STATISTICAL PROPERTIES OF THE RAUZY-VEECH-ZORICH MAP

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0. Introduction

In this note we will consider Rauzy-Veech-Zorich renormalization map for interval exchange maps. The special case of interval exchange transformations on two intervals simply corresponds to rotations on the unit circle, and in this case the corresponding renormalization map reduces to the usual continued fraction transformation. Thus, one might naturally view interval exchange maps on n intervals as generalizations of circle rotations; and the renormalization map as a generalization of the classical continued fraction transformation. It was shown by Masur and Veech that their original renormalization map $\mathcal{T}_0$ possesses an absolutely continuous ergodic invariant measure, and Zorich showed that for the accelerated version $\mathcal{T}_1$ there is a finite invariant measure.

A number of interesting statistical results already have already been established for the renormalization map, and related transformations (e.g., Central Limit Theorems and other Limit Theorems cf. [2], [1], [14]). The first aim of this paper is to present an alternative approach to some of these results, and to give some simple generalizations. Indeed, for dynamical systems in general there is a potential heirarchy of statistical properties that one may establish for such maps, beginning with ergodicity; central limits theorems; functional central limit theorems, and finally almost sure invariance principles. In this paper we will re-derive the central limit theorem, the stronger functional central limit theorem, and establish the almost sure invariance principle, from which the others then follow. A basic technique, familiar from other non-uniformly hyperbolic settings, is to induce a hyperbolic map $\mathcal{T}_2$ on a smaller set B in the domain of $\mathcal{T}_1$. In particular, statistical properties are typically easier to establish for $\mathcal{T}_2$, and these can then be "lifted" to the map $\mathcal{T}_1^2$. There is a well known application of related results to Teichmüller flows for abelian differentials, which can be modeled in terms of suspended flows over these maps. The corresponding statistical properties for Teichmüller flows can be deduced from the properties of $\mathcal{T}_2$ by applying the methods of Melbourne-Török.

Theorem A. *The transformations $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the functional central limit theorem. In particular, they satisfy the law of the iterated logarithm and the arcsine law.*

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The second aim of this paper is to describe a zeta function associated to $\mathcal{T}_2$. This is defined by analogy with the Ruelle zeta function for Axiom A diffeomorphisms. The poles of these zeta functions (and the residues of associated complex functions) encapsulate dynamical information about the maps. Moreover, when these invariants vanish then the zeta function takes a particularly trivial form.

We will initially follow Morita in studying a transfer operator associated to $\mathcal{T}_2$ acting on Lipschitz (or, more generally, Hölder) continuous functions [14]. This allows us to apply the method of MacKay and Tyran Kaminski [7,8], to give a direct proof of the (Functional) Central Limit Theorem, and the method of Philipp-Stout [16], as developed in the dynamical context by Melbourne and Nicol [12], to show the almost everywhere invariance principles. Subsequently, we will consider a transfer operator associated to $\mathcal{T}_2$ on a smaller space of an analytic functions and study the complex function

$$\eta_f(z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\mathcal{T}_2^n x = x} \frac{f^n(x)}{|\det(D\mathcal{T}_2^n)(x)|}.$$

where f is an Dirichlet function.

Theorem B. *The function $\eta_f(s)$ is entire.*

As a corollary, we can apply a powerful approach of Ruelle [17] (cf also Mayer [9,10]) based on Fredholm determinants to give an alternative expression for the the top Lyapunov exponent.

The methods in this note will work for other multidimensional continued fraction type algorithms, for which the (accelerated) Rauzy-Veech-Zorich algorithm forms a topical example. In particular, a suspension flow for this map forms a well known model for the Teichmüller flow.

In section 1 we recall results on interval exchanges and their renormalizations. In section 2 we introduce the transfer operator on Hölder continuous functions and recall the results of Morita on its spectra. In section 3 we prove the statistical properties for the induced map $\mathcal{T}_2$. In section 4 we derive the statistical properties for the Zorich map $\mathcal{T}_1$. In section 5 we study the transfer operator on the smaller space of analytic function, and in section 6 we use these results to study Lyapunov exponents and zeta functions. Finally, in section 7 we describe the extension to Teichmüller flows.

1. Interval exchange transformation

In this section we recall some of the basic constructions. We refer the reader to the excellent surveys [20] and [21] for further details.

Interval exchange transformations $T : [0,1] \to [0,1]$ are piecewise isometries of the unit interval. In the case of two intervals, this corresponds to a rotation of the circle, i.e., a translation of the interval (modulo one). More generally, assume that I is partitioned into n intervals $I_1, \cdots, I_n$ of lengths $\lambda_1, \cdots, \lambda_n$, respectively, upon each of which T acts isometrically. We can represent this partition as a vector λ in the standard $(n-1)$-dimensional simplex

$$\Delta = \{\lambda = (\lambda_1, \cdots, \lambda_n) : 0 < \lambda_1, \cdots, \lambda_n < 1 \text{ and } \lambda_1 + \cdots + \lambda_n = 1\}.$$

Thus the transformation T is completely determined by these lengths, and by order of the images of the original intervals. This latter information is encapsulated by a permutation π on $\{1, \cdots, n\}$. In particular, every interval exchange transformation corresponds to a pair (λ, π), where $\lambda \in \Delta$ and π is a permutation. Moreover, corresponding to the natural assumption that T doesn't have contain an invariant subsystem, we assume that π is irreducible if there is no $1 \le l < k$ such that $\pi(\{1, \cdots, l\}) = \{1, \cdots, l\}$.

The classical Keane Conjecture (proved by Masur and Veech, independently) states that the transformation T is uniquely ergodic for almost all $\lambda \in \Delta$. The method of proof lead to the development of an important renormalization scheme on such transformations, which we will briefly describe.

1.1 The Rauzy class of permutations. Given a permutation π, let us denote by $k = \pi^{-1}(n)$ (i.e., $\pi(k) = n$). A key idea of Rauzy was to replace the permutation π by one of two new permutations: either

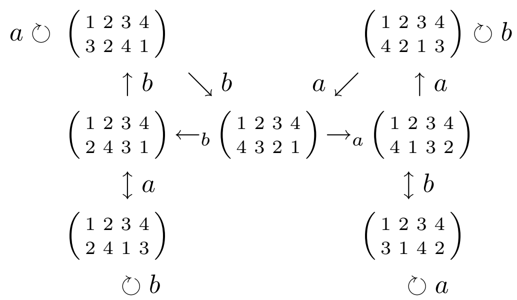
$$a\pi(j) := \begin{cases} \pi(j) & \text{if } 1 \le \pi(j) \le k \\ \pi(n) & \text{if } j = k+1 \\ \pi(j-1) & \text{if } k+2 \le j \le n \end{cases} \quad \text{or } b\pi(j) := \begin{cases} \pi(j) & \text{if } 1 \le j \le \pi(n) \\ \pi(j)+1 & \text{if } \pi(n) < \pi(j) < n \\ \pi(n)+1 & \text{if } j = k \end{cases}$$

If we start from a given permutation we do not necessarily get all permutations by these two operations. This leads to the following definition.

Definition. Given a permutation π the Rauzy class $\mathcal{R}$ consists of all permutations that can be derived from π by repeatedly applying these two operations.

Thus the irreducible permutations are a union of a finite number of Rauzy classes.

Example 1 ($n = 4$). The irreducible permutation $\pi_0 = \binom{1\,2\,3\,4}{4\,3\,2\,1}$ lies in a Rauzy class of 7 permutations. These are illustrated in the following diagram, where an arrow labeled by a goes from π to $a\pi$ (and an arrow labeled by b goes from π to $b\pi$).



There are excellent descriptions of this procedure in [20].

1.2 The Rauzy-Veech renormalization $\mathcal{T}_0$. Consider some given $1 \le k \le n$. We can then apply one of the following two operations on the vector $\lambda = (\lambda_1, \cdots, \lambda_n)$, to produce a new vector $\lambda' = (\lambda'_1, \cdots, \lambda'_n)$: Either

Case I ($\lambda_n > \lambda_k$): Let $\lambda \mapsto \lambda' = (\lambda_1, \cdots, \lambda_{n-1}, \lambda_n - \lambda_k)$; or

Case II ($\lambda_k > \lambda_n$): Let $\lambda \mapsto \lambda' = (\lambda_1, \cdots, \lambda_{k-1}, \lambda_k - \lambda_n, \lambda_n, \lambda_{k+1}, \cdots, \lambda_{n-1})$.

Firstly, we would like to make a particular choice of case such that vector λ' is strictly positive. The case $\lambda_k = \lambda_n$ is therefore ambiguous, but atypical, and shall be ignored. Secondly, we observe that the definition of λ' is such that it does not lie in the simplex Δ. However, this will soon be corrected by rescaling.

