Appendix

Spectral theory of unitary operators

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1 Two basic theorems

We shall base our account on two theorems; the first is due to Herglotz (later generalised to locally compact abelian groups by Bochner (cf. Loomis [1])) and the second is due to Wiener (cf. Halmos [4]; Plessner and Rohlin [1] for general accounts of spectral theory).

A sequence $\{r_n\}$ of complex numbers is said to be *positive definite* if $\sum_{n,m=0}^{N} r_{n-m}\, a_n\bar{a}_m \geq 0$ for all sequences $\{a_n\}$ and all non-negative integers N.

If U is a unitary operator on the Hilbert space H and $x \in H$, then $r_n = \langle U^n x, x\rangle$ $(n = 0, \pm 1, \pm 2, \ldots)$ is positive definite since

$$\sum_{n,m=0}^{N} \langle U^{n-m}x, x\rangle a_n\bar{a}_m = \left\langle \sum_{n=0}^{N} a_n U^n x, \sum_{m=0}^{N} a_m U^m x \right\rangle \geq 0.$$

1. Theorem (Herglotz; cf. Feller [1]) *If $\{r_n\}$ is a positive definite sequence then there is a unique finite non-negative measure μ on $K = \{z : |z| = 1\}$ (or on $[0, 1)$) such that*

$$r_n = \int_K z^n \,\mathrm{d}\mu = \int_0^1 \exp(2\pi i n x)\,\mathrm{d}\mu.$$

Conversely, if μ is a non-negative measure on K then r_n defined as above is a positive definite sequence.

Proof Clearly $r_0 \geq 0$ and for all complex λ, $(1 + |\lambda|^2)r_0 + r_n\lambda + r_{-n}\bar{\lambda} \geq 0$. Hence $r_n\lambda + r_{-n}\bar{\lambda}$ is real for all complex λ and it is easy to see that $r_{-n} = \bar{r}_n$. Put $\lambda = \theta\bar{r}_n$ to obtain

$$(1 + |\theta|^2|r_n|^2)r_0 + \theta|r_n|^2 + \bar{\theta}|r_n|^2 \geq 0$$

for all complex θ.

For real θ we have a quadratic in θ which is never negative. The discriminant of the quadratic shows that $|r_n| \leq r_0$ for all n. In

particular, the sequence $\{r_n\}$ is bounded. We can dismiss the trivial case $r_0 = 0$, so that without loss of generality we take $r_0 = 1$. Let $0 < s < 1$, then positive definiteness yields

$$f_s(z) = \sum_{n,m=0}^{\infty} r_{n-m} s^{n+m} z^{m-n} \geq 0$$

for all $|z| = 1$. But this sum equals

$$\sum_{n=-\infty}^{\infty} r_n z^{-n} \sum_{m=0}^{\infty} s^{|n|+2m} = \sum_{n=-\infty}^{\infty} r_n z^{-n} s^{|n|} \frac{1}{1-s^2}.$$

Hence

$$\int_K f_s(z) z^{-n}\,\mathrm{d}z = \frac{r_{-n}s^{|n|}}{1-s^2}.$$

Define μ_s by

$$\frac{\mathrm{d}\mu_s}{\mathrm{d}z} = (1 - s^2) f_s(z) \geq 0$$

so that

$$\int_K z^{-n}\,\mathrm{d}\mu_s = r_{-n}s^{|n|}, \quad \mu_s(K) = r_0 = 1.$$

Choose a sequence $s_m \to 1$ $(0 < s_m < 1)$; then $\int_K z^{-k}\,\mathrm{d}\mu_{s_m} \to r_{-k}$ for all $k = 0, \pm 1, \ldots$. Hence $\int_K p(z)\,\mathrm{d}\mu_{s_m}$ converges as $m \to \infty$ for all linear combinations $p(z)$ of functions z^k. Since such functions are dense in $C(K)$, we see that $\int_K f(z)\,\mathrm{d}\mu_{s_m}$ converges for all $f \in C(K)$ to, say, $J(f)$. But $\mathrm{d}\mu_s/\mathrm{d}z \geq 0$ implies $J(f) \geq 0$ when $f \geq 0$ and therefore $J(f) = \int f\,\mathrm{d}\mu$ for some probability μ on K. We conclude that

$$\int_K z^{-k}\,\mathrm{d}\mu = \lim_{m\to\infty} \int z^{-k}\,\mathrm{d}\mu_{sm} = r_{-k}$$

and the existence part of the theorem is complete.

