ESTIMATING VARIANCE FOR EXPANDING MAPS

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0. INTRODUCTION

In the study of the statistical properties of dynamical systems there are a number of important characteristics, e.g., entropy, Lyapunov exponents, etc. In this note we want to concentrate on the variance $\sigma^2 : C^\alpha(M) \to \mathbb{R}$ of Hölder functions.

The variance appears in statistical properties for both hyperbolic maps and flows. For a hyperbolic map $T : M \to M$, a Hölder continuous function $f : M \to \mathbb{R}$ and a Gibbs measure (for a Hölder continuous function g) we can define

$$\sigma^2(f) = \lim_{n\to+\infty} \frac{1}{n} \int \left(\sum_{i=0}^{n-1} f(T^i x) - n \int f d\mu \right)^2 d\mu(x), \tag{0.1}$$

where $f \in C^\alpha(M)$[18], [14]. The variance is easily seen to be invariant under adding a coboundary or a constant, i.e., if $u : X \to \mathbb{R}$ is continuous and $c \in \mathbb{R}$ then $\sigma^2(f) = \sigma^2(f + uT - u + c)$. Let $T : M \to M$ be an expanding map on a compact manifold of dimension d and let μ be the unique (ergodic) equilibrium state for a Hölder continuous (normalized) function $g : M \to \mathbb{R}$. Let $f : M \to \mathbb{R}$ be a Hölder continuous function. By the Birkhoff ergodic theorem we can write that for a.e. (μ) $x \in X$

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \to \int f d\mu, \text{ as } N \to +\infty,$$

as $N \to +\infty$. An important dynamical invariant is the variance $\sigma^2 = \sigma^2(f)$ of a function and measure. This appears in many statistical properties, such as the Central Limit Theorem. More precisely, we can write

$$\mu \left\{ x \in X : \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left(f(T^n x) - \int f d\mu \right) \le z \right\} \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{z} e^{-\sigma t^2} dt,$$

as $N \to +\infty$ (cf. [7], [14], [12], [15]).

In some familiar examples it is possible to use (0.1) to give explicit expressions for the variance. For example, let $T : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d/\mathbb{Z}^d$ be a linear hyperbolic toral automorphism. If a function on a torus is represented by a Fourier series $f(x) = \sum_{\underline{n}\in\mathbb{Z}^d} a_{\underline{n}} e^{2\pi i \langle \underline{n}, x\rangle}$ then in this particular case, a direct calculation gives

$$\sigma^2(f) = \sum_{\underline{n}\in\mathbb{Z}} |a_{\underline{n}}|^2. \tag{0.2}$$

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A similar simple argument works for any system with countable Lebesgue spectrum, although they may not always be so practical for the purposes of computation.

For simplicity, let us consider the case of expanding maps. Many of the results in the invertible case can be reduced to this setting using Markov partitions or Markov sections, etc. Let us denote $||f||_2 = \left(\int |f|^2 d\mu\right)^{1/2}$, $||f||_1 = \int |f| d\mu$ and $||f||_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|f(x)-f(y)|}{d(x,y)}$.

Proposition 1.

(1) *There is a simple upper bound for the variance given by*

$$\sigma^2(f) \leq ||f||_2^2 + C||f||_{L^1}(1 + b||f||_{\mathrm{Lip}}) \tag{0.3}$$

where $C, b > 0$ are independent of f.

(2) *Let $\mathcal{B}$ denote the Borel σ-algebra. Let $E(\cdot|T^{-1}\mathcal{B}) : L^2(M, \mathcal{B}) \to L^2(M, \mathcal{B})$ denote the conditional expectation operator. We can choose $u : X \to \mathbb{R}$ such that $\overline{f} := f + u \circ T - u$ satisfies*

$$E(\overline{f}|T^{-1}\mathcal{B}) = 0 \text{ and } \sigma^2(f) = ||\overline{f}||_2. \tag{0.4}$$

For completeness, we present the proof of Proposition 1 in sections 2 and 3. An explicit expression for C and b in Part (1) will be given later. We can consider the following simple example to illustrate Part (2).

Simple Example (Doubling map). The map $T : [0, 1] \to [0, 1]$ defined by $T(x) = 10x$ (mod 1) peserves the Lebesgue measure μ. Fix $i \in \{0, 1, \ldots, 9\}$ and let $f = \chi_{[\frac{i}{10}, \frac{(i+1)}{10}]}$. Then for almost every $x = 0 \cdot x_1 x_2 x_3 \cdots$ we have from the Birkhoff ergodic theorem that

$$\frac{1}{N} \mathrm{Card}\{0 \leq i \leq N-1 : x_n = i\} \to \frac{1}{10} \text{ as } N \to +\infty$$

for a.e. (μ) x, with respect to Lebesgue measure. In this case, the Central Limit Theorem holds and, since $E(f|T^{-1}\mathcal{B}) = \frac{1}{10}$, we have that

$$\sigma^2(f) = \int \left| \frac{1}{10} - \chi_{[\frac{i}{10}, \frac{(i+1)}{10}]} \right|^2 dx = \frac{9}{100}.$$

Unfortunately, typically $E(f|T^{-1}\mathcal{B}) \neq 0$ and we cannot expect the variance $\sigma^2(f)$ to have such a simple explicit expression as (0.4). Therefore, it is natural to find methods for obtaining estimates on variance which can give arbitrarily good results. Given $m \geq 1$, we can write $f^m(x) = f(x) + f(Tx) + \cdots + f(T^{m-1}x)$ and $g^m(x) = g(x) + g(Tx) + \cdots + g(T^{m-1}x)$. It is not difficult to show that there exists $0 < \theta < 1$ such that

$$\frac{1}{m} \sum_{T^m x = x} f^m(x) e^{g^m(x)} = \int f d\mu + O(\theta^m), \tag{0.5}$$

as $m \to +\infty$. This can be compared with Bowen's equidistribution theorem [3]. Moreover, this helps to motivate the following simple estimate.