We can define a map $\mathcal{T}_0$ from $\Delta \times \mathcal{R}$ to itself (modulo some codimension one planes, as described above, on which it is ambiguously defined). This will be a renormalization map, in the sense that it associates a new interval exchange map to an old one (with the same number of intervals, n). To be more precise, given $\pi \in \mathcal{R}$ we denote

$$\Delta_\pi^+ = \{(\lambda, \pi) \in \Delta \times \{\pi\} : \lambda_n > \lambda_{\pi^{-1}n}\} \text{ and}$$
$$\Delta_\pi^- = \{(\lambda, \pi) \in \Delta \times \{\pi\} : \lambda_n < \lambda_{\pi^{-1}n}\}.$$

We can define a transformation $\mathcal{T}_0 : \Delta \times \mathcal{R} \to \Delta \times \mathcal{R}$ a.e. by

$$\mathcal{T}_0(\lambda, \pi) = \left(\frac{\lambda'}{||\lambda'||_1}, \pi'\right) = \begin{cases} \left(\frac{(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n - \lambda_k)}{1-\lambda_k}, a\pi\right) & \text{if } \lambda \in \Delta_\pi^- \\ \left(\frac{(\lambda_1, \cdots, \lambda_{k-1}, \lambda_k - \lambda_n, \lambda_n, \lambda_{k+1}, \cdots, \lambda_{n-1})}{1-\lambda_n}, b\pi\right) & \text{if } \lambda \in \Delta_\pi^+ \end{cases}$$

where we divide by $||\lambda'||_1 = \sum_i \lambda_i'$ so as to rescale the image vectors to lie on the simplex Δ.

Example 1 revisited. Let $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. We can again consider the Rauzy class $\mathcal{R}$ of $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ as described above. We can then consider, say, the restriction of the map to the simplex labelled by $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$. Since $k = \pi^{-1}(4) = 3$ we have that

$$\mathcal{T}_0\left((\lambda_1, \lambda_2, \lambda_3, \lambda_4), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\right)$$
$$= \begin{cases} \left((\frac{\lambda_1}{1-\lambda_3}, \frac{\lambda_2}{1-\lambda_3}, \frac{\lambda_3}{1-\lambda_3}, \frac{\lambda_4-\lambda_3}{1-\lambda_3}), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\right) & \text{if } \lambda_4 > \lambda_3 \\ \left((\frac{\lambda_1}{1-\lambda_4}, \frac{\lambda_2}{1-\lambda_4}, \frac{\lambda_3-\lambda_4}{1-\lambda_4}, \frac{\lambda_4}{1-\lambda_4}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}\right) & \text{if } \lambda_4 < \lambda_3. \end{cases}$$

Unfortunately, these transformations aren't uniformly hyperbolic (as one can readily see since some of the boundaries of the simplicies remain fixed (e.g., the side $\lambda_3 = 0$ in the simplex). This will be partly remedied by replacing $\mathcal{T}_0$ by maps which are "more hyperbolic".

1.3 The Zorich accelerated remormalization $\mathcal{T}_1$. Following Zorich, one can consider a map $\mathcal{T}_1 : \Delta \times \mathcal{R} \to \Delta \times \mathcal{R}$ defined a.e. by $\mathcal{T}_1(\lambda, \pi) = \mathcal{T}_0^{n(\lambda,\pi)}(\lambda, \pi)$ where

$$n(\lambda, \pi) = \inf\{k > 0 : \mathcal{T}_0^k(\lambda, \pi) \in \Delta^\pm \times \mathcal{R} \text{ where } \lambda \in \Delta^\mp\}$$

and where we denote $\Delta^+ = \bigcup_{\pi \in \mathcal{R}} \Delta_\pi^+$ and $\Delta^- = \bigcup_{\pi \in \mathcal{R}} \Delta_\pi^-$.

The following elegant result was proved by Zorich.

Propostion 1.1 (Zorich). *The transformation $\mathcal{T}_1$ preserves a finite absolutely continuous invariant measure μ_1 (i.e., $\mu_1(\Delta \times \mathcal{R}) < +\infty$). Moreover, the restriction $\mathcal{T}_1^2 : \Delta^+ \to \Delta^+$ is ergodic (and $\mathcal{T}_1^2 : \Delta^- \to \Delta^-$ is ergodic).*

Previously, Masur and Veech had shown the existence of a sigma finite $\mathcal{T}_0$-invariant measure μ_0, which can be easily recovered from μ_1.

However, to gain more control over the distortion properties of the transformations one can induce on a smaller set, so as to get a transformation which has even stronger properties.

1.4 The induced map $\mathcal{T}_2$ on a smaller set. Let $\mathcal{P} = \{\Delta_\pi^+, \Delta_\pi^- : \pi \in \mathcal{R}\}$ be the natural finite partition of Δ then we can define the refinements

$$\mathcal{P}_n := \vee_{k=0}^{n-1} {\mathcal{T}_1}^{-k} \mathcal{P} = \{P_{i_1} \cap {\mathcal{T}_1}^{-1} P_{i_2} \cap \cdots \cap {\mathcal{T}_1}^{-n-1} P_{i_{n-1}} : P_j \in \mathcal{P}\}$$

for any $n \geq 1$. Following a now standard approach we can can choose $n_0 > 1$ and $B \in \mathcal{P}_{n_0}$, say, to be any image of an inverse branch of $\mathcal{T}^{n_0}$ which is a contraction.[1]

Finally, we can then consider the induced map $\mathcal{T}_2 : B \to B$ defined by $\mathcal{T}_2(\lambda, \pi) = {\mathcal{T}_1}^{\widehat{n}(\lambda,\pi)}(\lambda, \pi)$ where

$$\widehat{n}(\lambda, \pi) = \inf\{k > 0 : \mathcal{T}_1^k(\lambda, \pi) \in B\}.$$

The following is immediate from the observation that the composition of projective transformation remains projective.

Lemma 1.1. *The induced map $\mathcal{T}_2 : B \to B$ is piecewise projective expanding map of the general form*

$$(\lambda_1, \cdots, \lambda_n) \mapsto \left(\frac{\sum_{j=1}^d a_{1j}\lambda_j}{\sum_{i,j=1}^d a_{ij}\lambda_j}, \cdots, \frac{\sum_{j=1}^d a_{dj}\lambda_j}{\sum_{i,j=1}^d a_{ij}\lambda_{ij}} \right)$$

on each of the pieces $B_n := \{x \in B : \widehat{n}(\lambda, \pi) = n\}$, $n \geq 1$.

We are now in a position to use familiar techniques for the study of hyperbolic maps.

2. Transfer operators

Let ω denote the natural volume form on B. We can formally define a linear map $\mathcal{L} : L^1(B, \omega) \to L^\infty(B, \omega)$ associated to $\mathcal{T}_2 : B \to B$ by the identity

$$\int_B \mathcal{L}f(x) g(x) d\omega(x) = \int_B \mathcal{L}f(x) g(\mathcal{T}_2 x) d\omega(x), \text{ where } f \in L^1(B), g \in L^\infty(B)$$

and we denote $x = (\underline{\lambda}, \pi) \in B$. (The existence of such a $\mathcal{L}f(x) \in L^\infty(B)$ follows immediately from the Riesz representation theorem). Moreover, we can use the change of variables formula to formally write:

$$\mathcal{L}f(x) = \sum_{y \in {\mathcal{T}_2}^{-1} x} \frac{f(y)}{|\mathrm{Jac}(\mathcal{T}_2)(y)|} \text{ a.e..}$$

In fact, a simple calculation shows:

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[1] All of these transformations are projective, i.e., matrices act linearly on vectors, followed by normalizing. Such a transformation is contracting in the projective matrix when the simplex is mapped strictly inside itself, which happens when the matrix is strictly positive

Lemma 2.1. *Let A be the matrix such that $\mathcal{T}_2^k(x) = \frac{Ax}{||Ax||_1}$. We can write the Jacobian as $Jac(\mathcal{T}_2^k)(x) = ||Ax||_1^n$.*

From this explicit formula for the Jacobian one easily sees that $\mathcal{L}(C^0(B)) \subset C^0(B)$. In order to get stronger results on $\mathcal{T}_2$, we need to consider the operator acting on smaller Banach spaces than $C^0(B)$. In section 6 we will consider the operator acting on analytic functions. However, for the present we shall follow the more classical approach of studying the operator acting on Hölder continuous functions.

Given $\beta > 0$ and a function $w : B \to \mathbb{C}$, we define $||w|| = ||w||_\infty + ||w||_\beta$ where

$$||w||_\beta = \sup_{x \neq y} \frac{|w(x) - w(y)|}{||x - y||^\beta}$$

and let $C^\beta(B) = \{w : B \to \mathbb{C} : ||w|| < \infty\}$. When $\beta = 1$ these are simply the Lipschitz functions. The next result can be used to show that $\mathcal{L}$ preserves Hölder functions. Let $\mathcal{Q} = \{Q_n : k \geq 1\}$, where $Q_n := \{x : \widehat{n}(x) = k\}$, and let $\mathcal{Q}_k = \vee_{i=0}^{k-1} \mathcal{T}_2^{-i} Q$. The following result is basically due to Morita [14]

Lemma 2.2.

