $X = \cup_{|n|\le M}\sigma^{-n}V$*

Proof. Following [7], we can choose an enumeration $\{U_k\}_{k=1}^{\infty}$ of all the (non-empty) cylinder sets $[\alpha_{-L}, \cdots \alpha_L] := \{w \in X : w_{-L} = \alpha_{-L}, \cdots, w_L = \alpha_L\}$, where $\alpha_{-L}, \cdots, \alpha_L \in \{1, \cdots, N\}$ and $L \ge 0$. Let $X_1 = Y$. We then proceed inductively, given X_k we define

$$X_{k+1} = \begin{cases} X_k & \text{if } X \subset \cup_{n\in\mathbb{Z}}\sigma^{-n}U_k \\ X_k - (\cup_{n\in\mathbb{Z}}\sigma^{-n}U_k) & \text{otherwise} \end{cases}$$

We then define $X = \cap_{n=1}^{\infty}X_k$ which is closed, invariant and (by construction) a minimal set. For the second part, since X is minimal, $X = \cup_{n\in\mathbb{Z}}\sigma^{-n}V$ and the result then follows by compactness. □

Proof of van der Waerden's Theorem

We proceed following [2], [1] and [3]. It suffices to restrict our attention to a minimal subset $X \subset \Sigma$ and to prove the following:

Proposition 1 (Multiple Recurrence). *Let $V \subset X$ be an open set. $\forall k_0$, $\exists k \ge k_0$ and $d \ge 1$ such that*

$$V \cap \sigma^{-d}V \cap \sigma^{-2d}V \cap \cdots \cap \sigma^{-(k-1)d}V \neq \emptyset. \qquad (P(k))$$

Proof of Theorem (assuming Proposition 1). This follows from Lemma 1 and Proposition 1 (with the particular choice $V = [i]$, chosen so that $[i] \cap X_0 \neq \emptyset$). □

The importance of minimality of X is that it allows us (by part (2) of Lemma 2) to write $X = \cup_{|n|\le M}\sigma^{-n}V$, say.

The proof of Proposition 1 is now by induction on k. When $k = 1$ the result $P(1)$ is trivial. Assume we know that $P(k-1)$ holds, then we shall use the following lemma to extend this result to $P(k)$.

Lemma 3. *For each $l \ge 1$ we can choose points and open sets $x_j \in \sigma^{-n_j}V$ ($-M \le n_j \le M$) $j = 0, \cdots, l$ and natural numbers $N_1 < N_2 < \cdots < N_l$ such that $\forall 0 \le r \le s \le l$*

$$\sigma^{j(N_s-N_r)}x_r \in \sigma^{-n_s}V, \qquad (Q(l))$$

for $j = 0, \cdots, l$.

Proof. This is proved by induction on l (within the induction on k, for which we are currently assuming $P(k-1)$ holds). When $l = 0$ it is trivial that $Q(0)$. Assume we know $Q(l-1)$ holds. Let us choose a small neighbourhood $V_0 \ni x_{l-1}$. By $P(k-1)$ we can choose d such that there exists $y \in V_0 \cap \sigma^{-d}V_0 \cap \sigma^{-2d}V_0 \cap \cdots \cap \sigma^{-(k-2)d}V_0$. We set $x_l := \sigma^{-d}y$ and $N_l := N_{l-1} + d$. Moreover, we can choose some $\sigma^{-n_l}V_0 \ni x_l$ with $|n_l| \le M$ (by Lemma 2 (2)). In particular, for each $0 \le r < l$:

$$\sigma^{j(N_l-N_r)}x_l = \sigma^{j(N_{l-1}-N_r)}(\sigma^{(j-1)(d)}y) \in \sigma^{j(N_{l-1}-N_r)}(V),$$

for $j = 0, \cdots, k-1$ Moreover, providing V_0 is sufficiently small $Q(l)$ follows from $Q(l-1)$, the additional results for x_l coming from those for x_{l-1} and by continuity of $\sigma^{j(N_l - N_r)}$. □

To prove $P(k)$ holds it suffices to apply the Lemma 3 where $l = 2M+1$. By the pigeonhole principle, we can choose $0 \leq r < s \leq 2M+1$ such that $n_r = n_s \in \{-M, \cdots, M\}$. Then setting $x = x_r$ and now setting $d = N_s - N_r$ gives a point in the intersection for $P(k)$. This completes the inductive step, and thus the proof of Proposition 2. □

Some final comments

Observe that the proof of Proposition 1 holds for any homeomorphism of a compact topological space (without any assumption of a metric).

D. Birkhoff showed that for a homeomorphism $T : X \to X$ of a compact metric space there exists $x \in X$ and a sequence $n_l \to \infty$ such that $d(T^{n_l}x, x) \to 0)$. The following is a corollary of Proposition 1:

Proposition 2 (Multiple Birkhoff recurrence). *Let $T : X \to X$ be a homeomorphism of a compact metric space. For each $k \geq 1$ there exists $x \in X$ and a a sequence $n_l \to \infty$ such that $\max_{0 \leq i \leq k-1}\{d(T^{in_l}x, x)\} \to 0$.*

Proof. We can let $V_l = B(y_l, 2^{-l})$, for $l \geq 0$. By Proposition 1, there exists $x_l \in X$ with $\max_{0 \leq i \leq k-1}\{d(T^{in_l}x_l, x_l)\} \leq 2^{-l}$. Letting x be an accumulation point of $\{x_l\}$, the result follows.

Another application of Proposition 1 is to a higher dimensional analogue of van der Waerden's theorem for $\mathbb{Z}^D$, with $D \geq 1$.

Theorem (Higher Dimensional van der Waerden's Theorem). *Let $\mathbb{Z}^D = \cup_{i=1}^N C_i$ be a finite partition and let $F \subset \mathbb{Z}^D$ be a finite set. Then: $\forall k_0$, $\exists k \geq k_0$, $c \in \mathbb{Z}^d$ and $d \in \mathbb{Z}$ such that $c + jd \in C_i$ for $0 \leq j \leq k-1$).*

References

1. V. Bergelson, *Ergodic theory and Diophantine problems*, Topics in symbolic dynamics and applications (Temuco, 1997), London Math. Soc. Lecture Note Ser., 279, Cambridge University Press, Cambridge, 2000, pp. 167–205.
2. A. Blaszczyk, S. Plewik, and S. Turek, *Topological multidimensional van der Waerden theorem.*, Comment. Math. Univ. Carolin. **30** (1989), 783–787.
3. M. Brin and G. Stuck, *Introduction to dynamical systems*, Cambridge University Press, Cambridge, 2002.
4. H. Furstenberg and B. Weiss, *Topological dynamics and combinatorial number theory*, J. Analyse Math. **34** (1978), 61–85.
5. A. Khinchin, *Three pearls of number theory*, Dover Publications, Inc.,, Mineola, NY,, 1998..
6. B.L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw. Arch. Wisk. **15** (1927), 212216.
7. B. Weiss,, *Single orbit dynamics*, CBMS Regional Conference Series in Mathematics, 95., American Mathematical Society, Providence, RI,, 2000.

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