FORMULAE FOR RESIDUES OF DYNAMICAL ZETA FUNCTIONS.

Mark Pollicott

University of Warwick

0. Introduction

Let M be a compact surface with curvature $\kappa = -1$. Let τ denote a typical directed closed geodesic. Let $\lambda(\tau)$ denote the least period of τ. The Selberg zeta function is defined by

$$Z(s) = \prod_{\tau} \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)\lambda(\tau)}\right), \tag{0.1}$$

which converges for $Re(s) > 1$, and has an analytic extension to the entire complex plane. The value $s = 1$ is a simple zero.

Theorem 1. *We can write*

$$\frac{Z'(s)}{Z(s)} - \frac{1}{s-1} = \frac{1}{2} \frac{\sum_{k=1}^{\infty} a_k^{(2)}}{\sum_{k=1}^{\infty} a_k^{(1)}} \tag{0.2}$$

where $a_k^{(1)}$ and $a_k^{(2)}$ can be explicitly defined in terms of the lengths of the finitely many closed orbits having word length at most k.

In [9], Jorgenson and Kramer considered geometric estimates for (0.2) using spectral methods. By contrast, I used we used dynamical methods introduced by Ruelle. We present the proof of this theorem, along with the explicit expressions for $a_k^{(1)}$ and $a_k^{(2)}$ in section 2, based on a dynamical viewpoint contained in a fundamental article of Ruelle [13].

Let $f : M \to \mathbb{R}$ be a real analytic function (for example, an eigenfunction of the Laplacian) and write $\lambda_f(\tau) = \int_0^{\lambda(\tau)} f(\phi_t x_\tau) dt$ for the weighting of the orbit by f. In 1972, Bowen proved a natural equidistribution result for the closed geodesics that covers this case [3], i.e.,

$$\frac{\sum_{\lambda(\tau)\le T} \lambda_f(\tau)}{\sum_{\lambda(\tau)\le T} \lambda(\tau)} \to \int f d\mu, \text{ as } T \to +\infty,$$

where μ is the normalized Haar measure on M. Our second result shows that in the present special case the integral of f can be explicitly expressed as a convergent series in terms of all of the closed orbits.

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Theorem 2. *We can explicitly write $\int f d\mu$ as the ratio of two absolutely convergent series*

$$\int f d\mu = \frac{\sum_{k=1}^{\infty} b_k}{\sum_{k=1}^{\infty} c_k}$$

where:

1. *c_k can be explicitly defined in terms of the lengths of the finitely many closed orbits having word length at most k;*
2. *b_k can be explicitly defined in terms of the integral of the function f around these finitely many closed orbits.*

We present the proof of this theorem, along with the explicit expressions for b_k and c_k, in section 3.

The integral $\int f d\mu$ can also be interpreted in terms of the residue of a suitable function. More precisely, we can define a Dirichlet series by

$$L_f(s) = \sum_{\tau} \lambda_f(\tau) \frac{e^{-s\lambda(\tau)}}{1 - e^{-\lambda(\tau)}},$$

which converges to an analytic function for $Re(s) > 1$. The value $s = 1$ is a simple pole with residue $\mathrm{res}(L_f, 1) = \int f d\mu$, i.e., in a neighbourhood of $s = 1$ we can write

$$L_f(s) = \frac{\int f d\mu}{s-1} + \psi(s),$$

where $\psi(s)$ is analytic in a neighbourhood of $s = 1$. This characterization is a key ingredient in Parry's proof of Bowen's equidistribution result [11].

For surfaces of constant negative curvature, the location of other poles on the line $Re(s) = \frac{1}{2}$, or in the interval $[0, 1]$, is related to the spectrum of the Laplacian. The (negative) Laplacian $-\Delta : L^2(M) \to L^2(M)$ is a self-adjoint second order differential operator with countably many eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. The zeros s_n for $L_f(s)$, are closely related to the eigenvalues by $\lambda_n = \frac{1}{4} + r_n^2$ by $s_n = \frac{1}{2} \pm i r_n$. We can then carry out a similar analysis for the other residues. If we assume, for convenience, that the value $s = s_n$ is a simple pole with residue $\mathrm{res}(L_f, s_n)$ then we can write

$$L_f(s) = \frac{\mathrm{res}(L_f, s_n)}{s - \left(\frac{1}{2} \pm i r_n\right)} + \psi_n(s),$$

where $\psi_n(s)$ is analytic in a neighbourhood of $s = \frac{1}{2} \pm i r_n$.

Theorem 3. *We can explicitly write the residues of simple poles for $L_f(s)$ as the ratio of two absolutely convergent series*

$$\mathrm{res}(L_f, s_n) = \frac{\sum_{k=1}^{\infty} b_k^{(n)}}{\sum_{k=1}^{\infty} c_k^{(n)}}$$

where

1. *$b_k^{(n)}$ can be explicitly defined in terms s_n and the lengths of the finitely many closed orbits having word length at most k;*
2. *$c_k^{(n)}$ can be explicitly defined in terms of s_n and the integral of the function f around these finitely many closed orbits.*

We present the proof of this theorem, along with the explicit expressions for $b_k^{(n)}$ and $c_k^{(n)}$, in section 4.