(1) *There exists $C > 0$ and $\Theta > 1$ such that for any $n \geq 1$ and x, y in the same element of $\mathcal{Q}_n$ we have*

$$||\mathcal{T}_2{}^n x - \mathcal{T}_2{}^n y|| \geq C\Theta^n ||x - y||;$$

(2) *There exists $C > 0$ such that for any $n \geq 1$ and x, y lie in the same element of $\mathcal{Q}_n$ we have*

$$\left| \log \left(\frac{Jac(\mathcal{T}_2^n)(x)}{Jac(\mathcal{T}_2^n)(y)} \right) \right| \leq C ||\mathcal{T}_2{}^n x - \mathcal{T}_2{}^n y||.$$

(3) *There exists $D > 1$ such that for any $A \in \mathcal{Q}_n$ and any $x \in A$ we can estimate*

$$\frac{1}{D} \leq \frac{\mu(A)}{|Jac(\mathcal{T}_2^n)(x)|} \leq D.$$

Proof. These results are based on the basic observation that the first return map $\mathcal{T}_2 : B \to B$ must be of the form $\mathcal{T}_2(x) = \mathcal{T}_1^{\widehat{n}(\lambda,\pi)}(x) = \mathcal{T}_1^{\widehat{n}(\lambda,\pi)-n_0} \circ \mathcal{T}_1^{n_0}(x)$, where $\mathcal{T}_1^{\widehat{n}(\lambda,\pi)-n}$ does not contract distances and $\mathcal{T}_1^{n_0}$ definitely expands them. Full details can be found in [14]. $\square$

Corollary 2.2.1. *The operator $\mathcal{L}$ preserves the space of Hölder functions, i.e., $\mathcal{L} : C^\beta(B) \to C^\beta(B)$ is well defined.*

Many of the statistical results for $\mathcal{T}_2$ are related to the existence of a spectral gap for $\mathcal{L}$. In the case of the operator acting on analytic functions is essentially automatic since the operator is compact (as we will see later). However, in the present context of Hölder continuous functions it remains true.

Lemma 2.3. *The value* 1 *is a simple eigenvalue with a positive eigenfunction* $\rho > 0$. *The rest of the spectrum is contained in a disk of radius strictly smaller than* 1.

Proof. The proof follows a classical approach [15]. Given $g \in C^\beta(B)$, we can estimate for each $x \in B$, we can bound

$$|(\mathcal{L}^n g)(x)| \leq ||g||_\infty \left(\sum_{\mathcal{T}_2^n y = x} \frac{1}{\mathrm{Jac}(\mathcal{T}_2^n)(y)} \right) \leq D||g||_\infty$$

by part (3) of Lemma 2.2. Thus $||\mathcal{L}^n g||_\infty \leq D||g||_\infty$. Similarly, in the special case $g = 1$ we can see that $D^{-1} \leq \mathcal{L}^n(1) \leq D$, for all $x \in B$.

Given $x_1, x_2 \in B$, assume that $y_i \in (\mathcal{T}_2^n)^{-1} x_i$ $(i = 1, 2)$ are chosen in the same inverse branch. With this convention, we write that

$$\begin{aligned}
&(\mathcal{L}^n g)(x_1) - (\mathcal{L}^n g)(x_2) \\
&= \sum_{\mathcal{T}_2^n y_i = x_i} \left(\frac{1}{\mathrm{Jac}(\mathcal{T}_2^n)(y_1)} - \frac{1}{\mathrm{Jac}(\mathcal{T}_2^n)(y_2)} \right) g(y_1) + \sum_{\mathcal{T}_2^n y_2 = x_2} \frac{(g(y_1) - g(y_2))}{\mathrm{Jac}(\mathcal{T}_2^n)(y)}.
\end{aligned}$$

Thus we can bound

$$\begin{aligned}
&|(\mathcal{L}^n g)(x_1) - (\mathcal{L}^n g)(x_2)| \\
&\leq \sum_{\mathcal{T}_2^n y_2 = x_2} \frac{1}{|\mathrm{Jac}(\mathcal{T}_2^n)(y_2)|} \left(\left(e^{C\Theta^{-n} \mathrm{diam}(\Delta^n)} - 1 \right) ||g||_\infty + \Theta^{-n}.||x_1 - x_2||.||g||_\beta \right) \\
&\leq D \left(\frac{C||g||_\infty}{1 - \Theta^{-1}} + \frac{||g||_\beta}{\Theta^n} \right) ||x_1 - x_2||.
\end{aligned} \tag{2.1}$$

(This gives the well known Marinescu-Tulcea inequality)

In particular, the family $\{\frac{1}{N} \sum_{n=0}^{N-1} \mathcal{L}^n 1\}_{N=1}^\infty$ is equicontinuous and bounded, and thus has a uniform accumulation point $\rho \in C^\beta(B)$, say, where $D^{-1} \leq \rho(x) \leq D$, for all x. Clearly, $\mathcal{L}\rho = \rho$ is a positive eigenfunction for the eigenvalue 1. To see that 1 is a simple eigenvalue, assume that $\mathcal{L}\rho' = \rho'$, and then choose the largest $\epsilon > 0$ that the eigenfunction $\rho_\epsilon := \rho + \epsilon\rho' \geq 0$. Since we can find $x \in B$ with $\rho_\epsilon(x) = 0$, it then follows from $\mathcal{L}\rho_\epsilon = \rho_\epsilon$ that $\rho_\epsilon(y) = 0$, for all $y \in \mathcal{T}_2^{-1} x$. Proceeding inductively, we see that $\rho_\epsilon(y)$ vanishes on the dense set $y \in \cup_{n=0}^\infty \mathcal{T}_2^{-n} x$, and thus $\rho' = \epsilon\rho$, i.e., 1 is a simple eigenvalue. We can define $\widehat{\mathcal{L}} : C^\beta(B) \to C^\beta(B)$ by

$$\widehat{\mathcal{L}} w(x) = \frac{1}{\rho(x)} \mathcal{L}(w\rho)(x).$$

then $\widehat{\mathcal{L}}1 = 1$ (and $\widehat{\mathcal{L}}^* \mu = \mu$) and again the Marinescu-Tulcea inequality holds for $\widehat{\mathcal{L}}$, i.e., $||\widehat{\mathcal{L}}^n w||_\beta \leq C||w||_\infty + \Theta^{-n}||w||_\beta$. Moreover, since for any $w \in C^\beta(B)$ we have $w(x) \geq \widehat{\mathcal{L}} w(x) \geq \widehat{\mathcal{L}}^2 w(x) \geq \cdots$ we can deduce from the equicontinuity that there is a unique limit which, using that $\widehat{\mathcal{L}}1 = 1$, we conclude must be the constant $\int w d\mu$, i.e., $\widehat{\mathcal{L}}^n w \to \int w d\mu$ and $n \to +\infty$.

Finally, to show that the rest of the spectrum of $\mathcal{L}$ is contained strictly within the unit disc it suffices to show the same for $\widehat{\mathcal{L}}$ and, more particularly, $\widehat{\mathcal{L}} : C^\beta(B)/\mathbb{C} \to$

$C^\beta(B)/\mathbb{C}$ has spectral radius strictly smaller than 1. However, the convergence of $\widehat{\mathcal{L}}^n w$ implies that $||\widehat{\mathcal{L}}^n w + \mathbb{C}||_\infty \to 0$ as $n \to +\infty$ and thus two applications of the Marinescu-Tulcea inequality gives

$$\begin{aligned} ||\widehat{\mathcal{L}}^{2n} w||_\beta &\leq C\left(||\widehat{\mathcal{L}}^n w + \mathbb{C}||_\infty + \Theta^{-n}||\widehat{\mathcal{L}}^n w||_\beta\right) + \Theta^{-n}||\widehat{\mathcal{L}}^n w||_\beta \\ &\leq C\left(||\widehat{\mathcal{L}}^n w + \mathbb{C}||_\infty + \Theta^{-n}(C+1)\left(C||w||_\infty + ||w||_\beta\Theta^{-n}\right)\right) \\ &< 1 \end{aligned}$$

for large enough $n \geq 0$, uniformly on the unit ball of $C^\beta(B)/\mathbb{C}$. The result follows from the spectral radius theorem. □

As usual, the probability measure μ which is the eigenprojection associated to 1 (i.e., $\widehat{\mathcal{L}}\mu = \mu$) is the unique absolutely continuous $\mathcal{T}_2$-invariant probability measure on B.

Corollary 2.3.1.

(1) *The transformation $\mathcal{T}_2 : B \to B$ is exponentially mixing on Hölder functions, i.e., given $F, G \in C^\beta(B)$ there exists $0 < \tau < 1$ and $C > 0$ such that*
$$\left|\int F \circ \mathcal{T}_2^n.G d\mu\right| \leq C\tau^n \text{ for all } n \geq 0.$$

(2) *For almost all (μ) $x = (\lambda, \pi) \in B$ we have that*
$$\frac{1}{N}\sum_{n=0}^{N-1} F({\mathcal{T}_2}^n(x,\lambda)) = \int f d\mu + O\left(\frac{\log N}{\sqrt{N}}\right)$$

Proof. For the first part, ssume that the spectrum of $\widehat{\mathcal{L}} : C^\omega(B)/\mathbb{C} \to C^\omega(B)/\mathbb{C}$ is contained in a disk $\{z \in \mathbb{C} : |z| < \tau\}$, where $0 < \tau < 1$. Given $G \in C^\omega(B)$ with $\int F d\mu = \int G d\mu = 0$ can write $\int F \circ \mathcal{T}_2^n G d\mu = \int F \widehat{\mathcal{L}}^n G d\mu = O(\tau^n)$, showing the transformation is exponentially mixing.

