

THE UNIVERSITY OF WARWICK

FIRST YEAR EXAMINATION: January 2009

Analysis I

Time Allowed: **1.5 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 3 QUESTIONS.

If you have answered more than the required 3 questions in this examination, you will only be given credit for your 3 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

1. a) Give an example of a sequence which is not increasing, but tends to infinity. [2]
- b) State, without proof, whether the sequence $(\tan(n))$ is convergent, bounded or monotone. [3]
- c) Show that every sequence which is not bounded above has an increasing subsequence. [3]
- d) Show that if $\lim_{n \rightarrow \infty} a_n = l$ and $l > 0$, then a_n is eventually positive. [3]
- e) If $(a_n) \rightarrow a$ and $a_n > 0$ for all $n > 0$, then is it true that $a > 0$? Justify your answer. [3]
- f) Show that if $(a_n) \rightarrow 0$ then either there is a subsequence of $(1/a_n)$ which tends to $+\infty$, or there is a subsequence of $(1/a_n)$ which tends to $-\infty$. [3]
- g) If a is irrational and $b \neq 0$ is an integer, is $\frac{a}{b}$ rational or irrational? Justify your answer. [2]
- h) What is the value of $\lim_{n \rightarrow \infty} (3^n + 5^n + 2^n)^{1/n}$? [2]
- i) Find the value of $\inf\{\cos x \mid 0 < x < \pi\}$. [2]
- j) Give an example of a sequence bounded above, which does not have a convergent subsequence. [2]

Question 1 continued

- k) State, without proof, whether or not there can exist a Cauchy sequence (a_n) in \mathbb{R} , which does not converge in \mathbb{R} . [2]
- l) State, without proof, whether or not a bounded and monotonic sequence is always convergent. [2]
- m) State, without proof, whether or not the series $\sum_{n=1}^{\infty} \frac{\sin(n!)}{10^n}$ is convergent. [2]
- n) State, without proof, whether or not the series $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ is convergent. [2]
- o) State without proof whether or not the convergence of $\sum_{n=1}^{\infty} |a_n|$ implies that $\sum_{n=1}^{\infty} a_n$ is convergent. [2]
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2. a) Define what it means for a sequence (a_n) to converge to a limit a .
Define what it means for the sequence to be bounded.
Prove that every convergent series is bounded. [6]
- b) Which of the following sequences converge? Justify your answers. Find the limit for those sequences that converge to a real number.
- (i) $a_n = \frac{\sin(n)+\cos(n)}{\log(n)}$, for $n \geq 2$
- (ii) $b_n = \frac{\log(n)}{3+\sin(n)+\cos(n)}$, for $n \geq 2$
- (iii) $c_n = \frac{\sin(n)+2\log(n)}{\cos(n)+\log(n)}$, for $n \geq 2$ [6]
- c) Given a sequence (a_n) , recall that the n th term a_n is called a *floor term* if for every $m > n$ we have that $a_m \geq a_n$.
Using this notion, show that any sequence which is bounded below has either an increasing subsequence or a strictly decreasing subsequence. [8]
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3. a) Define what it means for a non-empty set B of real numbers
- (i) to be bounded below.
 - (ii) to have a greatest lower bound.

Use this definition to state a version of the completeness axiom for real numbers.

[6]

- b) Guess the infimum and supremum of the set

$$A = \left\{ 1 + \frac{1}{n^2} \mid n \in \mathbb{N} \right\}.$$

Now use the definition of infimum and supremum to prove your guesses.

[6]

- c) (i) Define what it means for a sequence (c_n) to be Cauchy.
 (ii) Use this definition to show that the sequence defined by

$$c_n = 2 - \frac{\sin n}{n}$$

is Cauchy.

- (iii) Show, using the definition, that if (a_n) and (b_n) are Cauchy sequences, then the sequence given by $c_n = 3a_n - b_n$ is also Cauchy.

[8]

4. a) Explain what is meant by a *partial sum* of a series $\sum_{n=1}^{\infty} a_n$. Explain what it means for the series $\sum_{n=1}^{\infty} a_n$ to be convergent.

[2]

- b) Using the definition of a convergent sequence to show that $\sum_{n=1}^{\infty} a^n$ is convergent if and only if $|a| < 1$.

[3]

- c) Is it true that $\sum_{n=1}^{\infty} a_n^2$ is convergent if $\sum_{n=1}^{\infty} a_n$ is convergent? Justify your answer.

[3]

- d) Write down an examples of convergent series and a divergent series both of which satisfy the condition $\frac{a_{n+1}}{a_n} \rightarrow 1$ and $a_n \geq 0$.

[4]

- e) Determine whether the following series are conditionally convergent, absolutely convergent, or divergent. Justify your answers.