As before, we can estimate $|b_k^{(n)}|, |c_k^{(n)}| \leq C_n \theta^{k^2}$, where $C > 0$ and $0 < \theta < 1$. As n increases there will be contributions to the residue from $b_k^{(n)}, c_k^{(n)}$ for larger k before the super exponential decay becomes evident. More precisely, the bounds on the constants C_n are of the form $|C_n| = O(e^{Dr_n})$ (cf. [7], [8]). More generally, the poles of higher multiplicity and the trivial poles of negative integers can be similarly treated and shown to have corresponding expressions.

1. BACKGROUND

The interpretation of the poles and residues in terms of the spectrum of the Laplacian involves a number of different ingredients. This includes he Helgason-Poisson transform for eigenvectors [6] and the Bowen-Series-Adler-Flatto [4], [15] [1] coding. However, in addition to the coding the main ingredients in the proofs is an approach to dynamical zeta functions due to Ruelle.

1.1 Symbolic dynamics and geodesics (following Bowen-Series, Adler-Flatto). Let $I = \coprod_{i=1}^k I_i$ be a disjoint union of closed intervals in the real line. Let $T : I \to I$ be a map which is real analytic on a neighbourhood of each component I_i. Assume that the map T is Markov, i.e., the image $T(I_i)$ is a union of intervals from $\{I_1, \cdots, I_n\}$ and expanding, i.e., $\inf_{x\in I} |T'(x)| > 1$.

The word length of a closed geodesic corresponds to the shortest presentation of an element in its conjugacy class in terms of a fixed symmetric set of generators. More precisely, let Γ_0 be a symmetric set of generators for the fundamental group $\pi_1(M)$. We define the word length of a curve $c \in \pi_1(M)$ by

$$|c| = \min\{k : c = g_1 \cdots g_k, \text{ where } g_1, \cdots, g_k \in \pi_1(M)\}.$$

We define the word length of a geodesic τ as the least word length of curves in the conjugacy class corresponding to τ.

The word length of closed geodesics is merely a particularly convenient way to explain the grouping of terms. In fact, since the proof uses symbolic dynamics there are many alternative (but less canonical) ways to form the terms b_k, c_k.

We begin by recalling the following useful result on coding geodesics.

Proposition 1.1. *To each surface we can associate a piecewise real analytic expanding Markov interval map $T : I \to I$ such that:*

1. *The closed geodesics τ correspond to periodic orbits $\{x, Tx, \cdots, T^{n-1}x\}$*
2. *The length of the closed geodesic is given by $\lambda(\tau) = \log |(T^n)'(x)|$*
3. *The word length is given by $|\tau| = n$.*

The correspondence is one-one, except possibly for finitely many (prime) periodic orbits $\{x, Tx, \cdots, T^{n-1}x\}$ intersecting the boundary of the intervals $\{I_i\}$. It may be necessary to double up on finitely many prime orbits or redesignate their lengths. We assume without loss of generality that this correspondence is a bijection on closed orbits.

For our purposes it would be sufficient to associate to τ the value $|\tau| = n$, corresponding to the period of x in part (1). The interpretation in terms of word length will play no further role in the proof.

1.2 The meromorphic extension of zeta functions (afrer Ruelle). Following Ruelle [13], we can formally define a dynamical zeta function[1] by

$$d(z,s,w) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{|(T^n)'(x)|^{-s}}{1 - |(T^n)'(x)|^{-1}} e^{wf^n(x)}\right),$$

where $z, s, w \in \mathbb{C}$. This converges providing $Re(s) > 1$, $|z| < 1$ and $|w|$ is sufficiently small.

To study the function $d(z,s,w)$ we need to introduce an appropriate linear operator on functions. Let $T_i : T_i(I) \to I_i$ denote the local inverse of $T : I_i \to I$ for $i = 1, \cdots, n$ (i.e., $T \circ T_i(x) = x$, for $x \in I_i$). Assume that there are complex neighbourhoods $U_i \supset I_i$ such that T_i has a complex analytic extension, which we still denote by $T_i : \coprod_{I_j \subset T(I_i)} U_j \to U_i$. Let $U = \coprod_{i=1}^n U_i$ denote the disjoint union of the neighbourhoods. By abuse of notation, we let $|T'(x)|$ represent the complex analytic extension to U of the real analytic function $|T'| : I \to \mathbb{R}$.