The second part follows immediately from the first part by a standard spectral result [6]. □

3. Statistical properties for $\mathcal{T}_2$

Let $d\mu = \rho(x)d\omega(x)$ be the unique absolutely $\mathcal{T}_2$-invariant probability measure on B. This measure μ is ergodic (cf. [2] or, alternatively, by part (1) of Corollary 2.3.1) and so we can apply the Birkhoff ergodic theorem gives that for any $f \in L^1(X, \mu)$ and for a.e.(μ) $x \in B$ we have that

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\mathcal{T}_2^j x) \to \int f d\mu, \text{ as } n \to +\infty,$$

converges in distribution. In this section we want to discuss various generalizations of this basic property.

3.1 The Central Limit Theorem and Functional Central Limit Theorem.

Definition. We say that $\mathcal{T}_2$ satsifies the Functional Central Limit Theorem whenever for a Hölder continuous function $h \in C^\beta(B, \mathbb{R})$ (not equal to a coboundary plus a constant) there exists $\sigma > 0$ such that for $0 \le t \le 1$,

$$w_n(t) = \frac{1}{\sigma\sqrt{n}} \left(\sum_{j=0}^{[nt]-1} h \circ \mathcal{T}_2^j + (nt - [nt]) h \circ \mathcal{T}_2^{[nt]} \right)$$

converges weakly to the Wiener measure on $C([0,1], \mathbb{R})$.

This is sometimes called a weak invariance principle, in reference to the topology of covergence.

The Central Limit Theorem could be deduced directly from the results on $\widehat{\mathcal{L}}$ in the previous section, but, with no additional work we can deduce the stronger Functional Central Limit Theorem.

Proposition 3.1. *The Functional Central Limit Theorem holds for $\mathcal{T}_2$.*

Proof. By a quite general result of Mackey and Tyran-Kaminska [7,8] (cf. also [18]) if $h_0 \in L^2(B, \mu)$ satisfies $\int |h_0|^2 d\mu = 0$ and $\mathcal{L}h_0 = 0$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sqrt{\int \left(\sum_{k=0}^{n-1} \mathcal{L}^k h_0 \right)^2 d\mu} < \infty,$$

then setting $\sigma^2 = \int |h_0|^2 d\mu$ gives

$$w_n^0(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{[nt]-1} h_0 \circ \mathcal{T}_2^j \to \sigma w(t), \text{ for } t \in [0,1].$$

(i.e., the Functional Central Limit Theorem for h_0). More generally, given a Hölder continuous function h with $\int h d\mu = 0$, we recall from Lemma 2.3 that there exists $0 < \tau < 1$ such that $||\mathcal{L}^n h|| = O(\theta^n)$, and therefore $u = \sum_{n=1}^\infty \mathcal{L}^n h$ converges in $C^\beta(B)$. Let $u = \sum_{n=1}^\infty \mathcal{L}^n h$ and set $h_0 := h - u \circ T + u$ then $\mathcal{L}(h_0) = \mathcal{L}h - u + \mathcal{L}u = 0$. Since h and h_0 are cohomologous we can bound $|w_n(t) - w_n^0(t)| \le 2||u||_\infty / \sqrt{n}$ and thus deduce the Functional Central Limit Theorem for h. □

The following are standard corollaries using the Continuous Mapping Theorem [4,5] beginning with the central limit theorem (where $t = 1$).

Corollary 3.1.1 (Central Limit Theorem). *For $y \in \mathbb{R}$ we have that*

$$\lim_{n \to +\infty} \mu \left\{ x \in B : \frac{1}{\sqrt{n}} \sum_{j=1}^n f(\mathcal{T}_2^j x) \le y \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^y e^{-t^2/2\sigma^2} dt$$

The Central Limit Theorem (and much more besides) has already been proved by Butetov [2] and Morita [14]. The approach of Bufetov involved studying the rate of mixing of $\mathcal{T}_2$; and the method of Morita involved perturbation theory of the transfer operator.

The following are other standard corollaries [4, 5].

Corollary 3.1.2. *For $y \geq 0$ we have that*

$$\lim_{n\to+\infty} \mu \left\{ x \in B \,:\, \frac{1}{\sqrt{n}} \max_{1\leq k\leq n} \sum_{j=1}^{k} f(\mathcal{T}_2 x) \leq y \right\} = \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \int_{-\infty}^{y} e^{-t^2/2\sigma^2} dt - 1$$

Corollary 3.1.3 (Arcsine Law). *For $0 \leq y \leq 1$ we have that*

$$\lim_{n\to+\infty} \mu \left\{ x \in B \,:\, \frac{N_n(x)}{n} \leq y \right\} = \frac{2}{\sqrt{\pi}} \sin^{-1} \sqrt{y}$$

where $N_n(x) = Card \left\{ 1 \leq k \leq n \;:\; \sum_{j=1}^{k} f(\mathcal{T}_2^j x) > 0 \right\}$

Corollary 3.1.4 (Law of the iterated logarithm). *For a.e. (μ_2) $x \in B$ we have*

$$\limsup_{n\to+\infty} \frac{\sum_{j=1}^{n} f(\mathcal{T}_2^j x)}{\sigma\sqrt{2n\log\log n}} = 1.$$

Remark. There are a number of other statistical results which could be considered. For example, Morita has shown that there is a local limit theorem for $\mathcal{T}_2$.

This completes the proof of Theorem A.

3.2 Almost Sure Invariance Principles. With only a little further work, we next establish a class of stronger results, from which the preceeding (and several others) can easily be deduced.

Given a Hölder continuous function $f : B \to \mathbb{R}$ with $\int f d\mu = 0$ we can associate the summation $f^n(x) := \sum_{i=0}^{n-1} f(\mathcal{T}^i x)$, for each $n \geq 1$.

Definition. We say that $\mathcal{T}_2 : B \to B$ satisfies the *Almost Sure Invariance Principle* if for any such function $f : B \to \mathbb{R}$ there exists a sequence of random variables S_n which for a.e. (μ_2) x agrees with $f^n(x)$ in distribution and thete exists $\epsilon > 0$ such that $S_n = W_n + O(n^{\frac{1}{2}-\epsilon})$ as $n \to +\infty$.

The following result is a strengthening of Proposition 3.1.

Theorem 1 (Almost sure invariance principle for $\mathcal{T}_2$). *If f is Hölder continuous then $\mathcal{T}_2 : B \to B$ satisfies the Almost Sure Invariance Principle.*

Proof. The standard approach is to deduce this from an application of a result of Phillipp and Stout [16] (cf. [12] for a dynamical reformulation). In particular, we only need to establish that the hypotheses there hold. More precisely, given a β-Hölder function $f : B \to \mathbb{R}$ with $\int f d\mu = 0$ we observe that:

(1) $f \in L^{2+\delta}(B)$, for any $\delta > 0$ (since v is automatically bounded);

(2) for any $n \geq 1$,

$$\int |f^n|^2 d\mu = n\sigma^2 + O(1)$$

(by expanding the Left Hand Side and bounding the cross terms using Part (1) of Corollary 2.3.1);

(3) for any $k \geq 0$,

$$E\left(|f - E(f|\vee_{i=0}^{k-1} \mathcal{T}_2^{-i}\mathcal{Q})|^{2+\delta})|\vee_{i=0}^{k-1} \mathcal{T}_2^{-i}\mathcal{Q}\right) \leq ||f - E(f|\vee_{i=0}^{k-1} \mathcal{T}_2^{-i}\mathcal{Q}))||_\infty^{2+\delta}$$
$$\leq (||f||_\beta \sup_{a\in\mathcal{Q}_k} \operatorname{diam}(a))^{2+\delta}$$
$$\leq (||f||_\beta \Theta^{-k})^{2+\delta},$$

(where, as usual, $E(\cdot|\vee_{i=0}^{k-1} \mathcal{T}_2^{-i}\mathcal{Q}) = \sum_{a\in\mathcal{T}_2^{-i}\mathcal{Q}} \frac{1}{\mu(a)} \int_a \cdot d\mu$); and, finally,

(4) given any $A_1 \in \vee_{i=0}^{k-1}\mathcal{T}_2^{-i}\mathcal{Q}$ and any Borel measurable set $A_2 \subset B$, and for any $n, k \geq 0$, we can bound

$$\begin{aligned}
&|\mu(A_1 \cap \mathcal{T}_2^{-(k+n)}A_2) - \mu(A_1)\mu(A_2)| \\
&= |\int \chi_{A_1}(\chi_{A_2} \circ \mathcal{T}_2^{k+n})d\mu - \int \chi_{A_1} d\mu. \int \chi_{A_2} d\mu| \\
&= |\int (\widehat{\mathcal{L}}^n \chi_{A_1})(\chi_{A_2} \circ \mathcal{T}_2^k)d\mu - \int \widehat{\mathcal{L}}^n \chi_{A_1} d\mu \int \chi_{A_2} \circ \mathcal{T}_2^k d\mu| \\
&= |\int \left[\widehat{\mathcal{L}}^k \chi_{A_1} - \int \widehat{\mathcal{L}}^k \chi_{A_1 2}\right] (\chi_{A_2} \circ \mathcal{T}_2^n)d\mu| \\
&= |\int \left[\widehat{\mathcal{L}}^k \chi_{A_1} - \int \widehat{\mathcal{L}}^k \chi_{A_1}\right] (\chi_{A_2} \circ \mathcal{T}_2^n)d\mu| \\
&= |\int \widehat{\mathcal{L}}^n \left[\widehat{\mathcal{L}}^k \chi_{A_1} - \int \widehat{\mathcal{L}}^k \chi_{A_1}\right] \chi_{A_2} d\mu| \\
&\leq \left(\int \left|\widehat{\mathcal{L}}^n \left[\widehat{\mathcal{L}}^k \chi_{A_1} - \int \widehat{\mathcal{L}}^k \chi_{A_1}\right]\right|^2 d\mu\right)^{\frac{1}{2}} \left(\int \chi_{A_2}^2 d\mu\right)^{\frac{1}{2}} \\
&\leq C\tau^n ||\widehat{\mathcal{L}}^k \chi_{A_1}||_\infty \mu(A_2)^{\frac{1}{2}},
\end{aligned}$$

for some $C > 0$, using the Cauchy-Schwartz inequality, that $\widehat{\mathcal{L}}\mu = \mu$ and (again) that $0 < \tau < 1$ is a bound on the modulus of the second eigenvalue of $\widehat{\mathcal{L}}$. Finally, we can observe that $\widehat{\mathcal{L}}^k \chi_{A_1} = 1$ and so the bound can be taken to be $C\tau^n$.

