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To motivate some of the ideas we will first consider a somewhat simpler setting (integrals), then progress to the more subtle case (Lyapunov exponents) and eventually dabble a little with entropy.

The underlying method is actually based on some ideas from Thermodynamic Formalism (basically Dynamical Systems, with a dash of Statistical Physics).
The scheme of the talk

1. Computing integrals
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2. Computing Lyapunov exponents
The scheme of the talk

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3. Some examples
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3. Some examples
4. The ingredients in the proof
The scheme of the talk

1. Computing integrals
2. Computing Lyapunov exponents
3. Some examples
4. The ingredients in the proof
5. Computing entropy (rates)
1. Computing integrals

We could try to approximate the integral of an analytic function $f : [0, 1] \rightarrow \mathbb{R}$ by a discrete summation over a finite set of points. For $n \geq 1$ we can naively write

$$\int_0^1 f(x) \, dx \approx \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right).$$
Classical approximations to integrals

There are various classical methods which use different combinations of the same values $f(k/n)$ to give better approximations to the integral (e.g., Newton-Cotes).

For example, the best known is Simpson’s rule

$$
\frac{1}{3n} \left( f(0) + 2 \sum_{k=1}^{n/2-1} f \left( \frac{2k}{n} \right) + 4 \sum_{k=1}^{n/2} f \left( \frac{2k - 1}{n} \right) + f(1) \right).
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$$

For each of these classical methods the error is $O(n^{-k})$ as $n \to \infty$ for some $k > 0$. 
Other approximations to integrals

Let us replace \( \{k/n : 1 \leq j \leq n \} \) by the following finite sets:

**Notation.** For each \( N \geq 1 \) we define a family of points

\[
P_N = \left\{ \frac{j}{(2^m - 1)} : 0 \leq j \leq 2^m - 1, 1 \leq m \leq N \right\}.
\]

This has cardinality \( |P_N| \approx 2^{N+1} \).
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This has cardinality \( |P_N| \approx 2^{N+1} \).

Let us consider the following question:

*Is there a method for combining the values \( \{f(x) : x \in P_N\} \) so as to get a “good” estimate for the integral?*
For $n \geq 0$, we define (for no obvious reason)

$$e_n = (-1)^n \frac{2^{-n(n+1)/2}}{\prod_{i=1}^{n}(1 - 2^{-i})}.$$
Some notation

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For each $1 \leq m \leq N$, define

$$\beta_m(f) = \frac{1}{2m} \sum_{j=0}^{2m-1} f \left( \frac{j}{2m-1} \right).$$
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\[
\beta_m(f) = \frac{1}{2m} \sum_{j=0}^{2^m-1} f \left( \frac{j}{2m-1} \right)
\]

and

\[
\mu_N(f) = \frac{\sum_{m=1}^{N} e_{N-m} \beta_m(f)}{\sum_{m=1}^{N} \frac{e_{N-m}}{1-2^{-m}}}.
\]
Some notation

For $n \geq 0$, we define (for no obvious reason)

$$ e_n = (-1)^n \frac{2^{-n(n+1)/2}}{\prod_{i=1}^{n} (1 - 2^{-i})}. $$

For each $1 \leq m \leq N$, define

$$ \beta_m(f) = \frac{1}{2m} \sum_{j=0}^{2^m-1} f \left( \frac{j}{2m-1} \right) $$

and

$$ \mu_N(f) = \frac{\sum_{m=1}^{N} e_{N-m} \beta_m(f)}{\sum_{m=1}^{N} e_{N-m}}. $$

In particular, $\mu_N(f)$ depends only on the values $f$ takes on the points in $P_N$. 
A good approximation

**Theorem 0:** For any analytic function \( f : [0, 1] \to \mathbb{R} \),

\[
\left| \int_0^1 f(x) \, dx - \mu_N(f) \right| = O\left(\exp(-cN^2)\right) \quad \text{as} \quad N \to \infty,
\]

for some \( c > 0 \).
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for some \( c > 0 \).