Let $\mathcal{H}$ denote the Hilbert square integrable functions analytic functions on $g : U \to \mathbb{C}$ with the inner product

$$\langle f_1, f_2 \rangle = \int_U f_1(x+iy)\overline{f_2(x+iy)}dxdy.$$

Defintion. We can define a *transfer operator* $\mathcal{L}_{-s\log|T'|+wf} : \mathcal{H} \to \mathcal{H}$ for each $s \in \mathbb{C}$ by

$$\mathcal{L}_{-s\log|T'|+wf}\, g(x) = \sum_{Ty=x} \frac{e^{wf(y)}}{|T'(y)|^s} g(y).$$

This operator is well defined since we can write the $y = T_i x$, for suitable preimages. The following result is central to our analysis is a special case of a quite general result of Ruelle.

Proposition 1.2.

(1) *The function $d(z,s,w)$ is analytic for all $z, s, w \in \mathbb{C}$. Moreover, we can expand*

$$d(z,s,w) = 1 + \sum_{k=1}^{\infty} d_k(s,w) z^k$$

where for on any bounded domain there exists $D > 0$ and $0 < \theta < 1$ such that $|d_k(s,w)| \le D\theta^{k^2}$;

(2) *The zeros of $z \mapsto d(z, -s\log|T'| + wf)$ are the reciprocals of the eigenvalues $\{\lambda(s,w)\}$ of $\mathcal{L}_{-s\log|T'|+wf}$;*

(3) *We can write*

$$d(z,s,w) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} tr(\mathcal{L}^m_{-s\log|T'|+wf})\right)$$

for $|z| < 1$, where

$$tr(\mathcal{L}^m_{-s\log|T'|+wf}) = \sum_{T^m x = x} \frac{e^{wf^m(x)}}{|T'(x)|^s} \frac{1}{(1 - |(T^n)'(x)|^{-1})}.$$

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[1] There is a possibility that the symbolic zeta function differs from the natural one by a finite number of primes orbits. However, these make no contribution to the poles and residues of interest.

Proof. All of these observations are special cases of work of Ruelle on transfer operators acting on analytic functions, applied to symbolic dynamics for Anosov flows with real analytic stable and unstable foliations [13] (cf. [7], [8]). □

In practise we can choose θ arbitrarily close to $(\inf_x |T'(x)|)^{-1}$. This, in turn, can be related to e^{-l}, where l is the length of the shortest closed geodesic. In particular, this has a direct bearing on the speed of convergence in the terms of the series.

2. Proof of Theorem 1.

The Selberg zeta function defined in (0.1) can be studied a a special case of Proposition 1.2. In particular, the symbolic approach to $Z(s)$ is based on the identification

$$Z(s) = d(1, s, 0), \tag{2.1}$$

which follows directly from Proposition 1.1.

By Part (2) of Proposition 1.2 the zeros for $Z(s)$ correspond to the poles for $L_{-s\log|T'|}$. To briefly recall the connection with the spectrum of the laplacian, the zeros for $s \mapsto Z(s) = d(1, s, 0)$ correspond to $\mathcal{L}_{-s\log|T'|}$ having 1 as an eigenvalue. Equivalently 1 is an eigenvalue for the dual operator $\mathcal{L}^*_{-s\log|T'|}$ on suitable distributions on I. Furthermore, I essentially represents the boundary of the Poincaré disk. In particular, the fixed points $\mathcal{L}^*_{-s\log|T'|}\nu_s = \nu_s$ correspond to the Helgason distributions associated to the eigenfunctions, suggesting the connection between the spectral theory and the symbolic dynamics. Therefore, at least in principle, we can use the convergent series

$$Z(s) = \sum_{m=0}^{\infty} \left(\sum_{|\tau_1|+\cdots+|\tau_m|=k} (-1)^m \frac{e^{-s(\lambda(\tau_1)+\cdots+\lambda(\tau_m))}}{(1-e^{-\lambda(\tau_1)})\cdots(1-e^{-\lambda(\tau_m)})} \right)$$

to characterize the zeros of the Selberg zeta function, and thus the eigenvalues of the Laplacian.

The zeta function has a simple zero at value $s = 1$ and so we can expand it locally as

$$Z(s) = (s-1)Z'(1) + O\left((s-1)^2\right).$$

The next result gives an explicit expression for $Z'(1)$ in terms of closed orbits.

Proposition 2.1. *We can write*

(a) $Z'(1) = \sum_{k=1}^{\infty} a_k^{(1)}$ *where*

$$a_k^{(1)} = \sum_{\substack{\{\tau_1,\cdots,\tau_m\} \\ |\tau_1|+\cdots+|\tau_m|=k}} (-1)^{m+1} \frac{(\lambda(\tau_1)+\cdots+\lambda(\tau_m))}{(e^{\lambda(\tau_1)}-1)\cdots(e^{\lambda(\tau_m)}-1)}; \text{ and}$$

(b) $Z''(1) = \sum_{k=1}^{\infty} a_k^{(2)}$ *where*

$$a_k^{(2)} = \sum_{\substack{\{\tau_1,\cdots,\tau_m\} \\ |\tau_1|+\cdots+|\tau_m|=k}} (-1)^{m+2} \frac{(\lambda(\tau_1)+\cdots+\lambda(\tau_m))^2}{(e^{\lambda(\tau_1)}-1)\cdots(e^{\lambda(\tau_m)}-1)},$$

where in each case the absolutely convergent summations are over all possible finite sets $\{\tau_1, \cdots, \tau_m\}$ *of closed orbits whose word lengths sum to* k. *Moreover,* $|a_k^{(1)}|, |a_k^{(2)}| \leq C\theta^{k^2}$, *where* $C > 0$ *and* $0 < \theta < 1$.