It is clear that if $\int_K z^k\,\mathrm{d}\mu = r_k$ for some finite positive measure μ then $\{r_k\}$ is positive definite:

$$\sum_{n,m=0}^{N} r_{n-m}a_n\bar{a}_m = \sum_{n,m=0}^{N} a_n\bar{a}_m \int_K z^{n-m}\,\mathrm{d}\mu = \int_K \left| \sum_{n=0}^{N} a_n z^n \right|^2 \mathrm{d}\mu \geq 0.$$

Finally the measure μ such that $\int_K z^k\,\mathrm{d}\mu = r_k$ is unique since $\int_K z^k\,\mathrm{d}\mu$

$= \int_K z^k \, d\nu$ for all $k = 0, \pm 1, \ldots$ implies $\mu \equiv \nu$. We simply note that $\int_K f(z)\, d\mu = \int_K f(z)\, d\nu$ for all finite linear combinations of $\{z^k\}$ and therefore for all $f \in C(K)$.

2. Theorem (Wiener) *Let m be a finite Borel measure defined on the circle K.*

If H is a closed subspace of $L^2(K, m)$ which is invariant with respect to the unitary operator V: $f(z) \to z f(z)$ (i.e. $VH = H$) then

$$H = \chi_B L^2(K, m) = \{ f \in L^2(K, m) : f = 0 \quad \text{on } B^c \}$$

for some Borel subset B.

Proof Let $1 = k + h$ be the orthogonal decomposition of 1 with respect to $H^\perp, H$ (i.e. $k \in H^\perp, h \in H$). Then $k \perp V^n h$ for all n; $\int k(z) \cdot \overline{h(z)} z^n \, dm = 0$, $n = 0, \pm 1, \ldots$. Therefore $k \ \bar{h} = 0$ (a.e.) and $1 = |k|^2 + |h|^2$ (a.e.). Since k, h have disjoint 'supports' ($k = \chi_A k$, $h = \chi_{A^c} h$), $|k| = 1$ on A and $|h| = 1$ on A^c. But $1 = k + h$ implies $k = 1$ on A, $h = 1$ on A^c. In other words $1 = \chi_A + \chi_{A^c}$ is the decomposition of 1 with respect to $H^\perp$, H. Hence $z^n \chi_{A^c}(z) \in H$ for $n = 0, \pm 1, \ldots$ and we conclude that $\chi_{A^c} L^2(K, m) \subset H$, $\chi_A L^2(K, m) \subset H^\perp$ i.e. $\chi_{A^c} L^2(K, m) = H$.

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2 Spectral multiplicity theorems

If $U_i (i = 1, 2)$ are isometries of the Hilbert spaces H_i then U_1 is said to be *unitarily* (or *spectrally*) *equivalent* to U_2 if there exists an isometry W of H_1 onto H_2 such that $WU_1 = U_2 W$. In this case we write $U_1 \simeq U_2$.

We wish to characterise isometries up to unitary equivalence. The first thing to note is that if U is an isometry acting on the Hilbert space H then $U|H_\infty$ is a unitary operator where $H_\infty = \bigcap_{n=0}^{\infty} U^n H$, since $UH_\infty = H_\infty$ (i.e. U is invertible). $U|H_\infty^\perp$ is very easy to characterise. In fact $H_\infty^\perp = V \oplus UV \oplus \cdots$ where $V = H \ominus UH$, and therefore $U|H_\infty^\perp$ is completely described (up to unitary equivalence) by the dimension of V. *For this reason we shall be concerned exclusively with unitary operators U on a Hilbert space H.*

Let U be a unitary operator on the Hilbert space H. For $x \in H$, $Z(x)$ denotes the *cycle* (or *cyclic subspace*) *generated by* x which is the closure of the linear span of $\{U^n x : n = 0, \pm 1, \ldots\}$.

$U|Z(x)$ is unitarily (spectrally) equivalent to

$$V_x : L^2(K, \tilde{x}) \to L^2(K, \tilde{x}) \text{ defined by } (V_x f)(z) = z f(z) \qquad \text{(A.1)}$$

where $\tilde{x}$ is the *spectral measure* of x (with respect to U) i.e. $\langle U^n x, x \rangle = \int_K z^n \, d\tilde{x}$ for all n.