We can then apply Theorem 7.1 in [16] (cf. Theorem A.1 in [12]) to deduce that the Almost Sure Invariance Principle holds for $\mathcal{T}_2$. □

4. Statistical properties for $\mathcal{T}_1$

The statistical properties of $\mathcal{T}_2$ described above can be used to establish analogous results for the original Zorich map $\mathcal{T}_1 : \Delta \times \mathcal{R} \to \Delta \times \mathcal{R}$, with respect to μ_1, by viewing it as a suspension. More precisely, we can associate to the map $\mathcal{T}_2 : B \to B$ and the return time $\widehat{n} : B \to \mathbb{Z}^+$ a suspension space

$$B^{\widehat{n}} := \{(x, k) \in B \times \mathbb{Z} : 0 \leq k \leq \widehat{n}(x) - 1\}$$

where we identify $(\lambda, \pi; \widehat{n}(x))$ and $(\mathcal{T}_2(\lambda, \pi); 0)$. We can also define the natural map $\mathcal{T}_2^{\widehat{n}} : B^{\widehat{n}} \to B^{\widehat{n}}$ on this suspension space by

$$\mathcal{T}_2^{\widehat{n}}(x, k) = \begin{cases} (x, k+1) & \text{if } 0 \leq k \leq \widehat{n}(x) - 2 \\ (\mathcal{T}_2 x, 0) & \text{if } k = \widehat{n}(x) - 1. \end{cases}$$

There is a natural $\mathcal{T}_2^{\widehat{n}}$-invariant measure $d\mu_2 \times d\mathbb{N}/\int \widehat{n} d\mu_2$, where $d\mathbb{N}$ corresponds to the usual counting measure. The following result is standard.

Lemma 4.1. *The map* $\Psi : B^{\widehat{n}} \to \Delta^+$ *defined by* $\Psi(x,k) = \mathcal{T}_1{}^k(x)$ *is:*

(1) *a semi-conjugacy, i.e.,* $\mathcal{T}_1 \circ \Psi = \Psi \circ \mathcal{T}_2^{\widehat{n}}$*, and*
(2) *an isomorphism (with respect to* $d\mu_2 \times d\mathbb{N}/\int \widehat{n} d\mu_2$ *and* $d\mu_1$*).*

We can deduce the almost sure invariance principle for the Zorich map $\mathcal{T}_1 : \Delta \to \Delta$, by applying a result given in a paper of Melbourne and Nicol [12] (which is formulated from the results of Melbourne-Torok [13]). The other statistical properties follow as a direct consequence.

The main technical condition we require is the following:

Lemma 4.2. *For any* $\delta > 0$ *we have that*

$$\sum_{k=1}^{\infty} \mu_2 \left\{ x = (\lambda, \pi) \in B \,:\, \widehat{n}(x) = k \right\} k^{2+\delta} < +\infty.$$

Proof. By an estimate of Avila-Bufteov [1, Lemma 1], there exists $C > 0$ and $0 < \theta < 1$ such

$$\mu \left\{ x \in B \,:\, \widehat{n}(\underline{\lambda}, \pi) \geq k \right\} \leq C\theta^n, \text{ for all } n \geq 1.$$

Thus $\sum_{k=1}^{\infty} \mu \left\{ x \in B \,:\, \widehat{n}(\underline{\lambda}, \pi) = k \right\}^{2+\delta} \leq C \sum_{k=1}^{\infty} \theta^k k^{2+\delta} < +\infty.$ □

We now describe a general class of function for which the results will be established. Let $f : \Delta \to \mathbb{R}$ be Hölder continuous and satisfy $\int f d\mu_1 = 0$. We can associate to f a function $\overline{f} : B \to \mathbb{R}$ defined a.e.(μ_2) by

$$\overline{f}(x) = \sum_{l=0}^{\widehat{n}(x)-1} f(\mathcal{T}_1^l x).$$

In particular, we have that $\int \overline{f} d\mu_2 = 0$. If, in the interests of expediency, we make the hypothesis that the function $\overline{f} : B \to \mathbb{R}$ is Hölder continuous, then we can lift the results for $\mathcal{T}_2$ in Theorem 3.2 (with respect to $\overline{f}$) to those for $\mathcal{T}_1$ (with respect to f). More generally, we can assume that f is Hölder continuous and the associated function $\overline{f}$ satisfies a weaker "local Hölder" condition that if $\widehat{n}(x) = \widehat{n}(y) = n$, say, then $|\overline{f}(x) - \overline{f}(y)| \leq n||f||_\beta ||x-y||^\beta$. However, following [12] we can then consider the slightly larger Banach space $\mathcal{B}$ with respect to the norm

$$||h||_{\mathcal{B}} = \sum_{n=1}^{\infty} n \sup_{\substack{\widehat{n}(x)=\widehat{n}(y)=n \\ x \neq y}} \frac{||f(x) - f(y)||}{||x-y||^\beta},$$

for which the proofs of Lemma 2.3 and Theorem 3.2 readily generalize.

To extend the almost sure invariance principle from $\mathcal{T}_2$ to $\mathcal{T}_1$ we need to check the hypotheses of the theorem of Melbourne and Torok [13]. In particular,

(1) by the Lemma 4.2, we can choose $\delta > 0$ so that $\widehat{n} \in L^{2+\delta}(B, \mu_2)$, and
(2) by the analogue of part (2) of Corollary 2.3.1 we have that

$$\frac{1}{N} \sum_{i=0}^{N-1} \widehat{n}(\mathcal{T}_2{}^i x) = \int \widehat{n} d\mu + O\left(\frac{1}{N^{1-\epsilon}} \right) \text{ a.e. } (\mu_2) \ x \in B.$$

In particular, we can now conclude that the almost sure invariance principle holds for $\mathcal{T}_1$ with variance $\sigma^2 = \widehat{\sigma}^2 / \int \widehat{n} d\mu_2$.

Theorem 2 (Almost sure invariance principle for $\mathcal{T}_1$). *The almost sure invariance principle holds for $\mathcal{T}_1$ and μ_1.*

This theorem has several consequences, including the the analogues of Corollaries 3.1.1- 3.1.4 for $\mathcal{T}_1$.

There is an immediate application to return times for $\mathcal{T}_2$. Given any set A we denote by $r_A : A \to \mathbb{N}$ the first return time to A, i.e., $r_A(x) = \inf\{n \geq 1 : \mathcal{T}_2^n x \in A\}$. In particular, the value defined inductively by $r_A^{(n)}(x) = r_A^{(n-1)}(x) + r_A(T^{r_A^{(n-1)}(x)})$ is the nth return time. Using Kac's theorem on return times we have that

$$\lim_{n\to+\infty} \frac{r_A^{(n)}(x)}{n} = \frac{1}{\mu(A)} \text{ for a.e. } (\mu_2).$$

For the particular choice $A = B$ we can consider the function $r_B(x) = \widehat{n}(\lambda, \pi)$ and by Kac's theorem $\int r_B d\mu_2 = 1/\mu_2(B)$. It is easy to see that the variance is non-zerp and thus this leads, for example, to the following corollary:

Corollary 2.1. *There exists $\sigma > 0$ such that*

$$\lim_{N\to+\infty} \mu_2 \left\{ x : \frac{1}{N} r_B^{(N)}(x) \leq y \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{y} e^{-t^2/2\sigma^2} dt$$

for $a < b$.

5. Transfer operators and analytic functions

To take advantage of the transformation $\mathcal{T}_2$ being piecewise analytic, we can also consider the transfer operator acting on a space C^ω of analytic functions. Let us denote $\lambda = (\lambda_1, \cdots, \lambda_n), \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. For sufficiently small $\epsilon > 0$ we denote

$$B_\epsilon^{\mathbb{R}} = \left\{ \underline{\lambda} \in \mathbb{R}^n : \sum_{j=1}^n \lambda_j = 1 \text{ and } |\lambda - B| < \epsilon \right\}$$

and consider a complexification of the form

$$B_\epsilon^{\mathbb{C}} = \left\{ \underline{\lambda} + i\underline{\xi} \in \mathbb{C}^n : |\underline{\lambda} - B| < \epsilon, \sum_{j=1}^n \lambda_j = 1, \sum_{j=1}^n \xi_j = 0 \text{ and } |\xi_j| \leq \epsilon \right\}.$$

Let $\mathcal{T}_2 : B_\epsilon^{\mathbb{C}} \to \mathbb{C}^n$ also denote the analytic extension from B to $B_{\mathbb{C}}$.