Proof. Rocha and the author gave the formula for the first derivative $Z'(1)$ in [12]. The formula for the second derivative $Z''(1)$ is proved similarly. □

If $Z(s) = \phi(s)(s-1)$ then the logarithmic derivative takes the form $Z'(s)/Z(s) = 1/(s-1) + \psi(s)$ where $\psi(s) = \phi'(s)/\phi(s)$. In particular, we can write

$$\frac{Z'(s)}{Z(s)} = \frac{1}{(s-1)} + \psi(s)$$

In [9], Jorgenson and Kramer estimates were given for $\psi(1)$ in terms of the geometry of the surface. Theorem 1 follows from the following explicit formula.

Corollary. *We can write*

$$\psi(1) = \frac{1}{2} \frac{\sum_{k=1}^{\infty} a_k^{(2)}}{\sum_{k=1}^{\infty} a_k^{(1)}}$$

Proof. Since $Z'(s) = \phi'(s)(s-1) + \phi(s)$ and $Z''(s) = \phi''(s)(s-1) + 2\phi'(s)$ we can write $Z'(1) = \phi(1)$ and $Z''(1) = 2\phi'(1)$. The result then follows from Proposition 2.1. □

In principle, the value of $Z'(1)$ and $\psi(1)$ can be estimated numerically using information on the length of closed orbits. The effectiveness of this method is restricted by the difficulty in manipulating the information on closed orbits lengths.

We can carry out a similar analysis for other zeros of $Z(s)$. Assume that $s_n = \frac{1}{2} + ir_n$ is another zero for $Z(s)$ of order $r \geq 1$ then we can write $Z(s) = (s - s_n)^r Z^{(r)}(s)/r! + O\left((s-s_n)^{r+1}\right)$. We can then write $Z^{(r)}(s_n) = \sum_{k=1}^{\infty} a_k^{(r)}$ where

$$a_k^{(r)} = \sum_{\substack{\{\tau_1, \cdots, \tau_m\} \\ |\tau_1| + \cdots + |\tau_m| = k}} (-1)^{m+r} \frac{(\lambda(\tau_1) + \cdots + \lambda(\tau_m))^r}{(e^{\lambda(\tau_1)} - 1) \cdots (e^{\lambda(\tau_m)} - 1)} e^{\left(\frac{1}{2} - ir_n\right)(\lambda(\tau_1) + \cdots + \lambda(\tau_1))}.$$

If $Z(s) = \phi(s)(s - s_n)^r$, where $\phi(s)$ is a non-zero analytic function in a neighbourhood of $s = s_n$, then the logarithmic derivative takes the form

$$\frac{Z'(s)}{Z(s)} = \frac{k}{(s - s_n)} + \psi(s)$$

where $\psi(s) = \phi'(s)/\phi(s)$. Since $Z^{(r)}(s_n) = k!\phi(s_n)$ and $Z^{(r+1)}(s_n) = 2k!\phi'(s_n)$. We can write

$$\psi(1) = \frac{1}{2} \frac{\sum_{k=1}^{\infty} a_k^{(r+1)}}{\sum_{k=1}^{\infty} a_k^{(r)}}.$$

Remark. A similar analysis applies in the case of the Modular surface [10]. In this case, the weights $e^{-\lambda(\tau)}$ take the form $\overline{[n_1, \cdots, n_{2l}]}$ and $|\tau| = 2l$. The main distinction is that the summations over orbits with $|\tau| = 2l$ are now infinite (but still convergent). We can write

$$Z'(1) = \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} a_k^{(1)}$$

where

$$a_k^{(1)} = \sum_{\substack{n_1, \cdots, n_{2(l_1+\cdots+l_m)} \\ 2(l_1+\cdots+l_m)=k}} \frac{\log \prod_{i=1}^m \overline{[n_{2l_i+1}, \cdots, n_{2l_{i+1}}]}}{\prod_{i=1}^m \left(\overline{[n_{2l_i+1}, \cdots, n_{2l_{i+1}}]}^{-1} - 1 \right)}; \text{ and}$$

$Z''(1) = \sum_{k=1}^{\infty} a_k^{(2)}$ where

$$a_k^{(2)} = \sum_{\substack{n_1, \cdots, n_{2(l_1+\cdots+l_m)} \\ 2(s_1+\cdots+s_m)=k}} \frac{\left(\log \prod_{i=1}^m \overline{[n_{2l_i+1}, \cdots, n_{2l_{i+1}}]} \right)^2}{\prod_{i=1}^m \left(\overline{[n_{2l_i+1}, \cdots, n_{2l_{i+1}}]}^{-1} - 1 \right)}.$$

3. Proof of Theorem 2

We can adapt the basic approach used in the previous sections to get the formula for the reside in Theorem 1.