In fact, if we define $W : U^n x \to z^n \in L^2(K, \tilde{x})$ then W is an isometry on $\{U^n x : n = 0, \pm 1, \ldots\}$ since $\langle U^m x, U^n x \rangle = \int z^m z^{-n} \, d\tilde{x}$. Hence W extends to an isometry of $Z(x)$ onto $L^2(K, \tilde{x})$. Clearly $WU = V_x W$, and this is what we mean by the unitary equivalence of $U|Z(x)$ and V_x.

$U|Z(x)$ is unitarily equivalent to $U|Z(y)$

$(U|Z(x) \simeq U|Z(y))$ if and only if $\tilde{x} \simeq \tilde{y}$. (A.2)

We have to show that $V_x \simeq V_y$ if and only if $\tilde{x} \simeq \tilde{y}$. Suppose $WV_x = V_y W$ for some isometry W and write $f(z) = W(1)$, then $WV_x^n 1 = V_y^n f$, i.e. $W(z^n) = f(z) z^n$. Hence W is the multiplication operator $g \to f \cdot g$ and χ_B in $L^2(K, \tilde{x})$ has the same norm as $f \cdot \chi_B$ in $L^2(K, \tilde{y})$, i.e. $\tilde{x}(B) = \int_B |f|^2 \, d\tilde{y}$. Therefore $\tilde{x} \leq \tilde{y}$. A similar argument shows that $\tilde{y} \leq \tilde{x}$ and hence $\tilde{x} \simeq \tilde{y}$.

On the other hand, if $\tilde{x} \simeq \tilde{y}$ define $W : L^2(K, \tilde{x}) \to L^2(K, \tilde{y})$ by $Wg = g \cdot (d\tilde{x}/d\tilde{y})^{1/2}$. W is an isometry and $WV_x = V_y W$.

If $x \in H$ and $\mu \leq \tilde{x}$ is a finite non-negative Borel measure on K then there exists $y \in Z(x)$ with $\tilde{y} = \mu$. (A.3)

It suffices to note the existence of $f \in L^2(K, \tilde{x})$ with $\tilde{f} = \mu$. In fact $f = (d\mu/d\tilde{x})^{1/2}$ satisfies

$$\langle V_x^n f, f \rangle = \int z^n \frac{d\mu}{d\tilde{x}} \cdot d\tilde{x} = \int z^n \, d\mu = \int z^n \, d\tilde{f}$$

for all n, i.e. $\mu = \tilde{f}$.

If $x, y \in Z(z)$ and $Z(x) \perp Z(y)$ then $\tilde{x} \perp \tilde{y}$.

If in addition $z = x + y$ then $Z(z) = Z(x) \oplus Z(y)$. (A.4)

Transferring to $L^2(K, \tilde{z})$ we show that $Z(f) \perp Z(g)$, $f, g \in L^2(K, \tilde{z})$ implies $\tilde{f} \perp \tilde{g}$. In fact $Z(f) = \chi_A \cdot L^2(K, \tilde{z})$ and $Z(g) = \chi_B L^2(K, \tilde{z})$ (by Wiener's theorem) and orthogonality ensures that $\tilde{z}(A \cap B) = 0$. Since $d\tilde{f} = |f|^2 d\tilde{z}$, $d\tilde{g} = |g|^2 d\tilde{z}$ we have $\tilde{f} \perp \tilde{g}$. If we assume now that $z = x + y$ then $1 = f + g$ and $Z(1) = L^2(K, \tilde{z}) = Z(f) + Z(g)$.

If $y \in Z(x)$ *then* $\tilde{y} \le \tilde{x}$ *with equivalence holding when and only when* $Z(y) = Z(x)$. (A.5)

Map $Z(x)$ to $L^2(K, \tilde{x})$ (by sending $U^n x$ to z^n) and let f denote the image of y. Then we have to show that $\tilde{f} \le \tilde{x}$ with equivalence holding when and only when $Z(f) = Z(1)$ (with respect to V_x). But $\langle V_x^n f, f\rangle = \int z^n \, d\tilde{f} = \int z^n |f|^2 \, d\tilde{x}$. Hence $d\tilde{f} = |f|^2 \, d\tilde{x} \le d\tilde{x}$. If $Z(y) = Z(x)$ then $U|Z(y) \simeq U|Z(x)$ and we have seen that $\tilde{x} \simeq \tilde{y}$. If $Z(y)$ is a proper subspace of $Z(x)$ then $Z(f)$ is a proper subspace of $L^2(K, \tilde{x})$ invariant under V_x. By Wiener's theorem $Z(f) = \chi_B L^2(K, \tilde{x})$ where $\tilde{x}(B) < \tilde{x}(K)$ and hence $\tilde{x}(B^c) > 0$, $\tilde{f}(B^c) = 0$ i.e. $\tilde{y}$, $\tilde{x}$ are not equivalent.