Theorem 1 (Numerical Estimates). *Let $f \in C^\alpha(X)$. We can estimate*

$$\sigma^2(f) = \frac{1}{m} \sum_{T^m x = x} \left(f^m(x) - m \int f d\mu \right)^2 e^{g^m(x)} + O\left(\theta^m\right), \tag{0.6}$$

as $m \to +\infty$.

Unfortunately, as will be illustrated in the next section, this approximation may not converge particularly quickly. However, it is possible to improve on this result in the case of analytic maps. More precisely, there is an alternative approximation to $\sigma^2(f)$, using the same information on weights of periodic orbits, which converges far more rapidly.

Theorem 2 (Numerical Estimates: C^ω case). *If all of the functions are real analytic then we can write*

$$\sigma^2(f) = B_m + O\left(\theta^{m^{(1+1/d)}}\right), \tag{0.7}$$

as $m \to +\infty$, where B_m is explicitly given in terms of the periodic points of period at most m.

The explicit form of B_n will be given later.

1. Examples

In this section we consider some simple examples which illustrate the estimates in Theorems 1 and 2.

Example 1.1. Consider the linear expanding map $T_0 : [0,1) \to [0,1)$ defined by $T_0(x) = 2x \pmod 1$. Let μ_0 denote the associated T_ϵ-invariant Lebesgue measure. In particular, we set $g(x) = -\log|T_0'(x)| = -\log 2$. Consider the test function $f(x) = \cos^2(2\pi x) - \frac{1}{2}$ then, in particular, we have that $\int f(x) d\mu_0(x) = 0$. It is easy to see using Proposition 1 that $\sigma^2(f) = \int |f(x)|^2 dx = \frac{1}{8}$.

The first five approximations to σ^2 using Theorem 1 are given in table 1. Similarly, the first five approximations using Theorem 2 are given in table 2.

N	N^{th} approximation to $\sigma^2(f)$
1	0.1250
2	0.1875
3	0.1093
4	0.1171
5	0.1210

Table 1. Successive approximations to σ^2

N	N^{th} approximation to $\sigma^2(f)$
1	-0.250
2	0.000
3	0.125
4	0.125
5	0.125

TABLE 2. Successive approximations to σ^2 using Theorem 2

In Example 1.1, the linearity gives untypically fast convergence in Table 2. The comparison is fairer in the following nonlinear example.

Example 1.2. Consider the non-linear expanding maps $T_\epsilon : [0,1) \to [0,1)$ defined by $T_\epsilon(x) = 2x + \epsilon \sin 2\pi x \pmod 1$. Provided $|\epsilon| < \frac{1}{2\pi}$ the map T_ϵ is expanding. Let μ_ϵ denote the associated T_ϵ-invariant probability measure. Consider the particular choice $\epsilon = \frac{1}{4\pi}$ and the map $T(x) := T_{\frac{1}{2\pi}}(x) = 2x + \frac{1}{4\pi}\sin(2\pi x)$ on the unit circle. The invariant density can be estimated by

$$\rho(x) = 1 + 0.052458 \cos 2\pi x + 0.004252 \cos 4\pi x + \cdots$$

(cf. [9]). Thus we set $g(x) = -\log|T'(x)| = -\log(2 + (1/2)\cos(2\pi x))$. We can consider a test function such as $f(x) = \cos^2(2\pi x) - \int \cos^2(2\pi x) d\mu$ for which $\int f(x)\rho(x)dx = 0$. The first five approximations to σ^2 using Theorem 1 are given in Table 3.

N	N^{th} approximation to $\sigma^2(f)$
1	0.0995
2	0.2463
3	0.0478
4	0.1767
5	0.1341

TABLE 3. Successive approximations to σ^2 using Theorem 1

Similarly, the first five approximations to σ^2 using Theorem given in Table 4.

N	N^{th} approximation to $\sigma^2(f)$
1	0.3485
2	0.3551
3	0.2430
4	0.2146
5	0.2029

TABLE 4. Successive approximations to σ^2 using Theorem 2

2. Proof of Proposition 1 (a)

We begin with a well known result.

Lemma 2.1. *We can write*

$$\sigma^2(f) = ||f||_2 + 2\sum_{n=1}^{\infty} \rho(n) \tag{2.1}$$

where $\rho(n) = \int f(T^n x) f(x) d\mu(x)$.

The proof of this can be found, for example, in [10]. If we were to consider the Fourier transform $\widehat{\rho}(z) = \sum_{n=-\infty}^{\infty} z^n \rho(n)$ then we see that (2.1) corresponds to the value $\widehat{\rho}(1)$.

To prove Part (a) of Proposition 1, it now suffices to get suitable explicit bounds of the form

$$\int f(T^n x) f(x) d\mu(x) \leq K ||f||_{L^1} (1 + b||f||_{\mathrm{Lip}})\Theta^n \tag{2.2}$$

for some $C > 0$ and $0 < \Theta < 1$. Then by (2.1) we have that

$$\sigma^2(f) \leq ||f||_2 + \frac{2K}{1-\Theta} ||f||_{L^1} (1 + b||f||_{\mathrm{Lip}})$$

Although general estimates of the form (2.2) are well known, we recall a method which gives particularly contrete estimates on the constants. For simplicity we shall restrict attention to a suitable expanding map $T : I \to I$ on the unit interval.