Theorem 2 (explicit form). *We can explicitly write $\int f d\mu$ as the ratio of two absolutely convergent series*

$$\int f d\mu = \frac{\sum_{k=1}^{\infty} b_k}{\sum_{k=1}^{\infty} c_k}$$

with

$$b_k = \sum_{\substack{\{\tau_1, \cdots, \tau_m\} \\ |\tau_1|+\cdots+|\tau_m|=k}} (-1)^m \frac{(\lambda_f(\tau_1) + \cdots + \lambda_f(\tau_m))}{(e^{\lambda(\tau_1)} - 1) \cdots (e^{\lambda(\tau_m)} - 1)} \quad \textit{and}$$

$$c_k = \sum_{\substack{\{\tau_1, \cdots, \tau_m\} \\ |\tau_1|+\cdots+|\tau_m|=k}} (-1)^m \frac{(\lambda(\tau_1) + \cdots + \lambda(\tau_m))}{(e^{\lambda(\tau_1)} - 1) \cdots (e^{\lambda(\tau_m)} - 1)}$$

where both summations are over all possible finite sets $\{\tau_1, \cdots, \tau_m\}$, with $m \geq 1$, of closed orbits whose word lengths sum to k.

Moreover, we can find $C > 0$ and $0 < \theta < 1$ such that $|b_k|, |c_k| \leq C\theta^{k^2}$, for all $k \geq 1$. In particular, as a consequence we can approximate

$$\frac{\sum_{k=1}^{N} b_k}{\sum_{k=1}^{N} c_k} = \int f d\mu + O\left(\theta^{N^2}\right), \text{ for } N \geq 1.$$

To begin the proof of Theorem 2, we can use Proposition 1.1 to write

$$d(z, s, w) = \prod_{\tau} \prod_{n=0}^{\infty} \left(1 - z^{|\tau|} e^{-(s+n)\lambda(\tau) + w\lambda_f(\tau)} \right). \tag{3.1}$$

and using Proposition 1.2 can expand

$$d(z,s,w) = 1 + \sum_{k=1}^{\infty} d_k(s,w) z^k. \tag{3.2}$$

We then get, by comparing coefficients in (3.1) and (3.2) for $|z| < 1$, that

$$d_k(s,w) = \sum_{|\tau_1|+\ldots+|\tau_m|=k} (-1)^m \frac{e^{-s(\lambda(\tau_1)+\ldots+\lambda(\tau_m))} e^{w(\lambda_f(\tau_1)+\ldots+\lambda_f(\tau_m))}}{(1-e^{-\lambda(\tau_1)})\cdots(1-e^{-\lambda(\tau_m)})}, \tag{3.3}$$

for any s, w. We can now use (3.2) to write

$$\frac{\partial}{\partial w} d(z,s,w)|_{w=0} = \sum_{k=1}^{\infty} d_k(s) z^n, \tag{3.4}$$

where we see from explicit expression for $d_k(s,w)$ in (3.3) to write that

$$\begin{aligned} d_k(s) &= \frac{\partial}{\partial w} d_k(s,w)|_{w=0} \\ &= \sum_{|\tau_1|+\ldots+|\tau_m|=k} (-1)^m \frac{(\lambda_f(\tau_1)+\ldots+\lambda_f(\tau_m))}{(1-e^{-\lambda(\tau_1)})\cdots(1-e^{-\lambda(\tau_m)})} e^{-s(\lambda(\tau_1)+\ldots+\lambda(\tau_m))}. \end{aligned} \tag{3.5}$$

We can use Cauchy's theorem (with $r > 0$ sufficiently small) to write

$$\begin{aligned} |d_k(s)| &= |\frac{\partial}{\partial w} d_k(s,w)|_{w=0}| \\ &= |\frac{1}{2\pi i} \int_{|\xi|=r} \frac{d_k(s,\xi)}{\xi^2} d\xi| \le \frac{D_0}{r} \theta^{k^2}, \end{aligned} \tag{3.6}$$

for suitable $D_0 > 0$, by Part (1) of Proposition 1.2.

The following lemma relates these series to the function $L_f(s)$.

Lemma 3.1.

$$\frac{\partial}{\partial w} \log d(1,s,w)|_{w=0} = L_f(s) + \eta(s)$$

where $\eta(s)$ is analytic on $Re(s) \ge \frac{1}{2}$ (except possibly at $s = \frac{1}{2}$), and thus $\frac{\partial}{\partial w} \log d(1,s,w)|_{w=0}$ and $L_f(s)$ share the same residue at $s = 1$.