If $\tilde{x} \perp \tilde{y}$ *(mutually singular) then* $Z(x) \perp Z(y)$. (A.6)

The converse is *not* always true. It is this fact which gives rise to multiplicity.

Write $y = y_0 + y_1$, with $y_1 \in Z(x)$, $y_0 \perp Z(x)$ so that $Z(y_0) \perp Z(x)$. $\langle U^n y, y\rangle = \langle U^n y_0, y_0\rangle + \langle U^n y_1, y_1\rangle$, that is $\int z^n \, d\tilde{y} = \int z^n \, d\tilde{y}_0 + \int z^n \, d\tilde{y}_1$. Hence $\tilde{y} = \tilde{y}_0 + \tilde{y}_1 \perp \tilde{x}$. But $y_1 \in Z(x)$ implies $\tilde{y}_1 \le \tilde{x}$. Therefore $\tilde{y}_1 = 0$ and we conclude that $y_1 = 0$, $y = y_0$ and $Z(x) \perp Z(y)$.

If $\tilde{x} \perp \tilde{y}$ *then* $\widetilde{x+y} = \tilde{x} + \tilde{y}$ *and*

$$Z(x+y) = Z(x) \oplus Z(y). \tag{A.7}$$

Since $Z(x) \perp Z(y)$,

$$\langle U^n(x+y), (x+y)\rangle = \langle U^n x, x\rangle + \langle U^n y, y\rangle,$$

i.e.

$$\int z^n \, d\widetilde{x+y} = \int z^n \, d\tilde{x} + \int z^n \, d\tilde{y} \quad \text{so that} \quad \widetilde{x+y} = \tilde{x} + \tilde{y}.$$

Now $d\tilde{x}/d\widetilde{x+y} \in L^2(K, \widetilde{x+y})$ so that for $\varepsilon > 0$ there exists a polynomial p in z, z^{-1} with

$$\int \left| \frac{d\tilde{x}}{d\widetilde{x+y}} - p(z) \right|^2 d\widetilde{x+y} < \varepsilon.$$

Hence $\|x - p(U)(x+y)\|^2$

$$\begin{aligned}
&= \langle x, x\rangle - 2\mathscr{R}\langle x, p(U)(x+y)\rangle + \|p(U)(x+y)\|^2 \\
&= \int 1 \, d\tilde{x} - 2\mathscr{R}\langle x, p(U)x\rangle + \int |p(z)|^2 \, d\widetilde{x+y} \\
&= \int 1 \, d\tilde{x} - 2\mathscr{R}\int p(z) \, d\tilde{x} + \int |p(z)|^2 \, d\widetilde{x+y} \\
&= \int 1 \, d\tilde{x} - \int \frac{d\tilde{x}}{d\widetilde{x+y}} \, d\tilde{x} + \int \left| \frac{d\tilde{x}}{d\widetilde{x+y}} - p(z) \right|^2 d\widetilde{x+y} \\
&= \int \left| \frac{d\tilde{x}}{d\widetilde{x+y}} - p(z) \right|^2 d\widetilde{x+y} < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary $x \in Z(x+y)$. In the same way $y \in Z(x+y)$. Therefore $Z(x+y) \supset Z(x) \oplus Z(y)$. If $v \in Z(x+y)$ and $v \perp Z(x) \oplus Z(y)$ then for $\varepsilon > 0$ there exists a polynomial p in z, z^{-1} with $\|v - p(U)(x+y)\|^2 < \varepsilon$. Hence $\|v\|^2 + \|p(U)(x+y)\|^2 < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $v = 0$. Hence $Z(x+y) = Z(x) \oplus Z(y)$.

A cyclic subspace $Z(x)$ is said to be *maximal* if it is contained in no larger cyclic subspace. Evidently $Z(x)$ is maximal if and only if $\tilde{x} \ge \tilde{y}$ for all $y \in H$. $\tilde{x}$ is then called (along with other $\tilde{y} \sim \tilde{x}$) a *maximal spectral type*.

If U *is a unitary operator of a* separable *Hilbert space* H, *then there exists a maximal cyclic subspace. In fact if* $x \in H$, *there is a maximal cyclic subspace containing* x. (A.8)

This is easily proved by using Zorn's lemma applied to all cyclic subspaces containing x.