Lemma 2.2 [13]**.** *Assume that* $\inf |T'| \geq \lambda > 2$*, say, then (2.2) holds with*

$$\begin{aligned}
A &= \sup_{\xi \in I} \left\{ \frac{|T''(\xi)|}{|T'(\xi)|^2} \right\} + 2 \sup_{\xi \in I} \{ |T'(\xi)|^{-1} \} \\
B &= \frac{A}{(1 - 2/\lambda)} \\
a &> \frac{A}{1 - 2/\lambda} \\
b &= \frac{1}{a - B} \\
N &= \left[\frac{\log(2a)}{\log \lambda} \right] + 1 \\
\sigma &= \frac{2}{\lambda} + \frac{A}{a} \\
\sigma_1 &= \left(\frac{2}{\lambda} \right)^N + \left(\frac{1 - \left(\frac{2}{\lambda}\right)^N}{1 - \left(\frac{2}{\lambda}\right)} \right) \frac{A}{a} \\
\Delta &= \frac{\max\{1 + \sigma_1, 1 + a\sigma_1\}}{\min\{1 - \sigma_1, 1/(2||T'||_\infty^N)\}} \\
\Theta &= \tanh\left(\frac{\Delta}{4} \right)^{1/N} \\
K &= \left(e^{\Delta \Theta^{-N}} \Theta^{-N} \right) \Theta ||g||_\infty
\end{aligned}$$

The proof of Lemma 2.2 comes from the classical estimates on Birkhoff Cones, a method which gives fairly explicit bounds on K and Θ. This is originally due to G. Birkhoff, and was used for transfer operators by Ferrero and Schmitt [6]. More recent accounts appear in [13], [1], [20].

Remark. There are other explicit bounds on C and Θ due to Rychlik [2] and Bandtlow-Jenkinson [12].

3. Proof of Proposition 1 (b)

Let us associate to the expanding map $T : M \to M$ the linear *transfer* operator given by

$$\mathcal{L}_g w(x) = \sum_{Ty=x} e^{g(y)} w(y).$$

We recall the following classical result due to Ruelle [18] (cf. [5]).

Proposition 3.1 (Ruelle Operator Theorem). *The operator $\mathcal{L}_g : C^\alpha(X) \to C^\alpha(X)$ has a maximal simple positive eigenvalue λ and associated positive eigenfunction $h > 0$.*

The maximal eigenvalue is often written as $\lambda = e^{P(g)}$, where $P(g)$ denotes the pressure of f. By replacing g by $\overline{g} = g + \log w - \log w \circ T) - \log \lambda$ we can assume without loss of generality that $P(g) = 0$ and $\mathcal{L}_{\overline{g}} 1 = 1$, i.e., 1 is a simple eigenvalue with constant functions as eigenfunctions. The dual operator satisfies $\mathcal{L}_{\overline{g}}^* \mu = \mu$.

Example (Gauss map). Consider the Gauss transformation $T : [0,1) \to [0,1)$ given by $Tx = \{1/x\}$, the fractional part of $1/x$, if $x \neq 0$, and $T(0) = 0$. Let $\overline{g} = \frac{(x+1)}{(x+n)(x+n+1)}$ then $\mathcal{L}_{\overline{g}} 1 = 1$ and μ is the Gauss measure.

Corollary. *The operator $\mathcal{L}_{\overline{g}} : C^\alpha(X) \to C^\alpha(X)$ is quasi-compact (i.e., $\mathcal{L}_{\overline{g}} = \mu + U_n$ where $\lim_{n \to +\infty} ||U_n||^{1/n} = \rho < 1$).*

The following alternative characterization of the variance is useful.

Proposition 3.2. *Let $\lambda(\overline{g} + t\overline{f})$ be the maximal eigenvalue of the operator $\mathcal{L}_{\overline{g}+t\overline{f}}$. The variance $\sigma^2 = \sigma^2(f)$ is equal to $\sigma^2 = \frac{d^2 \lambda(g+tf)}{dt^2}|_{t=0}$. Equivalently, $\sigma^2 = \frac{d^2\left(\lambda(g+tf)^{-1}\right)}{dt^2}|_{t=0}$*

Proof. It is well known that $\sigma^2(f) = \frac{\partial^2 P(t\overline{f}+\overline{g})}{\partial t^2}|_{t=0}$ [18], [14]. We can first write

$$\frac{\partial e^{P(t\overline{f}+\overline{g})}}{\partial t} = e^{P(t\overline{f}+\overline{g})} \frac{\partial P(t\overline{f}+\overline{g})}{\partial t}$$

and then

$$\frac{\partial^2 e^{-P(t\overline{f}+\overline{g})}}{\partial t^2} = e^{P(t\overline{f}+\overline{g})} \left(\frac{\partial P(t\overline{f}+\overline{g})}{\partial t} \right)^2 + e^{P(t\overline{f}+\overline{g})} \frac{\partial^2 P(t\overline{f}+\overline{g})}{\partial t^2}.$$

Since $\frac{\partial P(t\overline{f}+\overline{g})}{\partial t}|_{t=0} = \int \overline{f} d\mu = 0$ and $e^{P(\overline{g})} = 0$ we have that

$$\frac{\partial^2 e^{P(t\overline{f}+\overline{g})}}{\partial t^2}|_{t=0} = \frac{\partial^2 P(t\overline{f}+\overline{g})}{\partial t^2}|_{t=0}$$

and the result follows. The correspond result for $\frac{d^2\left(\lambda(g+tf)^{-1}\right)}{dt^2}|_{t=0}$ follows similarly. $\square$

We need the following result.