Since \( P_N = 2^{N+1} \) the Theorem tells us that error is super-polynomial in \( |P_N| \), i.e., for some \( C > 0 \),

\[
\left| \int_0^1 f(x) \, dx - \mu_N(f) \right| = O \left( \exp(-C \log^2 |P_N|) \right) .
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$$\left| \int_0^1 f(x) \, dx - \mu_N(f) \right| = O \left( \exp(-C \log^2 |P_N|) \right).$$

Aside: The value $c > 0$ can be chosen arbitrarily close to $\log 2$. 
An example (in lieu of a proof)

Example: We can apply the theorem to $\int_0^1 f(x) \, dx$ where

$$f(x) = \frac{5\pi}{2} (e^\pi - 2)^{-1} e^{\pi x} \cos(\pi x/2).$$

The integral is actually equal to 1.
Approximations $\mu_N(f)$ to the integral

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</tr>
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<td>15</td>
<td>0.999999999999999993061845</td>
</tr>
</tbody>
</table>
3. Lyapunov exponents

Let \( \{A_1, A_2\} \) be a fixed pair of \( 2 \times 2 \) matrices.
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The Lyapunov exponent \( \lambda \) is given by the limit

\[
\lambda := \lim_{n \to +\infty} \frac{1}{n} \sum_{i_1, \ldots, i_n} \frac{1}{2^n} \log ||A_{i_1} \cdots A_{i_n}||
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We could consider more matrices and other measures, but this setting illustrates the results while keeping the notation (relatively) simple.
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We could consider more matrices and other measures, but this setting illustrates the results while keeping the notation (relatively) simple.

For simplicity, in the examples we shall also assume that both the matrices \( A_1 \) and \( A_2 \) have determinant 1.
A classical result for typical points

By a famous result of Kesten and Furstenberg one has the following pointwise estimate with respect to the Bernoulli measure $\mu = (\frac{1}{2}, \frac{1}{2})^{\mathbb{Z}^+}$:

Theorem (Furstenberg-Kesten) For a.e. $(\mu) \ i \in \Sigma$ one has

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**Standing assumption**: Henceforth, we shall assume that the matrices are all positive (i.e., each of the entries of each of the matrices $A_1, A_2$ are strictly positive).
Some notation

Given a finite string \( \mathbf{i} = (i_1, i_2, \cdots, i_n) \in \{1, 2\}^n \) let \(|\mathbf{i}| = n\) denote its length and let \(\sigma \mathbf{i} = (i_2, \cdots, i_n, i_1)\) denote a cyclic permutation.
Some notation

Given a finite string $\bar{i} = (i_1, i_2, \ldots, i_n) \in \{1, 2\}^n$ let $|\bar{i}| = n$ denote its length and let $\sigma\bar{i} = (i_2, \ldots, i_n, i_1)$ denote a cyclic permutation. Let $A_{\bar{i}} = A_{i_1} \cdots A_{i_n}$ denote the product matrix and let $A_{\bar{i}}x_{\bar{i}} = \lambda_{\bar{i}}x_{\bar{i}}$, where $\lambda_{\bar{i}}$ and $x_{\bar{i}}$ are the maximal positive eigenvalue and eigenvector (by the Perron-Frobenius Theorem).
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Let \( A\underline{i} = A_{i_1} \cdots A_{i_n} \) denote the product matrix and let \( A\underline{i}x\underline{i} = \lambda\underline{i}x\underline{i} \), where \( \lambda\underline{i} \) and \( x\underline{i} \) are the maximal positive eigenvalue and eigenvector (by the Perron-Frobenius Theorem).
For \( j = 1, 2 \) and \( t \in \mathbb{R} \) we define a complex function

\[
d^{(j)}(z, t) = \exp \left( -\sum_{n=1}^{\infty} \frac{z^n}{2n^n} \sum_{|\underline{i}|=n} \frac{1}{1 - \lambda_{\underline{i}}^{-2}} \left( \prod_{l=0}^{n-1} \frac{||A_{j}x_{\sigma^{l}\underline{i}}||}{||x_{\sigma^{l}\underline{i}}||} \right)^t \right)
\]

which converges to an analytic function for \( |z| \) small.
The “explicit” expression

We can deduce the following.