Proof. Considering the derivative on the domain of convergence (and using the Taylor expansion for $\log(1-x)$) we can use (3.1) to write

$$\begin{aligned} \frac{\partial}{\partial w} \log d(z,s,w)|_{w=0} &= \frac{\partial}{\partial w} \sum_{\tau} \sum_{n=0}^{\infty} \log\left(1 - z^{|\tau|} e^{-(s+n)\lambda(\tau)+w\lambda_f(\tau)}\right)|_{w=0} \\ &= \frac{\partial}{\partial w} \sum_{\tau} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} z^{|\tau|m} e^{-(s+n)m\lambda(\tau)+wm\lambda_f(\tau)}|_{w=0} \\ &= \frac{\partial}{\partial w} \sum_{\tau} \sum_{m=1}^{\infty} z^{|\tau|m} \frac{e^{-sm\lambda(\tau)+wm\lambda_f(\tau)}}{1-e^{-m\lambda(\tau)}}|_{w=0} \\ &= \sum_{\tau} \sum_{m=1}^{\infty} z^{|\tau|m} m\lambda_f(\tau) \frac{e^{-sm\lambda(\tau)}}{1-e^{-m\lambda(\tau)}} \\ &= \sum_{\tau} z^{|\tau|} \lambda_f(\tau) \frac{e^{-s\lambda(\tau)}}{1-e^{-\lambda(\tau)}} + \eta(s) \end{aligned} \tag{3.7}$$

Setting $z = 1$ completes the identity. It only remains to observe that since the geodesic flow is weak mixing there can be no poles for $\eta(s)$ on $Re(s) = \frac{1}{2}$, except possibly at $s = \frac{1}{2}$. □

Using (3.2) and (3.4) we can write the logarithmic derivative

$$\frac{\partial}{\partial w} \log d(z,s,w)|_{w=0} = \frac{\frac{\partial}{\partial w} d(z,s,w)|_{w=0}}{d(z,s,0)} = \frac{\sum_{k=1}^{\infty} z^k d_k(s)}{1 + \sum_{k=1}^{\infty} z^k d_k(s,0)}. \tag{3.8}$$

In particular, by Lemma 3.1 the pole at $s = 1$ for $L_f(s)$ is a zero for $s \mapsto 1 + \sum_{k=1}^{\infty} d_k(s,0)$. Furthermore, since $s = 1$ is a simple pole we can write $L_f(s) = \psi(s)/(s-1)$, where $\psi(s)$ is analytic and non-zero in a neigbourhood of $s = 1$. The residue for $L_f(s)$ at $s = 1$ is therefore $\psi(1)$ or, equivalently, the derivative of $L_f(s)^{-1}$ at $s = 1$ is $\psi(1)^{-1}$. In particular, by Lemma 3.1 and (3.8) we can write

$$\begin{aligned} \psi(1) &= \left(\frac{\partial}{\partial s} \left(\frac{1}{L_f(s)} \right) |_{s=1} \right)^{-1} \\ &= \left(\frac{\partial}{\partial s} \left(\frac{1 + \sum_{k=1}^{\infty} d_k(s,0)}{\sum_{k=1}^{\infty} d_k(s)} \right) |_{s=1} \right)^{-1} \\ &= \left(\frac{\sum_{k=1}^{\infty} d_k}{\sum_{k=1}^{\infty} d_k(1)} \right)^{-1} \\ &= \left(\frac{\sum_{k=1}^{\infty} d_k(1)}{\sum_{k=1}^{\infty} d_k} \right) \end{aligned} \tag{3.9}$$

where we can use (3.3) to explicitly write

$$\begin{aligned} d_k &= \frac{\partial}{\partial s} d_k(s,0)|_{s=1} \\ &= \sum_{|\tau_1| + \ldots + |\tau_m| = k} (-1)^m \frac{(\lambda(\tau_1) + \ldots + \lambda(\tau_m))}{(1 - e^{-\lambda(\tau_1)}) \cdots (1 - e^{-\lambda(\tau_m)})} e^{-(\lambda(\tau_1) + \ldots + \lambda(\tau_m))}. \end{aligned} \tag{3.10}$$

As before, we can use Cauchy's theorem (providing $r > 0$ is sufficiently small) and (3.3) to bound

$$\begin{aligned} d_k &= \frac{\partial}{\partial s} d_k(s,0)|_{s=1} \\ &= \frac{1}{2\pi i} \int_{|\xi - 1| = r} \frac{d_k(\xi, 0)}{\xi^2} d\xi \leq \frac{D_1}{r^2} \theta^{m^2}, \end{aligned}$$

for suitable $D_1 > 0$ by Part (1) of Proposition 1.2. Comparing (3.9), (3.5) and (3.1) completes the proof of Theorem 2.

4. Proof of Theorem 3

The method from the previous section can be applied at other poles for $L_f(s)$ on $Re(s) = \frac{1}{2}$.