Proposition 3.3. *Assume that* $\int f d\mu = 0$.

(1) *We can find a Hölder continuous function* $u : M \to \mathbb{R}$ *such that* $\overline{f} := f + u - u \circ T$ *satisfies* $\mathcal{L}_{\overline{g}} \overline{f} = 0$,
(2) $\sigma^2(f) = \sigma^2(\overline{f}) = \int |\overline{f}|^2 d\mu$

Proof. For the proof of part (1), we can define $u = \sum_{n=1}^{\infty} \mathcal{L}_{\overline{g}}^n f$. This converges because of the spectral properties of the operator (i.e., the spectral radius of $\mathcal{L}_g - \mu$ is strictly small than $\rho < 1$, say, and thus there exists $C > 0$ such that $||\mathcal{L}_{\overline{g}}^n v||_\alpha \leq C(\frac{\rho+1}{2})^n ||v||_\alpha$, for $n \geq 0$). Moreover, from the definition of u we have that

$$\mathcal{L}_{\overline{g}} u - u = -\mathcal{L}_{\overline{g}} f \tag{3.1}$$

Since $\mathcal{L}_{\overline{g}} 1 = 1$ we have that $\mathcal{L}_{\overline{g}} U_T = I$, where $U_T f = f \circ T$. In particular, $\mathcal{L}_{\overline{g}} \overline{f} = \mathcal{L}_{\overline{g}} f + \mathcal{L}_{\overline{g}}(u - U_T u)$, but since $\mathcal{L}_{\overline{g}}(u - U_T u) = \mathcal{L}_{\overline{g}} u - u = -\mathcal{L}_{\overline{g}} f$, by (3.1), we see that $\mathcal{L}_{\overline{g}} \overline{f} = 0$.

Part (2) follows from differentiating the eigenvalue equation $\mathcal{L}_{\overline{g}+t\overline{f}} w(t) = \lambda(t) w(t)$ and and using the characterization of the variance in Proposition 1.2. Let us fix the normalization $\int w(t) d\mu = 1$. The first derivative is:

$$\mathcal{L}_{\overline{g}+t\overline{f}}(\overline{g} w(t) + w'(t)) = \lambda'(t) w(t) + \lambda(t) w'(t). \tag{3.2}$$

At $t = 0$, we have $\lambda'(0) = \int \overline{f} d\mu = 0$, $w(0) = 1$ and $\lambda(0) = 1$ and so (3.1) reduces to

$$\mathcal{L}_{\overline{g}}(w'(0)) = \mathcal{L}_{\overline{g}}(\overline{f} + w'(0)) = w'(0), \tag{3.3}$$

using part (1). Since 1 is a simple eigenfunction for $\mathcal{L}_{\overline{g}}$, with constant functions as eigenfunctions, $w'(0)$ is a constant function. Moreover, the normalization implies that $\int w'(0) d\mu = 0$, i.e., $w'(0) = 0$. Differentiating (3.1) again gives that

$$\mathcal{L}_{\overline{g}+t\overline{f}}(\overline{f}^2 w(t) + 2\overline{f} w'(t) + w''(t)) = \lambda''(t) w(t) + 2\lambda'(t) w'(t) + \lambda(t) w''(t). \tag{3.4}$$

We can first evaluate this second expression at $t = 0$ and then integrate both sides with respect to μ. Since $\mu = \mathcal{L}_{\overline{g}}^* \mu$ we have that

$$\begin{aligned} &\int (\overline{f}^2 w(0) + 2\overline{f} w'(0) + w''(0)) d\mu \\ &= \int (\lambda''(0) w(0) + 2\lambda'(0) w'(0) + \lambda(0) w''(0)) d\mu \end{aligned} \tag{3.5}$$

Since $\lambda(0) = 1$ we can cancel the last terms on each side. since $w'(0)$ is constant we know (by considering the first order term) that $w'(0) = 0$, which eliminates an extra term on each side and leaves:

$$\lambda''(0) = \frac{\int \overline{f}^2 w(0) d\mu}{\int w(0) d\mu} \tag{3.6}$$

However, by hypothesis we have that $w(0) = 1$, thus (3.6) gives the result. □

We recall that $E(w|T^{-1}\mathcal{B}) = (\mathcal{L}_{\overline{g}} w) \circ T$ ([Wa], [Le]), from which Proposition 1 (b) now follows.

Example 3.1. Consider again the family of maps $T_\epsilon : [0,1) \to [0,1)$ defined by $T_\epsilon(x) = 2x + \epsilon \sin(2\pi x) \pmod 1$ on the unit circle, where $|\epsilon| < \frac{1}{2\pi}$. There is an associated a conjugating homeomorphism $\pi_\epsilon : [0,1] \to [0,1]$ such that $\pi_\epsilon \circ T_0 = T_\epsilon \circ \pi_\epsilon$. The conjugating map has an analytic dependence on ϵ and so can write $\pi_\epsilon(x) = x + \epsilon \pi^{(1)}(x) + O(\epsilon^2)$. The function $\pi^{(1)}(x)$ therefore satisfies $\pi^{(1)}(2x) = 2\pi^{(1)}(x) + \sin(2\pi\pi^{(1)}(x))$ and thus takes the form

$$\pi^{(1)}(x) = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sin(2\pi 2^n x).$$