**Theorem 1:** For each $j = 1, 2$, the function $d^{(j)}(z, t)$ is analytic for $z \in \mathbb{C}$ and $t \in \mathbb{R}$ and we can write

$$\lambda = \frac{\partial d^{(1)}}{\partial t} (1, 0) \frac{\partial d^{(1)}}{\partial z} (1, 0) + \frac{\partial d^{(2)}}{\partial t} (1, 0) \frac{\partial d^{(2)}}{\partial z} (1, 0).$$
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\]

Moreover, expanding

\[
d^{(j)}(z, t) = 1 + \sum_{n=1}^{\infty} a^{(j)}_n(t) z^n
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Moreover, expanding $d^{(j)}(z, t) = 1 + \sum_{n=1}^{\infty} a_{n}^{(j)}(t)z^{n}$ we have:

(i) the $a_{n}^{(j)}$ depend only on the maximal eigenvectors and the eigenvalues of the matrices $A_{i}$, with $|i| \leq n$; and
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\]

Moreover, expanding \( d^{(j)}(z, t) = 1 + \sum_{n=1}^{\infty} a^{(j)}_n(t) z^n \) we have:

(i) the \( a^{(j)}_n \) depend only on the maximal eigenvectors and the eigenvalues of the matrices \( A_i \), with \( |i| \leq n \); and

(ii) we can bound \( |a^{(j)}_n| = O(\exp(-Cn^2)) \).
Approximating $\lambda$

**Theorem 2:** There are approximations $\lambda_n$ to $\lambda$ defined in terms of the maximal eigenvectors and the eigenvalues of the $2^{n+1}$ matrices $A_i$, with $|i| \leq n$. Moreover, for some $C > 0$

$$|\lambda - \lambda_n| = O \left( \exp \left( -Cn^2 \right) \right).$$
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Theorem 2 shows that $\lambda_n \to \lambda$ faster than any exponential.
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Theorem 2 shows that $\lambda_n \rightarrow \lambda$ faster than any exponential.

*We can deduce Theorem 2 from Theorem 1, by setting*

$$\lambda_n = \frac{\sum_{k=1}^{n} \frac{\partial a_k^{(1)}}{\partial t}(0)}{\sum_{k=1}^{n} (k + 1) a_k^{(1)}(0)} + \frac{\sum_{k=1}^{n} \frac{\partial a_k^{(2)}}{\partial t}(0)}{\sum_{k=1}^{n} (k + 1) a_k^{(2)}(0)}.$$
3. Example 1

If we consider the matrices

\[ A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \]

then \( \lambda_{n=9} \) this gives an approximation

\[ \lambda = 1.1433110351029492458432518536555882994025 \cdots \]

to the Lyapunov exponent \( \lambda \).

This is presented to the number of decimal places to which it empirically appears to be accurate (i.e., by comparison with the other terms \( \{ \lambda_n \} \) in the approximation).
The convergent sequence

Using the complex functions to estimate $\lambda$ we get:

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<th>$\lambda_n$</th>
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</tr>
<tr>
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<td>1.14331103510294924584325185365555882994025</td>
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</tbody>
</table>

For $n = 9$ this is empirically accurate to the 40 decimal places presented.
Super exponential convergence

This plots the number of decimal places in which $\lambda_n$ and $\lambda_{n+1}$ agree, against $n$. 
Example 2