If U_i *are unitary operators on* H_i $(i = 1, 2)$ *such that* $U_1 \simeq U_2$ *and* $U_1|Z(x) \simeq U_2|Z(y)$ *then*

$$U_1|Z(x)^\perp \simeq U_2|Z(y)^\perp. \tag{A.9}$$

The problem can be transferred to one space. It becomes: if $U|Z(x) \simeq U|Z(y)$ then $U|Z(x)^\perp \simeq U|Z(y)^\perp$. It will suffice to show that

$$U|\overline{Z(x) + Z(y)} \ominus Z(x) \simeq U|\overline{Z(x) + Z(y)} \ominus Z(y)$$

since $U|\overline{Z(x) + Z(y)} \simeq U|\overline{Z(x) + Z(y)}$ (using the identity). In other words we may assume $\overline{Z(x) + Z(y)} = H$. Let $y = y_0 + y_1$, $y_0 \perp Z(x)$, $y_1 \in Z(x)$; then $H = Z(x) \oplus Z(y_0)$. Write $Z(x) = Z(x_0) \oplus Z(y_1)$ (some x_0): then

$$H = \overline{Z(x) + Z(y)} = Z(x_0) \oplus Z(y_0) \oplus Z(y_1).$$

$$\tilde{x}_0 + \tilde{y}_1 = \tilde{x} \sim \tilde{y} = \tilde{y}_0 + \tilde{y}_1, \quad \tilde{x}_0 \perp \tilde{y}_1, \quad \tilde{y}_0 \perp \tilde{y}_1$$

and therefore $\tilde{x}_0 \sim \tilde{y}_0$. We conclude that there is an isometry from $Z(x_0)$ onto $Z(y_0)$ conjugating $U|Z(x_0)$ to $U|Z(y_0)$ and the proof is easily completed.

If U *is a unitary operator on a* separable *Hilbert space* H *then* H *can be decomposed into an orthogonal sum of cyclic subspaces* $H = \sum_n \oplus Z(x_n)$ *with* $\tilde{x}_1 \ge \tilde{x}_2 \ge \cdots$. (A.10)

Such a decomposition is called a *canonical* decomposition into 'decreasing' cycles. In fact let $\{y_n\}$ be dense in H. Choose a maximal cyclic space containing y_1 – say $Z(x_1)$. Now let $\{y_n^1 : n = 2, 3, \dots\}$ be the projections of $\{y_n : n = 2, 3, \dots\}$ onto $Z(x_1)^{\perp}$. Choose a maximal cyclic subspace (with respect to $U|Z(x_1)^{\perp}$) of $Z(x_1)^{\perp}$ containing y_2^1 – say $Z(x_2)$. Note that $y_1 \in Z(x_1)$, $y_2 \in Z(x_2) \oplus Z(x_1)$ (part is in $Z(x_1)$ and part in $Z(x_2)$). Repeat this process indefinitely. This process ensures that $y_n \in Z(x_1) \oplus \cdots \oplus Z(x_n)$. Since the $\{y_n\}$ are dense, $H = \sum_n \oplus Z(x_n)$. By maximality $\tilde{x}_1 \geq \tilde{x}_2 \cdots$.

The decomposition above is summarised by $\tilde{x}_1$ and the sets A_2, $A_3, \dots$ where A_2 is the support of $\mathrm{d}\tilde{x}_2/\mathrm{d}\tilde{x}_1$ (with respect to $\tilde{x}_1$) etc., $K = A_1 \supset A_2 \supset \cdots$. Hence the decomposition is summarised by $\tilde{x}_1 = \mu_U$ and $M_U = \sum_{n=1}^{\infty} \chi_{A_n}$. M_U is called the *multiplicity function* and is defined a.e. with respect to μ_U.

3. Theorem *If* $U_1|H_1 \simeq U_2|H_2$ *then* $\mu_{U_1} \sim \mu_{U_2}$ *and* $M_{U_1} = M_{U_2}$.

We need only show that canonical decompositions $H_1 = \sum_{n=1}^{\infty} \oplus Z(x_n)$, $H_2 = \sum_{n=1}^{\infty} \oplus Z(y_n)$ satisfy $\tilde{x}_1 \sim \tilde{y}_1, \tilde{x}_2 \sim \tilde{y}_2, \dots$. We know $\tilde{x}_1 \sim \tilde{y}_1$. Now consider $U_1|Z(x_1)^{\perp}$ and $U_2|Z(y_1)^{\perp}$ which we know to be spectrally equivalent. Clearly $Z(x_1)^{\perp} = \sum_{n=2}^{\infty} \oplus Z(x_n)$ and $Z(y_1)^{\perp} = \sum_{n=2}^{\infty} \oplus Z(y_n)$. Again we know $\tilde{x}_2 \sim \tilde{y}_2$. The proof is easily completed by induction.