In particular, we can then write

$$\begin{aligned}
&-\log|T_\epsilon(\pi_\epsilon(x))| = -\log(2 + 2\pi\epsilon\cos(2\pi\pi_\epsilon(x)))\\
&= -\log 2 + \log\left(1 + \pi\epsilon\cos(2\pi[x + \epsilon\pi^{(1)}(x) + O(\epsilon^2)])\right)\\
&= -\log 2 + \log(1 + O(\epsilon^2))\\
&= -\log 2 + O(\epsilon^2).
\end{aligned}$$

We can fix a C^2 function $f : [0,1) \to \mathbb{R}$ then

$$\begin{aligned}
f(\pi_\epsilon(x)) &= f(x + \epsilon\pi^{(1)}(x) + O(\epsilon^2))\\
&= f(x) + f'(x)(\epsilon\pi^{(1)}(x) + +O(\epsilon^2)\\
&= f(x) + \epsilon\left(f'(x)\pi^{(1)}(x)\right) + O(\epsilon^2).
\end{aligned}$$

We can expand

$$\begin{aligned}
&P(-\log T'_\epsilon \circ \pi_\epsilon + tf \circ \pi_\epsilon)\\
&= P\left(-\log 2 + t\left[f(x) + \epsilon\left(f'(x)\pi^{(1)}(x)\right) + O(\epsilon^2)\right]\right)\\
&= t\left[f(x) + \epsilon\left(f'(x)\pi^{(1)}(x)\right) + O(\epsilon^2)\right].
\end{aligned} \tag{3.7}$$

In particular, if $\sigma_\epsilon^2(f \circ \pi_\epsilon)$ is the associated variance for T_ϵ then it comes from second derivative of (3.7), from which we see that

$$\sigma_\epsilon^2(f \circ \pi_\epsilon) = \sigma^2(f) + 2\epsilon\sigma^2(f, \pi^{(1)} f') + O(\epsilon^2).$$

For example, if $f(x) = \cos^2(2\pi x) - \frac{1}{2}$ then we have

$$f'(x).\pi^{(1)}(x) = 2\pi\sin(4\pi x)\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sin(2\pi 2^n x)$$

and thus

$$\begin{aligned}
\sigma^2(f, \pi^{(1)} f') &= 2\pi \int \sin(4\pi x)\left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sin(2\pi 2^n x)\right) dx\\
&= \frac{\pi}{2} \int \sin^2(4\pi x) dx = \frac{\pi}{4}.
\end{aligned}$$

Thus we conclude that

$$\sigma_\epsilon^2(f \circ \pi_\epsilon) = \sigma^2(f) + \frac{\pi}{2}\epsilon + O(\epsilon^2).$$

4. Proof of Theorem 1

Theorem 1 follows easily from The Ruelle Operator Theorem. We can use Proposition 3.1 and analytic perturbation theory to write that

$$\mathcal{L}_{g+tf}^m = e^{mP(g+tf)}\mu_{g+tf} + O(\theta^m)$$

for $m \geq 1$, where we choose θ to satisfy $\limsup_{n\to+\infty} ||U_n||^{1/n} < \theta < 1$. Moreover, the implied constant in the error term is uniform in $|t| \leq \delta$, say. The following easily follows.

Lemma 4.1. *We can write that*

$$e^{mP(f+tg)} = \left(\sum_{T^m x = x} e^{(g+tf)^m(x)}\right)(1 + O(\theta^n)) \tag{4.1}$$

where the implied constant in the error term is uniform in $|t| \leq \delta$.

We can differentiate the left hand side of (4.1) twice to write

$$\frac{\partial^2 e^{mP(f+tg)}}{\partial t^2} = \left(\left(m\frac{\partial P(f+tg)}{\partial t^2}\right)^2 + m\frac{\partial^2 P(f+tg)}{\partial t^2}\right) e^{mP(g+tf)}. \tag{4.2}$$

Evaluating (4.1) at $t = 0$ and using that $\frac{\partial P(f+tg)}{\partial t}|_{t=0} = 0$ gives

$$\begin{aligned}\frac{\partial^2 e^{mP(f+tg)}}{\partial t^2}|_{t=0} &= m\frac{\partial^2 P(f+tg)}{\partial t^2}|_{t=0} e^{mP(g+tf)}\\ &= m\sigma^2(f)e^{mP(g+tf)}\end{aligned}$$

The second derivative of $Z_m(t) = \sum_{T^m x = x} e^{(g+tf)^m(x)}$ is

$$\frac{\partial^2 Z_m(t)}{\partial t^2} = \sum_{T^m x = x} (f^m(x))^2 e^{(g+tf)^m(x)} \tag{4.3}$$

We can deduce from (4.2) and (4.3) and the fact that $P(g) = 0$ that

$$\begin{aligned}&\left|m\sigma^2(f) - \sum_{T^m x = x} (g^m(x))^2 e^{g^m(x)}\right|\\ &= \left|\frac{\partial^2 e^{mP(f+tg)}}{\partial t^2}|_{t=0} - \frac{\partial^2 Z_m(t)}{\partial t^2}|_{t=0}\right|\\ &= \left|\frac{1}{\pi i}\int_{|\xi|=\delta}\left(\frac{e^{mP(f+\xi g)}}{\xi^3} - \frac{Z_m(\xi)}{\xi^3}\right)d\xi\right|\\ &\leq \frac{C}{\delta^2}\theta^m\end{aligned} \tag{4.4}$$

for suitable $0 < \delta < 1$ and $C > 0$. In particular, we can deduce from (4.4) that

$$\sigma^2(f) = \frac{1}{m}\sum_{T^m x = x} (f^m(x))^2 e^{g^m(x)} + O(\theta^m).$$

This completes the proof of Theorem 1.