Consider the matrices

\[ A_1 = \begin{pmatrix} 10 & 1 \\ 9 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 9 & 1 \\ 8 & 1 \end{pmatrix} \]

then the approximations to \( \lambda \) are:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \lambda_n )</th>
</tr>
</thead>
<tbody>
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</tr>
</tbody>
</table>

For \( n = 9 \) this is empirically accurate to the 91 decimal places presented.
Example 3

Consider the matrices

\[
A_1 = \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix}.
\]

The approximations to \( \lambda \) are:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \lambda_n )</th>
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For \( n = 9 \) this is empirically accurate to the 84 decimal places presented.
4. Overview of the proof

The proof draws on old ideas from the work of Ruelle [1976] (based on even earlier work of Grothendieck [1955]).

Ultimately we need to connect $\lambda$ to the complex functions $d^{(j)}(z, t)$, which play a rôle roughly akin to characteristic polynomials for matrices.

The role of matrices is replaced by (Ruelle) transfer operators defined on spaces of (analytic) functions.
Strategy of the proof (of Theorem 1)

Step 1: Consider action of $A_1$, $A_2$ on projective space
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Step 2: Associate (transfer) operators and show they are trace class
Strategy of the proof (of Theorem 1)

- Step 1: Consider action of $A_1$, $A_2$ on projective space
- Step 2: Associate (transfer) operators and show they are trace class
- Step 3: Relate Lyapunov exponents to transfer operators
Strategy of the proof (of Theorem 1)

- Step 1: Consider action of $A_1$, $A_2$ on projective space
- Step 2: Associate (transfer) operators and show they are trace class
- Step 3: Relate Lyapunov exponents to transfer operators
- Step 4: Relate the transfer operators to the complex functions $d^{(j)}(z, t)$ and show that the terms in the expansion of $z \mapsto d^{(j)}(z, t)$ converge quickly.
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Steps 1 and 3 are standard.
Grothendieck and Ruelle
Step 1: Projective actions

Consider the real projective space

$$\mathbb{R}P^2 = (\mathbb{R}^2 - \{(0, 0)\})/\sim$$

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It is often convenient to think of $\mathbb{R}P^2$ in terms of representative vectors $x \in \mathbb{R}^2$ with $||x|| = 1$

The projective action $\overline{A}_j : \mathbb{R}P^2 \to \mathbb{R}P^2$ takes the form

$$\overline{A}_j(x) = \frac{A_j x}{||A_j x||},$$

for $j = 1, 2$. 
Let $\Delta \subset \mathbb{R}P^2$ correspond to the positive quadrant.
The projective actions as contractions

The one-dimensional set $\Delta$ can be easily identified with the interval $[0, 1]$, say.
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In these coordinates we can choose a neighbourhood \([0, 1] \subset U \subset \mathbb{C} \) (the complexification) such that we have \( \overline{A_1(U)}, \overline{A_2(U)} \subset \text{closure}(U) \).
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Step 2: Ruelle transfer operator

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We can define a bounded linear operator $\mathcal{L} : C^\omega(U) \to C^\omega(U)$ by

$$\mathcal{L}w(x) = \frac{1}{2}w(A_1 x) + \frac{1}{2}w(A_2 x).$$

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Crucially, this operator is trace class (on the analytic functions $C^\omega(U)$). Moreover, each $\text{tr}(\mathcal{L}^n)$ can be explicitly expressed in terms of the matrices $A_\|i\|$ with $|\|i\|| = n$. 
Step 3: transfer operators and $\lambda$

The maximal eigenvalue $1$ for $\mathcal{L}$ has an eigenprojection $\mu$
i.e., $\mathcal{L}^* \mu = \mu$. 
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Define functions $f_j : \mathbb{RP}^2 \rightarrow \mathbb{R}$ by

$$f_j(x) := \log \left( \frac{||A_j x||}{||x||} \right), \text{ for } j = 1, 2.$$
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Lemma (Lyapunov exponent from transfer operator).