4. Theorem *If* $U_1|H_1$ *and* $U_2|H_2$ *have maximal spectral types* $\mu_{U_1} \sim \mu_{U_2}$ *and multiplicities* $M_{U_1} = M_{U_2}$, *then* $U_1 \simeq U_2$.

This follows from the obvious fact: if $U|H$ has a canonical decomposition $H = \sum_{n=1}^{\infty} \oplus Z(x_n)$ then $U \simeq \sum_{n=1}^{\infty} \oplus V_{x_n} \simeq \sum_{n=1}^{\infty} \oplus V_{\mu_n}$ where $\mathrm{d}\mu_n/\mathrm{d}\mu_1 = \chi_{A_n}, \mu_n \sim \tilde{x}_n$.

3 Decompositions

Let U be a unitary operator on the Hilbert space H. An eigenvector is an element $x \in H$ such that $Ux = \lambda x$ for some $\lambda \in \mathbb{C}$: λ is the eigenvalue corresponding to $x (|\lambda| = 1)$. The discrete spectrum subspace of H, denoted by V_d, is the closure of the linear span of all eigenvectors. Clearly $UV_d = V_d$. Each $x \in V_d$ can be written as an orthogonal sum $x = \sum a(i) x_i$ with x_i an eigenvector with, say, eigenvalue λ_i. Hence

$U^n x = \sum a(i) \lambda_i^n x_i$ and

$$\langle U^n x, x \rangle = \sum |a(i)|^2 \lambda_i^n \|x_i\|^2 = \int_K z^n \, \mathrm{d}\tilde{x}$$

so that $\tilde{x}$ is the *atomic* measure which assigns measure $|a(i)|^2 \|x_i\|^2$ to $\lambda_i \in K$.

On the other hand, if $x \in H$ and if $\tilde{x}$ is purely atomic, then for each $\lambda_i \in K$ with $\tilde{x}(\lambda_i) > 0$ we have a measure μ_i such that $\mu_i(\lambda_i) = \tilde{x}(\lambda_i)$ and $\mu_i(K - \{\lambda_i\}) = 0$. Hence $\mu_i \leq \tilde{x}$ and $\mu_i = \tilde{x}_i$ for some $\tilde{x}_i \in Z(x)$. Since $\tilde{x}_i$ is concentrated on the single point λ_i, we have

$$\langle U^n x_i, x_i \rangle = \int z^n \, \mathrm{d}x_i = \lambda_i^n x(\lambda_i) = \lambda_i^n \|x_i\|^2.$$

The converse of Schwarz's inequality shows that $Ux_i = \lambda_i x_i$. Moreover $\tilde{x} = \sum \tilde{x}_i$ and therefore $x \in V_d$. We have proved

For each $x \in H$, x *is purely atomic if and only if* $x \in V_d$.

As an immediate consequence we have

For each $x \in H$, $\tilde{x}$ *is a non-atomic (i.e. continuous) measure if and only if* $x \in V_c = V_d^{\perp}$.

Exercise **1** Let $V_c = V_l \oplus V_l^{\perp}$ where $UV_l = V_l$ and $x \in V_l$ if and only if $\tilde{x}$ is absolutely continuous with respect to Lebesgue measure. Show that $x \in V_l^{\perp}$ if and only if $\tilde{x}$ is a continuous singular (with respect to Lebesgue) measure.

Exercise **2** Show that $\tilde{x}$ is equivalent to Lebesgue measure if and only if there exists $y \in Z(x)$ with $Z(x) = Z(y)$ and $\langle U^n y, y \rangle = 0$ for all $n \neq 0$.

Exercise **3** We say that x is a weak-mixing vector if $(1/N) \sum_{n=0}^{N-1} |\langle U^n x, x \rangle| \to 0$. Show that x is weak-mixing if and only if $\tilde{x}$ is non-atomic or, equivalently, $(\tilde{x} \times \tilde{x})(D) = 0$ where $D = \{(\lambda, \lambda) : \lambda \in K\} \subset K \times K$.