In order to show that $\mathcal{L}$ preserves a space of analytic functions on this space we can use the following simple lemma.

Lemma 5.1. *Providing $\epsilon > 0$ is sufficiently small we have that $\overline{\mathcal{T}_2^{-1} B_\epsilon^{\mathbb{C}}} \subset \text{int}(B_\epsilon^{\mathbb{C}})$. Moreover, for $x = \underline{\lambda} + i\underline{\xi} \in B_\epsilon^{\mathbb{C}}$ we have that*

$$\left| \sum_{\mathcal{T}_2^{-1} y = x} \frac{1}{\left(\sum_i (Ay)_i\right)^n} \right| < +\infty$$

Proof. Since the inverse branches of $\mathcal{T}_2 : B \to B$ are uniformly contracting, we can choose $\epsilon > 0$ sufficiently small and $0 < \theta < 1$ such that the $\mathcal{T}_2^{-1} B_{\theta\epsilon}^{\mathbb{R}} \subset B_\epsilon^{\mathbb{R}}$. We can show that their compexifications have a similar property with respect to $B_{\mathcal{C}}$ To begin, observe that the linear action of any of the positive matrices A corresponding to an inverse branch of $\mathcal{T}_2$ act on both the real and imaginary coordinates independently, and the complexification of the linear action is again a linear action:

$$(\lambda_1, \cdots, \lambda_n) + i(\xi_1, \cdots, \xi_n) \mapsto A(\lambda_1, \cdots, \lambda_n) + iA(\xi_1, \cdots, \xi_n).$$

The image under the projective action comes from dividing by $\sum_j (A\underline{\lambda})_j + i\sum_j (A\underline{\xi})_j$ (i.e., the complexification of $\|A\underline{\lambda}\|$) to get:

$$\frac{A\underline{\lambda} + iA\underline{\xi}}{\sum_j (A\underline{\lambda})_j + i\sum_j (A\underline{\xi})_j} = \frac{A\underline{\lambda}}{\sum_j (A\underline{\lambda})_j} - \left(\frac{A\underline{\lambda}\frac{(\sum_j (A\underline{\xi})_j)^2}{(\sum_j (A\underline{\lambda})_j)}}{\left(\sum_j (A\underline{\lambda})_j\right)\left(\sum_j (A\underline{\lambda})_j + \frac{(\sum_j (A\underline{\xi})_j)^2}{(\sum_j (A\underline{\lambda})_j)}\right)} \right) + i\frac{\left(A\underline{\xi}\left(1 - i\frac{\sum_j (A\underline{\xi})_j}{\sum_j (A\underline{\lambda})_j}\right) - A\underline{\lambda}\frac{\sum_j (A\underline{\xi})_j}{\sum_j (A\underline{\lambda})_j}\right)}{\sum_j (A\underline{\lambda})_j + \frac{(\sum_j (A\underline{\xi})_j)^2}{(\sum_j (A\underline{\lambda})_j)}}.$$

In particular, for $\theta' = (1+\theta)/2$ and $\epsilon > 0$ sufficiently small we can deduce that $\mathcal{T}_2^{-1} B_\epsilon^{\mathbb{C}} \subset B_{\theta'\epsilon}^{\mathbb{C}}$. This completes the proof of the first part of the lemma.

For the second part of the lemma, we firt observe that for $\underline{\lambda} + i\underline{\xi} \in B_\epsilon^{\mathbb{C}}$ we that

$$\begin{aligned} \frac{1}{(\sum_j (A\underline{\lambda})_j + i\sum_j (A\underline{\xi})_j)^n} &= \frac{1}{(\sum_j (A\underline{\lambda})_j)^n} \frac{1}{\left(1 + i\frac{(\sum_j (A\underline{\xi})_j)^n}{(\sum_j (A\underline{\lambda})_j)^n}\right)} \\ &= \left(\frac{1}{(\sum_j (A\underline{\lambda})_j)^n}\right)(1 + O(\epsilon)). \end{aligned} \tag{5.1}$$

Moreover, given $\underline{\lambda}, \underline{\lambda}' \in B_\epsilon^{\mathbb{R}}$ we can estimate

$$\begin{aligned} \left| \frac{1}{(\sum_j (A\underline{\lambda})_j})^n - \frac{1}{(\sum_j (A\underline{\lambda}')_j})^n \right| &\le \left| 1 - \frac{(\sum_j (A\underline{\lambda})_j)^n}{(\sum_j (A\underline{\lambda}')_j)^n} \right| \cdot \left| \frac{1}{(\sum_j (A\underline{\lambda})_j)^n} \right| \\ &\le C \left| \frac{1}{(\sum_j (A\underline{\lambda})_j)^n} \right| < \infty \end{aligned} \tag{5.2}$$

for some $C > 0$. However, by the formula for the transfer operator we know that for $\underline{\lambda} \in B$:

$$\left| \frac{1}{(\sum_j (A\underline{\lambda})_j)^n} \right| < \infty \tag{5.3}$$

Comparing (5.1), (5.2) and (5.3) completes the proof. □

We can apply the lemma to deduce that the operator $\mathcal{L} : C^\omega(B_\epsilon^{\mathbb{C}}) \to C^\omega(B_\epsilon^{\mathbb{C}})$ is well defined . In particular, that the series expression for $\mathcal{L}w(x)$ converges to an analytic function for $z \in B_\epsilon^n\mathbb{C}$ follows from Montel's theorem.

The main importance of this is that we can now consider the operator $\mathcal{L} : C^\omega(B_\epsilon^{\mathbb{C}}) \to C^\omega(B_\epsilon^{\mathbb{C}})$ on the Banach space of bounded analytic functions on $B_\epsilon^{\mathbb{C}}$ with respect to the supremum norm $||f|| = \sup_{B_\epsilon^{\mathbb{C}}} |f(z)|$.

In particular, it leads to the following result.

Definition. Any bounded linear operator $L : B \to B$ on a Banach space B with norm $\|\cdot\|$ is called *nuclear* (of order α)if there exist:

(i) vectors $u_n \in B$ (with $||u_n|| = 1$);
(ii) linear functionals $l_n \in B^*$ (with $||l_n|| = 1$); and
(iii) a sequence (ρ_n) such that $\sum_{n=0}^{\infty} |\rho_n|^\alpha < +\infty$ such that

$$L(v) = \sum_{n=0}^{\infty} \rho_n l_n(v) u_n, \quad \text{for all } v \in B.$$

We say that L has order zero, if property holds for any $\alpha > 0$.

In particular, a nuclear operator is automatically a compact operator, for which the non-zero eigenvalues are of finite multiplicity (and the dual spaces are of finite multiplicity).

Proposition 5.1. *The operator $\mathcal{L} : H(B_\epsilon^{\mathbb{C}}) \to H(B_\epsilon^{\mathbb{C}})$ is nuclear (of order zero).*

Proof. The proof follows the same lines as that in [10,11] Let $H(B_\epsilon^{\mathbb{C}})$ denote the Frechet space of analytic functions on $B_\epsilon^{\mathbb{C}}$, with the compact open topology. We observe that $\mathcal{L} : H(B_\epsilon^{\mathbb{C}}) \to C^\omega(B_\epsilon^{\mathbb{C}})$ is a bounded linear operator and recall that the space $H(B_\epsilon^{\mathbb{C}})$ is nuclear [3]. In particular, the operator $\mathcal{L}$ is nuclear (or order zero) [3] (cf. [10, proof of Lemma 3]). Finally, we can compose $\mathcal{L}$ with the continuous inclusion $H(B_\epsilon^{\mathbb{C}}) \subset C^\omega(B_\epsilon^{\mathbb{C}})$ to deduce that $\mathcal{L} : H(B_\epsilon^{\mathbb{C}}) \to H(B_\epsilon^{\mathbb{C}})$ is nuclear (of order zero). □

Many of the statistical results for $\mathcal{T}_2$ are related to the existence of a spectral gap for $\mathcal{L}$. In the analytic context this is essentially automatic since the operator is compact. Moreover, one can apply a result of Mayer [10, p.12] to recover that the value 1 is a simple eigenvalue of maximal modulus, and that eigenfunction ρ is real analytic.

We can recover the following:

Corollary 5.1.1. *The invariant density of $\mathcal{T}_2$ (and thus $\mathcal{T}_1$) is real analytic.*

We can again define $\widehat{\mathcal{L}} : C^\omega(B) \to C^\omega(B)$ by

$$\widehat{\mathcal{L}}w(x) = \frac{1}{\rho(x)} \mathcal{L}(w\rho)(x).$$

then $\widehat{\mathcal{L}}1 = 1$ and $\widehat{\mathcal{L}}^*\mu = \mu$.