In practise, we can estimate the pressure, and thus the derivatives of the pressure, by using piecewise constant functions f_n and g_n to approximate f and g. More precisely, let $\mathcal{A} = \{A_1, \cdots, A_k\}$ be a Markov Partition for the expanding map $T : M \to M$. For $n \geq 1$, we can consider the refinement $\mathcal{A}^{(n)} = \mathcal{A} \vee T^{-1}\mathcal{A} \vee \cdots \vee T^{-(n-1)}\mathcal{A}$. We then denote

$$||f||_\infty = \sup_{x \in \coprod_i A_i} |f(x)|$$
$$||f||_\lambda = \sup_{\substack{x,y \in A \\ x \neq y \\ A \in \mathcal{A}^{(n)}}} \frac{|f(x) - f(y)|}{\lambda^n},$$

where $1 < \lambda \leq ||D_x T(v)||$, for $x \in M$ and $v \neq 0$. The following result follows easily from an estimate of Ruelle [18] (and an account appears in [14]).

Lemma 4.2. *We can approximate f, g by locally constant functions f_n, g_n such that*

(1) $||(f + tg) - (f_n + tg_n)||_\infty \leq ||f_n + tg_n||_\lambda \lambda^{-n}$; *and*
(2) $||(f + tg) - (f_n + tg_n)||_{\lambda'} \leq ||f + tg||_\lambda \left(\frac{\lambda'}{\lambda}\right)^n$, *for* $1 < \lambda' < \lambda$.

We can then consider the operator $\mathcal{L}_{f_n + tg_n}$ as a matrix and compute its maximal positive eigenvalue $e^{P(f_n + tg_n)}$ and so approximate $e^{P(f+tg)}$. Finally, using analytic perturbation theory we see that similar estimates hold for the derivatives. We first recall the following result.

Lemma 4.3. *The map $w \mapsto P(w)$ is analytic in a neighbourhood $\{w : ||w - f||_{\lambda'} < \epsilon\}$ (of the complexification).*

In particular, we see that $|P(f + tg) - P(f_n + tg_n)| \leq C \left(\frac{\lambda'}{\lambda}\right)^n$. Moreover, since the analytic domain $0 \in U \subset \mathbb{C}$ of $t \mapsto f_n + tg_n, f + tg$ can be chosen uniformly we can deduce that

$$\begin{aligned}
|\sigma^2(f) - \sigma^2(f_n)| &= \left| \frac{\partial^2 P(f + tg)}{\partial t^2} - \frac{\partial^2 P(f_n + tg_n)}{\partial t^2} \right| \\
&= \left| \frac{1}{\pi i} \int_{|\xi| = \delta} \left(\frac{P(f + tg)}{\xi^3} - \frac{P(f_n + tg_n)}{\xi^3} \right) d\xi \right| \\
&\leq \frac{C}{\delta^2} \left(\frac{\lambda'}{\lambda} \right)^n
\end{aligned}$$

for suitable $\delta > 0$ and $C > 0$. Thus we conclude that we can estimate the variance using the locally constant approximations.

5. Proof of Theorem 2

The proof of Theorem 2 is based on estimates that arise in the study of certain analytic functions. It is based on a cycle expansion method (a la Cvitanovic) which was previously used by Jenkinson and the present author in the context of estimates on Hausdorff Dimension and Lyapunov exponents [8].

We can formally define

$$D(z,t) := \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{T^n x = x}\frac{e^{g^n(x)+tf^n(x)}}{1-|(T^n)'(x)|^{-1}}\right)$$
$$=\prod_{\tau}\prod_{n=0}^{\infty}\left(1-z^{|\tau|}e^{-\lambda(\tau)+t\lambda_f(\tau)+n\lambda_{\log|T'|}(\tau)}\right)$$

The following result is well known.

Proposition 5.1 (Ruelle [17]). *The function $D(z,t)$ converges to an analytic function in a neighbourhood of the disk $|z| \le e^{P(tf+g)}$. Moreover, there is a simple zero for $z \mapsto \zeta(z,t)$ at $z = z(t) := e^{-P(tf+g)}$.*

By Proposition 2.1 the zero $z(t) := e^{-P(g+tf)}$ satisfies $D(z(t),t) = 0$. Differentiating in t gives that

$$0 = \frac{\partial}{\partial t}D(z(t),t) = \frac{\partial D}{\partial z}(z(t),t)\frac{\partial z(t)}{\partial t} + \frac{\partial D}{\partial t}(z(t),t)$$

Differentiating again in t gives

$$\begin{aligned}0 = \frac{\partial^2}{\partial t^2}D(z(t),t) &= \frac{\partial^2 D}{\partial z^2}(z(t),t)\left(\frac{\partial z(t)}{\partial t}\right)^2 + 2\frac{\partial^2 D}{\partial t\partial z}(z(t),t)\left(\frac{\partial z(t)}{\partial t}\right)\\ &+ \frac{\partial D}{\partial z}(z(t),t)\frac{\partial^2 z(t)}{\partial t^2} + \frac{\partial^2 D}{\partial t^2}(z(t),t)\end{aligned} \tag{5.1}$$

Let $b_n(t) = \frac{1}{n}\sum_{T^n x = x} e^{g^n(x)+tf^n(x)}$, then from the definition of $D(z,t)$ we have that

$$\frac{\partial D}{\partial z}(z,t) = D(z,t)\left(-\sum_{n=1}^{\infty} b_n(t) n z^{n-1}\right)$$
$$\frac{\partial^2 D}{\partial t\partial z}(z,t) = D(z,t)\left(-\sum_{n=1}^{\infty} b_n'(t) z^{n}\right)$$
$$\frac{\partial^2 D}{\partial t^2}(z,t) = D(z,t)\left(\left(\sum_{n=1}^{\infty} b_n'(t) z^{n}\right)^2 - \sum_{n=1}^{\infty} b_n''(t) z^{n}\right),$$

where in the second expression we use $\frac{\partial D}{\partial t} = 0$. The convergence of these series is easily checked (as in the proof of Proposition 5.3, below).