Theorem 3 (explicit form). *We can explicitly write the residues of simple poles for $L_f(s)$ as the ratio of two absolutely convergent series*

$$\mathrm{res}(L_f, s_n) = \frac{\sum_{k=1}^{\infty} b_k^{(n)}}{\sum_{k=1}^{\infty} c_k^{(n)}}$$

with

$$b_k^{(n)} = \sum_{\substack{\{\tau_1, \cdots, \tau_m\} \\ |\tau_1| + \cdots + |\tau_m| = k}} (-1)^m \frac{(\lambda_f(\tau_1) + \cdots + \lambda_f(\tau_m))}{2\sinh(\lambda(\tau_1)) \cdots \sinh(\lambda(\tau_m))} e^{-ir_n(\lambda(\tau_1) + \cdots + \lambda(\tau_m))} \text{ and}$$

$$c_k^{(n)} = \sum_{\substack{\{\tau_1, \cdots, \tau_m\} \\ |\tau_1| + \cdots + |\tau_m| = k}} (-1)^m \frac{(\lambda(\tau_1) + \cdots + \lambda(\tau_m))}{2\sinh(\lambda(\tau_1)) \cdots \sinh(\lambda(\tau_m))} e^{-ir_n(\lambda(\tau_1) + \cdots + \lambda(\tau_m))}$$

where both summations are over all possible finite sets $\{\tau_1, \cdots, \tau_m\}$, with $m \geq 1$, of closed orbits whose word lengths sum to k.

Moreover, there exists $0 < \theta < 1$ and, for each n a constant $C_n > 0$, such that we can estimate $|b_k^{(n)}|, |c_k^{(n)}| \leq C_n \theta^{k^2}$, for $k \geq 1$. As n increases one expects the contributions to the residue from $b_k^{(n)}, c_k^{(n)}$ for larger k before the super exponential decay becomes evident. (More precisely, the bounds on the constants C_n are of the form $|C_n| = O(e^{Dr_n})$ (cf. [7], [8]).)

To begin the proof of the theorem, we recall from (3.7) that we can write

$$\frac{\partial}{\partial w} \log d(1, s, w)|_{w=0} = \frac{\frac{\partial}{\partial w} \log d(1, s, w)|_{w=0}}{d(z, s, 0)} = \frac{\sum_{k=1}^{\infty} d_k(s)}{1 + \sum_{k=1}^{\infty} d_k(s, 0)} \tag{4.1}$$

for all $s \in \mathbb{C}$. In particular, the poles $s_n = \frac{1}{2} + ir_n$ for $L_f(s)$ correspond to the zeros for $s \mapsto 1 + \sum_{k=1}^{\infty} d_k(s, 0)$. If s_n is a simple pole then we can write $L_f(s) = \psi(s) / \left(s - \left(\frac{1}{2} + ir_n\right)\right)$, where $\psi(s)$ is analytic and non-zero in a neighbourhood of $s = s_n$. By analogy with (3.8) the residue of $L_f(s)$ at $s = s_n$ is

$$\begin{aligned} \psi\left(\frac{1}{2} + ir_n\right) &= \left(\frac{\partial}{\partial s}\left(\frac{1}{L_f(s)}\right)|_{s = \frac{1}{2} + ir_n}\right)^{-1} \\ &= \left(\frac{\partial}{\partial s}\left(\frac{1 + \sum_{k=1}^{\infty} d_k(s, 0)}{\sum_{k=1}^{\infty} d_k(s)}\right)|_{s = \frac{1}{2} + ir_n}\right)^{-1} \\ &= \frac{\sum_{k=1}^{\infty} d_k(s_n)}{\sum_{k=1}^{\infty} e_k^{(n)}} \end{aligned} \tag{4.2}$$

where we can use (3.3) to explicitly write

$$\begin{aligned} e_k^{(n)} &= \frac{\partial}{\partial s} d_k(s, 0)|_{s = \frac{1}{2} + ir_n} \\ &= \sum_{|\tau_1| + \ldots + |\tau_m| = k} (-1)^{m+1} \frac{(\lambda(\tau_1) + \ldots + \lambda(\tau_m))}{(1 - e^{-\lambda(\tau_1)}) \cdots (1 - e^{-\lambda(\tau_m)})} e^{-(\frac{1}{2} + ir_n)(\lambda(\tau_1) + \ldots + \lambda(\tau_m))} \end{aligned} \tag{4.3}$$

As in the previous section, we can use Cauchy's theorem (provided $r > 0$ is sufficiently small) and (3.3) to bound

$$\begin{aligned} e_k^{(n)} &= \frac{\partial}{\partial s} d_k(s,0)|_{s=\frac{1}{2}+ir_n} \\ &= \frac{1}{2\pi i}\int_{|\xi - (\frac{1}{2}+ir_n)|=r} \frac{d_k(\xi)}{\xi^2} d\xi \le \frac{D_2}{r^2}\theta^{k^2}, \end{aligned} \tag{4.4}$$

for suitable $D_2 > 0$, by Part (1) of Propostion 1.2. Comparing (4.2), (4.3) and (3.5) completes the proof of Theorem 3.