6. Zeta functions and Lyapunov Exponents

We begin by considering the largest Lyapunov exponent Θ for these transformations. Let E_{ij} denote the $m \times m$ matrix with entries 1 on the diagonal and in the (i, j)th place and let P_π denote the permutation matrix associated to π. Consider the matrices $A(\pi, a) = (I + I_{\pi^{-1}m,m}).P(\tau^{\pi^{-1}(m)})$ and $A(\pi, a) = E + I_{m,\pi^{-1}m}$. We then define a matrix valued function $B(\lambda, \mu)$ on $\cup_{\pi\in\mathcal{R}}\Delta^+ \cup \Delta^-$ by

$$B(\lambda, \pi) = A(\lambda, \pi) A\mathcal{T}_0(\lambda, \pi) \cdots A\mathcal{T}_0^{\widehat{n}(\lambda,\pi)-1}(\lambda, \pi).$$

The leading Lyapunov exponent for this matrix is

$$\Theta = \inf_{n\geq 1} \left\{ \frac{1}{n} \int \log ||B(\lambda, \pi) B\mathcal{T}_1(\lambda, \pi) \cdots B\mathcal{T}_1^n(\lambda, \pi)|| d\mu_1 \right\}.$$

We recall the following elegant result of Zorich [22, Theorem 4].

Proposition 6.1 (Zorich).

$$\begin{aligned}\Theta &= \sum_{\pi\in\mathcal{R}} \int_\Delta (\log(1-\lambda_n) - \log(1-\lambda_{\pi^{-1}n}))\, d\mu_1(\lambda) \\ &= \frac{1}{m}\sum_{\pi\in\mathcal{R}} \int_\Delta \log|\det D\mathcal{T}_1| d\mu_1(\lambda).\end{aligned}$$

In view of the induced transformation $\mathcal{T}_2 : B \to B$ we can immediately rewrite this as

$$\Theta = \frac{1}{m}\sum_{\pi\in\mathcal{R}} \int_B \log|\det D\mathcal{T}_2| d\mu_2(\lambda)$$

We shall now describe an approach to the Lyapunov exponents using complex functions. The connection between zeta functions and the both the standard and multidimensional continued fraction transformations was explored by Mayer in [10] (cf. also [11])

Defintion. We can associate to $\mathcal{T}_2$, and an analytic function $f(x)$, a complex function $d(z,s)$ in two variables defined by

$$d(z,s) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{\mathcal{T}_2^n x = x} |\det(D\mathcal{T}_2^n)(x)|^{-1} e^{sf^n(x)}\right)$$

where we interpret the periodic points as points in the disjoint union. This converges for $|z|$ and $Re(s)$ sufficiently small.

The main technical result on such functions is the following.

Proposition 6.2.

(1) *If $|s|$ is sufficiently small, then $d(z,s)$ is an entire function in z;*
(2) *Moreover, if we expand $d(z,s) = 1 + \sum_{n=1}^{\infty}{}_n a_n(s) z^n$, then there exists $c > 0$ such that $|a_n| = O(e^{-cn^{1+1/(d-1)}})$;*
(3) *The zeros z_0 for $d(z,1)$ correspond to eigenvalues $\lambda = 1/z_0$. In particular, 1 is the zero of smallest modulus; and*
(4) *We can write*

$$\frac{\frac{\partial d(1,s)}{\partial s}|_{s=0}}{\frac{\partial d(z,0)}{\partial z}|_{z=1}} = \int f(x) d\mu_2(x).$$

Proof. This follows from the method of Ruelle [17] and Grothendieck [3]. The only additional feature is that the operator has infinitely many inverse branches but, as in [10,11], this presents no additional complications to the proof. □

We can apply the theorem with $f(x) = \log|\det D\mathcal{T}_2|$. In principle, this gives an alternative expression for Lyapunov exponent.

Corollary 6.2.1. *We can write*

$$\Theta = \frac{\sum_{n=1}^{\infty} b_n}{\sum_{n=1}^{\infty} c_n}$$

where $|b_n| = O(e^{-cn^{1+1/(d-1)}})$ *and* $|c_n| = O(e^{-cn^{1+1/(d-1)}})$.

Proof. We can write

$$\Theta = \frac{\frac{\partial d(1,s)}{\partial s}|_{s=0}}{\frac{\partial d(z,0)}{\partial z}|_{z=1}} = \frac{\sum_{n=1}^{\infty} a_n'(0) z^n}{\sum_{n=1}^{\infty} {}_n n a_n(0)}$$

Using the expansion $\exp(z) = \sum_{n=1}^{\infty} z^n/n!$ we can write that

$$a_n(s) = \sum_{k_1+\cdots+k_r=n} \frac{(-1)^r}{k_1!\cdots k_r!} \sum_{\mathcal{T}_2^{k_i} x_i = x_i} \frac{\exp\left(s\sum_{i=1}^r f^{k_i}(x_i)\right)}{\left(\prod_{i=1}^k |\det D\mathcal{T}_1^{k_1}(x_i)|\right)}$$

and thus

$$b_n = n a_n(0) = n \sum_{k_1+\cdots+k_r=n} \frac{(-1)^r}{k_1!\cdots k_r!} \sum_{\mathcal{T}_2^{k_i} x_i = x_i} \frac{1}{\left(\prod_{i=1}^k |\det D\mathcal{T}_1^{k_1}(x_i)|\right)}$$

and

$$c_n = a_n'(0) = \sum_{k_1+\cdots+k_r=n} \frac{(-1)^r}{k_1!\cdots k_r!} \sum_{\mathcal{T}_2^{k_i} x_i = x_i} \frac{\sum_{i=1}^r f^{k_i}(x_i)}{\left(\prod_{i=1}^k |\det D\mathcal{T}_1^{k_1}(x_i)|\right)}$$

Using the bounds on $a_n(s)$ we get bounds on $a_n'(0)$ using Cauchy's theorem, i.e., $|a_n'(0)| \leq \frac{1}{2\pi}|\int_{|\xi|=\epsilon} a_n(\xi)\xi^{-2}d\xi| = O(e^{-cn^{1+1/(d-1)}})$ and the bounds on $|a_n(0)|$ also serve to bound b_n. $\square$

We can define formally define

$$\eta_f(z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\mathcal{T}_2^n x = x} \frac{f^n(x)}{|\det(D\mathcal{T}_2^n)(x)|}.$$

By the estimate in Part (2) of Proposition 6.2 we see that $d(z,1)$ is an entire function of order 1. In particular, if $\{z_n(t)\}$ are poles of $d(z,t)$ then by the Hadamard Weierstrauss theorem we can write

$$d(z,t) = e^{A(t)z+B(t)} \prod_n \left(1 - \frac{z}{z_n(t)}\right) e^{\frac{z}{z_n(t)}}.$$

In particular, we observe that $\eta_f(z) = \frac{\partial \log d(z,t)}{\partial t}|_{t=0}$ and then we can write

$$\eta_f(z) = B'(0) + \sum_n \frac{\frac{z z_n'(0)}{z_n(0)}}{(z_n(0)-z)} + z\left(A'(0) + \frac{z_n'(0)}{[z_n(0)]^2}\right).$$

for which the poles are $\{z_n\}$ and the residues are $\frac{z_n'(0)}{z_n(0)}$. This completes the proof of Theorem B. Moreover, the poles also correspond to the values $\mu_n(f)$, where μ_n are eigendistributions associated with the eigenvalue 1 for the transfer operator associated to z. This leads to the following simple observation.

Corollary 6.2.2. *Assume that* $\mu_n(f) = 0$ *for every eigendistribution* μ_n *then* $\eta_f(z) = 0$ *for all* $z \in \mathbb{C}$.

Of course, this is equivalent to $\frac{1}{n}\sum_{\mathcal{T}_2^n x = x} \frac{f^n(x)}{|\det(D\mathcal{T}_2^n)(x)|} = 0$ for each $n \geq 1$.

7. An Application: Teichmüller flows

For completeness, we briefly describe in this final section the relationship with Teichmüller flows.

There is a close connection between interval exchange maps and flat metrics on surfaces. A particularly convenient presentation of a flat surface is as a union of into n rectangles based on the intervals I_i and of height l_i, for $i = 1, \cdots, n$. Thus the information we need to reconstruct the flat torus begins with

(a) The lengths λ_i of the intervals I_i $(i = 1, \cdots, n)$;
(b) The heights h_i of the rectangles $(i = 1, \cdots, n)$.

Since we will assume that the surface has unit area we can write that $\lambda_1 h_1 + \cdots + \lambda_n h_n = 1$. In addition in order to attach the tops of the rectangles back to X in the correct order we need:

(c) The permutation π on $\{1, \cdots, n\}$ which tells the change in order in which we reattach the tops of the rectangles.

In addition, to define the flow and invariant measure it is convenient to introduce two other coordinates (which obviously depend on those above):

(c) $\delta_i = a_{i-1} - a_i$, for $i = 1, \cdots, n$ with the convention $a_0 = a_{n+1} = 0$

and the heights of other singularities (which lie in the sides of the rectangles); and

(d) $a_0, \cdots, a_n$, which are actually dependent on the other variables by $h_i - a_i = h_{\pi^{-1}(\pi(i)+1)} - a_{\pi^{-1}(\pi(i)+1)-1}$ for $i = 0, \cdots, n$.