We can write $\frac{\partial z(t)}{\partial z} = z(t)\frac{\partial P(g+tf)}{\partial t}$. Recall that $\frac{\partial z(t)}{\partial t}|_{t=0} = \int f d\mu = 0$ and $z(0) = 1$, thus evaluating (5.1) at $t = 0$ gives

$$0 = \left(-\sum_{n=1}^{\infty} b_n(0) n\right)\left(\frac{\partial^2 z}{\partial t^2}|_{t=0}\right) + \left(\left(\sum_{n=1}^{\infty} b_n'(0)\right)^2 - \sum_{n=1}^{\infty} b_n''(0)\right). \tag{5.2}$$

The following result is essentially contained in [17]. A detailed account of the ideas appears in [8].

Proposition 5.2. *We can write $D(z,t) = 1 + \sum_n b_n(t) z^n$ where*

(1) *We can explicitly write*

$$b_n(t) = \sum_{|\tau_1|+\cdots+|\tau_k|=n} e^{\lambda_g(\tau_1)+t\lambda_f(\tau_1)} \cdots e^{\lambda_g(\tau_k)+t\lambda_f(\tau_k)}$$

(2) *There exists $C > 0$ and $0 < \theta < 1$ such that $|a_n(t)| \leq C\theta^{n^{(1+\frac{1}{d})}}$ (and, moreover, these hold uniformly in a δ-neighbourhood of t in the complex plane).*

In particular, we see that $z \mapsto D(z,t)$ is an entire function.

We can now deduce the following.

Proposition 5.3. *We can write*

$$\sigma^2(f)m = -\left(\frac{\partial^2 z}{\partial t^2}|_{t=0}\right) = -\frac{\left(\left(\sum_{n=1}^\infty b_n'(0)\right)^2 - \sum_{n=1}^\infty b_n''(0)\right)}{\left(\sum_{n=1}^\infty b_n(0)n\right)}. \tag{5.3}$$

where

$$\begin{aligned}
b_n(0) &= \sum_{|\tau_1|+\cdots+|\tau_k|=n} (-1)^k e^{\lambda_g(\tau_1)} \cdots e^{\lambda_g(\tau_k)} \\
b_n'(0) &= \sum_{|\tau_1|+\cdots+|\tau_k|=n} (-1)^k (\lambda_f(\tau_1) + \cdots + \lambda_f(\tau_k)) e^{\lambda_g(\tau_1)} \cdots e^{\lambda_g(\tau_k)} \\
b_n''(0) &= \sum_{|\tau_1|+\cdots+|\tau_k|=n} (-1)^k (\lambda_f(\tau_1) + \cdots + \lambda_f(\tau_k))^2 e^{\lambda_g(\tau_1)} \cdots e^{\lambda_g(\tau_k)}
\end{aligned}$$

and there exist $C > 0$ and $0 < \theta < 1$ such that $|a_n|, |b_n|, |c_n| \leq C\theta^{n^2}$.

Proof. The identity (5.3) comes from rearranging (5.2). The convergence of the series in (5.3) uses the bounds on $b_n(t)$ and Cauchy's theorem, i.e.,

$$\begin{aligned}
|b_n'(0)| &= \left| \frac{1}{2\pi i} \int_{|\xi|=\delta} \frac{b_n(\xi) d\xi}{\xi^2} \right| \leq \frac{\sup_{|\xi|=\delta} |a_n(\xi)|}{\delta}, \text{ and} \\
|b_n''(0)| &= \left| \frac{1}{\pi i} \int_{|\xi|=\delta} \frac{b_n(\xi) d\xi}{\xi^3} \right| \leq \frac{2\sup_{|\xi|=\delta} |a_n(\xi)|}{\delta^2}
\end{aligned}.$$

The result follows. □

Proof of Theorem 2. By truncating the series in (2.1) we can write

$$\begin{aligned}
\left(\frac{\partial^2 z}{\partial t^2}|_{t=0}\right) &= -\frac{\left(\left(\sum_{n=1}^N b_n'(0) + O(\theta^{N^{1+\frac{1}{d}}})\right)^2 + \sum_{n=1}^N b_n''(0) + O(\theta^{N^{1+\frac{1}{d}}})\right)}{\left(\sum_{n=1}^N b_n(0)n + O(\theta^{N^{1+\frac{1}{d}}})\right)} \\
&= -\frac{\left(\left(\sum_{n=1}^N b_n'(0)\right)^2 + \sum_{n=1}^N b_n''(0)\right)}{\left(\sum_{n=1}^N b_n(0)n\right)} + O(\theta^{N^{1+\frac{1}{d}}})
\end{aligned} \tag{5.4}$$

Writing B_n for the first term on the right hand side of (5.4) the result follows. □

6. Final remarks

Let us first consider the connection between uniformly expanding maps and other systems.