$$\lambda = \frac{1}{2} \mu(f_1) + \frac{1}{2} \mu(f_2).$$

(“Furstenberg’s formula”).
Step 4: Determinants

We can define determinants by

\[ d(z) := \exp \left( - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr}(L^n) \right) \]
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**Theorem (Grothendieck, Ruelle).** We can expand

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d(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \text{ where } |a_n| = O(e^{-cn^2}).
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*If we replaced \( L \) by a matrix then \( d(z) \) would just be a polynomial.*

The complex functions \( d^{(j)}(z, t) \) have similar properties. The rest of the proof is then routine stuff.
5. Speculation on entropy (rates)

We might like to try to adapt the previous approach to describe the entropy rate of some Hidden Markov Processes.
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However, it seems like a better idea to try to approach estimating integrals (with respect to the Blackwell measure)
Two transformations

Fix $\epsilon > 0$ and $0 < p < 1$. 
Two transformations

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Consider the transformations $f_1 : \mathbb{R}^+ \to \mathbb{R}^+$ to $f_2 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$f_1(x) = \frac{(1 - \epsilon) \, px + (1 - p)}{\epsilon} \frac{(1 - p) \, x + p}{(1 - p) \, x + p}$$

and

$$f_2(x) = \frac{\epsilon}{(1 - \epsilon)} \frac{(1 - p) \, x + p}{px + (1 - p)}.$$
A measure

We can also associate weight functions $r_1 : \mathbb{R}^+ \to \mathbb{R}$ and $r_2 : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$r_1(x) = \frac{((1 - \epsilon)p + \epsilon(1 - p))x + (1 - \epsilon)(1 - p) + \epsilon p}{x + 1}$$

$$r_2(x) = \frac{\epsilon p + (1 - \epsilon)(1 - p)x + \epsilon)(1 - p) + (1 - \epsilon)p}{x + 1}.$$
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**Definition.** We can define the (Blackwell) probability measure $\mu$ by the identity

$$\mu(A) = \int_{f_1^{-1}A} r_1(x)d\mu(x) + \int_{f_2^{-1}A} r_2(x)d\mu(x)$$
Consider a binary symmetric channel with crossover probability $\epsilon > 0$ and input Markov chain with transition matrix

$$
\begin{pmatrix}
p & 1 - p \\
1 - p & p
\end{pmatrix}.
$$

**Theorem (Han-Marcus).** The entropy rate is given by the following:

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H = - \int (r_1(x) \log r_1(x) + r_2(x) \log r_2(x)) \, d\mu(x).
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This identity has an explicit integrand and a mysterious measure. But one can hope to try estimate this integral using the same method as before.
A transfer operator

We can consider the associate transfer operator $\mathcal{L}$ defined on analytic functions on $\mathbb{R}^+$ by

$$\mathcal{L}w(x) = w(f_1 x)r_1(x) + w(f_2 x)r_2(x).$$
A transfer operator

We can consider the associate transfer operator $L$ defined on analytic functions on $\mathbb{R}^+$ by

$$Lw(x) = w(f_1 x) r_1(x) + w(f_2 x) r_2(x).$$

By analogy with the approach for Lyapunov exponents, we might try to estimate the integral for the entropy rate using complex functions defined in terms of the fixed points

$$f_{i_1} \cdots f_{i_n} x = x$$

of the maps $f_{i_1} \cdots f_{i_n} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, where $i_1, \cdots, i_n \in \{1, 2\}$ and $n \geq 1$, and weights

$$r_{i_1}(x) r_{i_2}(f_{i_1} x) \cdots r_{i_n}(f_{i_{n-1}} \cdots f_{i_1} x).$$
Consider the specific case of $\epsilon = 0.01$ and $p = 0.3$. We get the following approximations to the entropy rate $H$:

<table>
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<th>$n$th approximation to $H$</th>
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<td>7</td>
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