- (i) $\sum_{n=1}^{\infty} \left(\frac{\sin n}{2^n + 1} + \frac{1}{n^2} \right)$.
- (ii) $\sum_{n=1}^{\infty} 1 + (-1)^n$
- (iii) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{2\sqrt{n+3}}$
- (iv) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

[8]

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Course Title: Analysis I

Model Solution No: 1

- a) The sequence $(a_n) = (n + 2(-1)^n)$ tends to infinity, but is not increasing.
- b) None of these.
- c) Let (a_n) sequence which is not bounded above. For any $C > 0$ there is a there is a natural number n_1 such that $a_{n_1} > C$. Next choose $C = a_{n_1}$, then there is a number n_2 such that $a_{n_2} > a_{n_1}$. Continuing inductively we have an increasing subsequence.
- d) Take $\epsilon = \ell/2$ then there is an N such that when $n > N$, $a_n > \ell - \epsilon = \ell/2 > 0$.
- e) No. Consider $a_n = \frac{1}{n} > 0$, then $a = 0$.
- f) If there are infinitely many terms $a_{n_k} > 0$ then the sequence $1/a_{n_k} \rightarrow \infty$. Otherwise, there are infinitely many terms $a_{n_k} < 0$ then the sequence $1/a_{n_k} \rightarrow -\infty$.
- g) It is irrational, since otherwise there would be integers p and q ($q \neq 0$) such that $\frac{a}{b} = \frac{p}{q}$. But this is equivalent to $a = \frac{bp}{q}$ which is rational, a contradiction.
- h) $\lim_{n \rightarrow \infty} (3^n + 5^n + 2^n)^{1/n} = 5$.
- i) $\{\cos x \mid 0 < x < \pi\} = (-1, 1)$, so its infimum is -1 .
- j) The sequence $(a_n) = (-n)$ is bounded above and every subsequence tends to $-\infty$.
- k) No, since in \mathbb{R} every Cauchy sequence converges by completeness.
- l) Yes, it is convergent.
- m) Yes, it is convergent.
- n) No, it is not convergent.
- o) Yes, it is convergent.

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Course Title: Analysis I

Model Solution No: 2

- a) The sequence (a_n) converges to a if for each $\epsilon > 0$ we can choose $N > 0$ such that for any $n > N$ we have that $|a - a_n| < \epsilon$.

The sequence is bounded if we can find L, U such that $L \leq a_n \leq U$ for all $n \geq 0$.

Let $\epsilon = 1$, say. Since (a_n) converges to a we can choose $N \in \mathbb{N}$ such that $|a_n - a| < 1$ for $n > N$. In particular, for $n > N$ we have that

$$a - 1 < a_n < a + 1.$$

We therefore see that for all $n > 0$ we have that:

$$L = \min\{a_1, \dots, a_N, a - 1\} \leq a_n \leq \max\{a_1, \dots, a_N, a + 1\} = U.$$

In particular, (a_n) is bounded.

- b) (i) This converges to 0 by comparison with $2/\log(n)$. Clearly, for any $\epsilon > 0$ we can choose $N > \exp(2/\epsilon)$. Then for all $n > N$ we have that $|a_n| < 2/\log(n) < 2/\log(N) < \epsilon$.
- (ii) This doesn't converge by comparison with $\log(n)$, since $b_n > \log(n)/5$. Clearly, for any $C > 0$ we can choose $N > \exp(5C)$. Then for all $n > N$ we have that $|b_n| > \log(n)/5 > \log(N)/5 > C$. Hence $(b_n) \rightarrow \infty$.
- (iii) We can bound the sequence by

$$\frac{2 \log(n) - 1}{\log(n) + 1} \leq c_n \leq \frac{2 \log(n) + 1}{\log(n) - 1}$$

or equivalently

$$\frac{2 - \frac{1}{\log(n)}}{1 + \frac{1}{\log(n)}} \leq c_n \leq \frac{2 + \frac{1}{\log(n)}}{1 - \frac{1}{\log(n)}}$$

Since $\lim_{n \rightarrow +\infty} \frac{1}{\log(n)} = 0$ the sequence converges to 2 by the sandwich theorem (and the sum rule and quotient rule for sequences).

- c) If (a_n) has infinitely many floor term then $a_{n_1}, a_{n_2}, a_{n_3} \dots$ then $(a_{n_k})_{k=1}^{\infty}$ forms an increasing subsequence, by definition.

If (a_n) has only finitely many floor terms, then we can choose m such that a_n is not a floor term for $n \geq m$. Let us write $n_1 = m$. Then by definition we can choose $n_2 > n_1$ such that $a_{n_2} < a_{n_1}$. Since a_{n_2} isn't a floor term we can similarly choose $n_3 > n_2$ with $a_{n_3} < a_{n_2}$. Proceeding inductively, we have a strictly decreasing sequence $(a_{n_k})_{k=1}^{\infty}$.