6.1 Geodesic and Anosov Flows. The results for discrete maps can be applied to geodesic flows on surfaces of negative curvature, and other Anosov flows. Let $\phi_t : M \to M$ denote the flow. For a Gibbs measure μ and a Hölder function $F : M \to \mathbb{R}$ we can associate the variance

$$\sigma^2 = \lim_{T \to +\infty} \frac{1}{T} \int \left(\int_0^T F(T^i x) - T \int F d\mu \right)^2 d\mu(x).$$

Example 6.1. Let M be a compact surface of constant negative curvature. We can identify the unit tangent bundle $SM = SL(2,\mathbb{R})/\Gamma$, where Γ is a discrete subgroup. We can write $L^2(M) = \oplus_{n=1}^{\infty} H_n$, where H_n correspond to irreducible components of the canonical representation $g \mapsto R_g : L^2(M) \to L^2(M)$ given by $R_g f(h) = f(g^{-1}h)$. In particular, for $f, g \in C^\infty(M)$ we can write $f = \sum_n f_n$, $g = \sum_n g_n$ and then

$$\rho_{f,g}(t) = \sum \rho_{f_n, g_n}(t).$$

Let μ be the normalized Haar measure. We thus deduce that

$$\sigma^2(f) = \sum_n \sigma^2(f_n).$$

However, Moore showed that $|\rho_{f,g}(t)| \leq C_n ||f||.||g|| e^{-\alpha_n t}$ and then we deduce that $\sigma^2(f_n) \leq C_n ||f||^2 2 \int_0^\infty e^{-\alpha_n t} dt$

More generally, for any Anosov flow we can associate an expanding map $T : X \to X$, as follows. Consider Markov sections $N = \cup_i N_i$ for the flow and associate the Poincaré map $S : N \to N$ on the sections. Using the local product we can write each N_i as the local product of a piece of stable manifold S_i and unstable manifold U_i. The Poincare map induces an expanding map on $X = \cup_i U_i$. Let $R : N \to \mathbb{R}$ correspond to the return time function. This induces a function $r : X \to \mathbb{R}$.

The following results are well known [4], [16], [17].

Lemma 6.1.

(1) *If ϕ is a geodesic flow on a surface of constant negative curvature then the associated expanding map T and function r are analytic*
(2) *If ϕ is a geodesic flow on a surface of variable negative curvature then the associated expanding map T and function r are C^1.*

We can associate associate a suspension space defined by

$$X^r = \{(x, u) : 0 \leq u \leq r(x)\}$$

where we identify $(x, r(x))$ and $(Tx, 0)$. Let $\Psi : X^r \to X^r$ be the semi-flow defined by $\psi_t(x, u) = (x, u + t)$, for $t > 0$, subject to the identifications. Let $F : X^r$ be a function defined on the suspension space. We can associate a function $f : X \to \mathbb{R}$ by

$$f(x) = \int_0^{r(x)} F(x, u) du.$$

If $dm = d\mu \times dt / \int r d\mu$ then we can relate the variance of the flow to that of the expanding map.

Lemma 6.2. *We can relate*

$$\sigma^2(F) = \frac{\sigma^2(f)}{\int r d\mu}.$$

Example 6.2. We recall that an Anosov flow synchronised flow if the measure of maximal entropy μ is equal to the Sinai-Ruelle-Bowen measure (i.e., $h(\mu) = \int E^u(x) d\mu(x)$, where $E^u(x) := \lim_{t\to 0} \log ||D\phi_t||/t) d\mu$. Any C^2 hyperbolic flow can be reparameterised to give a C^α flow which is synchronised (where C^α is the regularity of the stable foliation). For example, for a geodesic flow on a surface of negative curvature we can always assume the foliation is $C^{1+\alpha}$. In the particular case of such synchronised flows we can consider the associated $C^{1+\alpha}$ expanding map and let $r(x) = \log |T'(x)|$.

Let us then consider the special choice $F : M \to \mathbb{R}$ given by $F = E^u$ and let $\sigma^2(F)$ denote the associated variance. For the associated interval map let $f(x) = g(x) = -\log |T'(x)|$ and then let $\sigma^2(f)$ denote the associated variance. Then by Lemma 6.2 we can write

$$\sigma^2(E^u) = \frac{\sigma^2(f)}{\int E^u d\mu} = \frac{\sigma^2(f)}{h(\mu)}.$$

6.2 Induced maps and Rohlin Towers. Closely related to the idea of suspended flows is that of Rohlin Towers. We will illustrate the general principle with a specific example. Fix $0 < \alpha < 1$. Consider the Manneville Pomeau map $T : [0, 1) \to [0, 1)$ defined by

$$Tx = x^{1+\alpha} + x \pmod 1.$$

Let $0 < b < 1$ be the largest value such that $b^{1+\alpha} + b = 1$. We can associate an expanding map $S : [b, 1) \to [b, 1)$ defined by $S(x) = T^{n(x)}(x)$ where $n(x) = \inf\{n > 0 : T^n x \in [b, 1)\}$.

This map has the advantage that it is uniformly hyperbolic, although it has the disadvantage that it has infinitely many branches. There is a natural S-absolutely continuous invariant measure μ_0 supported on $[0, 1)$ which gives rise to an absolutely continuous T-invariant measure μ supported on $[0, 1]$.

Given a C^∞ function $F : [0, 1] \to \mathbb{R}$ we can associate a function $f : [b, 1] \to \mathbb{R}$ defined by $f(x) = \sum_{i=0}^{n(x)-1} F(T^i x)$. We can relate the variances of these two functions, for the respective transformations, by

$$\sigma^2(f) = \frac{\sigma^2(F)}{\int n(x) d\mu_0}.$$

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