Let $\Omega_{\mathcal{R}}$ denote the space of all unit area (zippered) rectangles for which the associated permutation $\pi \in \mathcal{R}$, say. This a natural volume $d\lambda_1 \cdots d\lambda_n d\delta_1 \cdots d\delta_n$. Let μ denote the normalized measure. A Teichmüller flow $T_t : \Omega_{\mathcal{R}} \to \Omega_{\mathcal{R}}$ is defined locally by $T_t(\lambda, h, a, \pi) = (e^t \lambda, e^{-t} h, e^{-t} a, \pi)$ (i.e, flattening the rectangles from above) and this preserves the volume. There is a natural projection from $\Omega_{\mathcal{R}}$ to the moduli space of flat metrics $\mathcal{M}$ and the corresponding semi-conjugate flow $S_t : \mathcal{M} \to \mathcal{M}$ is the *Teichmüller flow*. We can consider the cross section

$$\mathcal{Y} = \left\{ (\lambda, h, a, \pi) \in \Omega_{\mathcal{R}} : \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

to the flow T_t. Under the natural identification on $\Omega_{\mathcal{R}}$ corresponding to different presentations of surfaces as rectangles: the return time function to $\mathcal{Y}$ corresponds to the natural extension of the map $\mathcal{T}_0$ and the return time function is simply $r(\lambda, \pi) = \log(1 - \min\{\lambda_n, \lambda_{\pi^{-1}n}\})$. In particular, the Teichmüller flow T_t is a finite-to-one factor of the natural extension of the suspended semi-flow associated to the map $\mathcal{T}_0$ and the function r, i.e., let

$$(\Delta \times \mathcal{R})^r = \left\{ (\underbrace{\lambda, \pi}_{=:x}, u) \in \Delta \times \mathcal{R} \times \mathbb{R} : 0 \le u \le r(\lambda, \pi) \right\}$$

where we identify $(x, r(x)) = (\mathcal{T}_0(x), 0)$ and we define the semi-flow $(\mathcal{T}_0)_t^r : (\Delta \times \mathcal{R})^r \to (\Delta \times \mathcal{R})^r$ locally by $(\mathcal{T}_0)_t^r(x, u) = (x, u + t)$, subject to the identifications.

Since inducing on B gives the map $\mathcal{T}_2 : B \to B$, we can also represent this semi flow as a suspension semiflow over $\mathcal{T}_2 : B \to B$ with respect to a function

$r_2 : B \to \mathbb{R}$, i.e., let $B^{\overline{r}_2} = \{(x,u) \in B \times \mathbb{R} : 0 \le u \le r(x)\}$ where we identify $(x, r_2(x)) = (\mathcal{T}_2(x); 0)$ and we define $(\mathcal{T}_2)_t^{\overline{r}_2} : B^r \to B^r$ locally by $(\mathcal{T}_2)_t^{\overline{r}_2}(x,u) = (x, u+t)$, subject to the identifications.

The following lemma was established by Bufetov [2].

Lemma 7.1.

(1) $r_2 \in L^\gamma(B, \mu_2)$, *for every* $\gamma > 1$*; and*
(2) *if* $F : \Omega_{\mathcal{R}} \to \mathbb{R}$ *is Hölder and* $f : B \to \mathbb{R}$ *is defined by* $f(x) := \int_0^{r_2(x)} F(S_t x) dt$ *then there exists* $\delta > 0$ *such that* $f \in L^{2+\delta}(B, \mu_2)$.

We now recall the continuous analogue of the Almost Sure Invariance Principle.

Definition. A flow $\psi_t : X \to X$ is said to satisfy the *Almost Sure Invariance Principle* with respect to an invariant probability μ if for a Hölder function $\Psi : X \to \mathbb{R}$ such that $\int \Psi d\mu = 0$ there is a $\epsilon > 0$ and a random variable $\{S_t\}$ and a Brownian motion B with variance σ^2 such that $\int_0^t \Phi(\psi_t) d\mu$ is equal in distribution to random variables S_t and $S_t = B_t + O(t^{1/2-\epsilon})$.

The result for Teichmüller flows corresponding to Theorem 1 is the following.

Theorem 3. *The Teichmüller flow satisfies the almost sure invariance principle.*

Proof. It suffices to show the result for the associated semi-flow (the result for the natural extension requiring a standard argument involving changing functions by a coboundary). Let $\Phi : B^{\overline{r}_2} \to B^{\overline{r}_2}$ be a Hölder function with $\phi(x) = \int_0^{r(x)} \Phi(\psi_t x)) dt$.

(i) $\overline{r}_2 \in L^{2+\beta}(B, \mu_2)$, for some $\beta > 1$ (by part (1) of Lemma 7.1)
(ii) $\phi \in L^{2+\delta}(B, \mu_2)$, for some $\delta > 0$ (by part (2) of Lemma 7.1); and
(iii) $\mathcal{T}_2 : B \to B$ satisfies the Almost Sure Invariance Principle (by Theorem 1)

The Teichmüller flow then satisfies the Almost Sure Invariance Principle by the results of [13]. □

The following are standard corollaries.

Corollary 3.1.1 (Central Limit Theorem). *For* $y \in \mathbb{R}$ *we have that*

$$\lim_{T \to +\infty} \mu \left\{ x : \frac{1}{\sqrt{T}} \int_0^T \Phi(T_t x) dt \le y \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^y e^{-t^2/2\sigma^2} dt$$

The central limit theorem for Teichmüller flows was proved by Bufetov [2].

Corollary 3.1.2. *For* $y \ge 0$ *we have that*

$$\lim_{T \to +\infty} \mu \left\{ x : \frac{1}{\sqrt{T}} \max_{1 \le t \le T} \int_0^t \Phi(\psi_t x) dt \le y \right\} = \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \int_{-\infty}^y e^{-t^2/2\sigma^2} dt - 1$$

Corollary 3.1.3 (Arcsine Law). *For* $0 \le y \le 1$ *we have that*

$$\lim_{T \to +\infty} \mu \left\{ x : \frac{N_T(x)}{T} \le y \right\} = \frac{2}{\sqrt{\pi}} \sin^{-1} \sqrt{y}$$

where $N_T(x) = Leb \left\{ 0 \le t \le T \ : \int_0^t \Phi(\psi_t x) d\mu > 0 \right\}$

Corollary 3.1.4 (Law of the iterated logarithm). *For a.e. (μ) x we have*

$$\limsup_{T \to +\infty} \frac{\int_0^T \Phi(\psi_t x) d\mu}{\sigma \sqrt{2T \log \log T}} = 1.$$

In the study of the Teichmüller flow one can consider the suspended flow over the transformation $\mathcal{T}_2$ using the functon $f(x) = \log |\det D\mathcal{T}_2|$.

References

1. A. Avila and A. Bufetov, *Exponential decay of correlations for the Rauzy-Veech-Zorich induction map*, Preprint.
2. A. Bufetov, *Decay of correlations for the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations and the central limit theorem for the Teichmller flow on the moduli space of abelian differentials*, J. Amer. Math. Soc. **19** (2006), 579–62.
3. A. Grothendieck, *Produits tensoriels topologiques et espaces nuclaires.*, Mem. Amer. Math. Soc. , 16, Amer. Math. Soc., Providence, 1955.
4. P. Hall and C. Heyde, *Martingale Limit Theory and Its Application.*, Academic Press, New York, 1980.
5. C. Heyde, *Invariance Principles in Statistics.*, International Statistical Review **49** (1981), 143-152.
6. A.G. Kachurovskiĭ, *Rates of convergence in ergodic theorems*, Russian Math. Surveys **51** (1996), 653–703.
7. M. Mackay and M. Tyran Kaminska, *Central limit theorems for non-invertible measure preserving maps*, to ppear, Colloquium Mathematicum.
8. M. Mackey and M. Tyran-Kaminska, *Deterministic Brownian motion: the effects of perturbing a dynamical system by a chaotic semi-dynamical system*, Phys. Rep. **422** (2006), 167222..
9. H. Masur, *Interval exchange transformations and measured foliations*, Annals of Math. **115** (1982), 169-200.
10. D. Mayer, *On a ζ function related to the continued fraction transformation*, Bull. Soc. Math. France **104** (1976), 195–203.
11. D. Mayer, *Approach to equilibrium for locally expanding maps in $\mathbb{R}^k$*, Comm. Math. Phys. **95** (1984), 1-15.
12. I. Melbourne and M. Nicol, *Almost sure invariance principle for nonuniformly hyperbolic systems*, Comm. Math. Phys. **260** (2005), 131–145..
13. I. Melbourne and A. Tör̈k, *Statistical limit theorems for suspension flows*, Israel J. Math. **144** (2004), 191–209.
14. T. Morita, *Renormalized Rauzy inductions*, Advanced Studies in Pure Mathematics **43** (2005), 1–25.
15. W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic systems*, Asterisque **187-188** (1990), 1-268.
16. W. Philipp and W. Stout, *Almost sure invariance principles for partial sums of weakly dependent random variables*, Mem. Amer. Math. Soc., 161, Amer. Math. Soc., Providence, RI..
17. D. Ruelle, *Zeta-functions for expanding maps and Anosov flows*, Invent. Math. **34** (1976), 231–242.
18. M. Tyran-Kamińska, *An invariance principle for maps with polynomial decay of correlations*, Comm. Math. Phys. **260** (2005), 1–15.
19. W. Veech, *The Teichmüller geodesic flow*, Annals of Math. **124** (1986), 441-530.
20. J.-C. Yoccoz, *Lecture Notes at the College de France.*
21. A. Zorich, *Flat surfaces*, Frontiers in number theory, physics, and geometry. I,, Springer, Berlin, 2006, pp. 437–583.
22. A. Zorich, *Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents*, Ann. Inst. Fourier **46** (1996), 325–370.

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