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Course Title: Analysis I

Model Solution No: 3

- a) (i) A non-empty set B of real numbers is bounded below, if there exists a real number L such that $b \geq L$ for all $b \in B$. L is then called a lower bound for B .
- (ii) A non-empty set B has a greatest lower bound l , if l is a lower bound for B and $l \geq L$ for any other lower bound L for B .

Using this definition, we can state the following version of the completeness axiom for real numbers: Every non-empty set of real numbers which is bounded below has a greatest lower bound.

- b) $\sup A = 2$: For all $n \in \mathbb{N} : 1 + \frac{1}{n^2} \leq 2$. Hence $U = 2$ is an upper bound. Since for $n = 1$ we have $1 + \frac{1}{n^2} = 2$, no $u < 2$ can be an upper bound for A , hence $U = 2$ is the least upper bound.

$\inf A = 1$: For all $n \in \mathbb{N} : 1 + \frac{1}{n^2} \geq 1$. Hence $L = 1$ is a lower bound. Assume that for some $\epsilon > 0$ we have $l = 1 + \epsilon$ is a lower bound. Then for $n > \frac{1}{\sqrt{\epsilon}}$ we have $1 + \frac{1}{n^2} < 1 + \epsilon$. But then l could not have been a lower bound for A , hence $L = 1$ is the greatest lower bound.

- c) (i) A sequence (c_n) is Cauchy if for each $\epsilon > 0$ there exists a natural number N such that $|c_m - c_n| < \epsilon$ for all $n, m > N$.
- (ii) Let $\epsilon > 0$. Then

$$\begin{aligned} |c_m - c_n| &= \left| 2 - \frac{\sin m}{m} - \left(2 - \frac{\sin n}{n} \right) \right| = \left| \frac{\sin n}{n} - \frac{\sin m}{m} \right| \\ &\leq \left| \frac{\sin n}{n} \right| + \left| \frac{\sin m}{m} \right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

if $n, m > N := \frac{2}{\epsilon}$, hence (c_n) is Cauchy.

- (iii) Let $c_n = 3a_n - b_n$ and $\epsilon > 0$. Since (a_n) and (b_n) are Cauchy there exists a natural number N_a such that for all $n, m > N_a : |a_m - a_n| < \frac{\epsilon}{6}$ and a natural number N_b such that for all $n, m > N_b : |b_m - b_n| < \frac{\epsilon}{2}$. Then for all $n, m > N := \max\{N_a, N_b\}$:

$$\begin{aligned} |c_m - c_n| &= |3a_m - b_m - (3a_n - b_n)| = |3(a_m - a_n) - (b_m - b_n)| \\ &\leq 3|a_m - a_n| + |b_m - b_n| < 3 \cdot \frac{\epsilon}{6} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

hence (c_n) is Cauchy.

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Course Title: Analysis I

Model Solution No: 4

- a) The numbers $S_N = \sum_{k=1}^N a_k$ are called partial sums. The series is convergent if the sequence (S_n) of partial sums is convergent.
- b) If $a = 1$ then $S_N = N \rightarrow \infty$. If $a \neq 1$, then we can write

$$S_N = \sum_{n=1}^N a^n = \frac{a(1 - a^N)}{1 - a}$$

which converges precisely when $|a| < 1$.

- c) No. Consider the counter example $\sum (-1)^n \frac{1}{\sqrt{n}}$.
- d) Consider the divergence series $\sum \frac{1}{n}$ and the convergent series $\sum \frac{1}{n^2}$. The corresponding limits of ratios of consecutive terms are

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + 1/n} \right) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + 1/n} \right)^2 = 1$$

- e) (i) Since $|\frac{\sin n}{2^{n+1}}| \leq \frac{1}{2^n}$, the series $\sum \frac{\sin n}{2^{n+1}}$ is absolutely convergence by the comparison theorem. Moreover, the series $\sum \frac{1}{n^2}$ converges absolutely. Thus the series converges absolutely by the algebra of series.
- (ii) This series is divergent as the sequence $1 + (-1)^n$ does not converge to 0.
- (iii) Note that $\frac{1}{2\sqrt{n+3}} \geq 0$ is a monotone decreasing sequence. Thus the series is convergent by the alternative series test. However, it is not absolutely convergent since $\sum_n \frac{1}{2\sqrt{n+3}} \geq \sum_n \frac{1}{5\sqrt{n}}$ diverges by the integral test, since the integral $\int \frac{1}{\sqrt{x}} dx$ diverges. Thus the original series is conditionally convergent.
- (iv) Using the ratio test we have that:

$$\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \frac{3^{n+1}n!}{(n+1)!3^n} = \frac{3}{n+1} \rightarrow 0 < 1$$

as $n \rightarrow \infty$.

Thus the series converges absolutely.