Remark. More generally, we can consider the case that $s = s_n = \frac{1}{2} + ir_n$ is a pole of order $r \geq 1$. The poles s_n for $L_f(s)$ correspond to the zeros for $s \mapsto 1 + \sum_{k=1}^\infty d_k(s,0)$. Let

$$L_f(s) = \psi(s) / \left(s - \left(\frac{1}{2} + ir_n\right)\right)^r$$

The residue is $\psi(s_n)$ and we can write

$$\begin{aligned} \psi\left(\frac{1}{2} + ir_n\right) &= \left(\frac{1}{r!}\frac{\partial^r}{\partial s^r}\left(\frac{1}{L_f(s)}\right)|_{s=\frac{1}{2}+ir_n}\right)^{-1} \\ &= \left(\frac{1}{r!}\frac{\partial^r}{\partial s^r}\left(\frac{1+\sum_{k=1}^\infty d_k(s,0)}{\sum_{k=1}^\infty d_k(s)}\right)|_{s=\frac{1}{2}+ir_n}\right)^{-1} \\ &= r!\frac{\sum_{k=1}^\infty d_k(s_n)}{\sum_{k=1}^\infty \frac{\partial^r}{\partial s^r}\left(d_k^{(n)}(s,0)|_{s=\frac{1}{2}+ir_n}\right)} \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} &\frac{\partial^r}{\partial s^r}\left(d_k^{(n)}(s,0)|_{s=\frac{1}{2}+ir_n}\right) \\ &= \sum_{|\tau_1|+\ldots+|\tau_m|=k} (-1)^{m+r}\frac{(\lambda(\tau_1)+\ldots+\lambda(\tau_m))^r}{(1-e^{-\lambda(\tau_1)})\cdots(1-e^{-\lambda(\tau_m)})} e^{-(\frac{1}{2}+ir_n)(\lambda(\tau_1)+\ldots+\lambda(\tau_m))} \end{aligned}$$

5. Generalizations

5.1 Schottky groups. The principle restriction to the method we have described is in choosing a suitable symbolic dynamics. For any convex cocompact hyperbolic manifold we can formulate analogues results to Theorems 1,2 and 3, if we replace the word length by the somewhat less satisfactory notion of the number of times a closed orbits crosses a Poincaré section in the associated symbolic dynamics for the geodesic flow. Since the Poincaré sections are not canonical, neither is this particular ordering of closed geodesics.

However, a particularly simple setting in which the same analysis applies is that of Schottky groups. More precisely, we can choose disjoint circles $C_1, \cdots, C_{2n}$, $n \geq 2$, in $\mathbb{C}$ and linear fractional transformations g_i which map the interior of C_i to the exterior of C_{n+i}. Let Γ be the group of isometries of three dimensional hyperbolic space $\mathbb{H}^3$ corresponding to the free group generated by g_i, $i = 1, \ldots, n$.

The cyclically reduced words $g_{i_1} \cdots g_{i_n}$ are in one-one correspondence with closed geodesics τ on the infinite volume manifold $\mathbb{H}^3/\Gamma$. We then define $|\tau| = n$, the number of generators. The same method of proof described in this paper applies. The only difference is that we now consider the transfer operator on the space $\mathcal{H}$ of analytic functions on a neighbourhood of the complexification of $\coprod_{i=1}^n \mathrm{int}(D_i)$. The corresponding Selberg Zeta function now has its largest real zero at $s = \delta$, where δ is the Hausdorff dimension of the limit set. The analogues of Theorem 1 and 2 hold in this case. The main technical difference is that in this case the terms in the convergence series are $O(\theta^{k^{3/2}})$, cf. [5], [7], [8].

5.2 Real analytic metrics. Theorem 1 has an extension to the case of real analytic metrics, using an approach of Rugh. In this case we can consider a real analytic surface M and a geodesic flow $\phi_t : SM \to SM$ on the unit tangent bundle of M. Let $T_1, \cdots, T_k$ be a family of Markov sections for the flow. The transfer operator described by Rugh on analytic distributions on $\coprod_{i=1}^k T_i$ is of trace class and its determinant naturally leads to the study of zeta functions of the general form

$$\zeta(s) = \prod_\tau \left(1 - \frac{e^{-s\lambda(\tau)}}{\det\left(D\phi_{\lambda(\tau)}(x_\tau) - I\right)} \right)^{-1}$$

where x_τ denotes any point on τ (and $D\phi_{\lambda(\tau)}(x_\tau)$ corresponds to the determinant of the Poincaré map on a transverse section) [Ru].

In particular, the residues for poles of this zeta function can be analyzed using the same methods we described earlier. However, in this case the value $|\tau|$ is described in terms of the number of times the orbit traverses a section, which is less than canonical.

5.3 Relation with Quantum Chaos. There is apparently an interesting connection between these residues and problems in Quantum Chaos. Let ϕ_n denote a normalized eigenfunction for $-\Delta$, i.e., $\Delta\phi_n = -\lambda_n\phi_n$, for $n \geq 0$, with $\int |\phi_n|^2 d\mu = 1$. Consider the Wiener measures defined by $dW_n = |\phi_n|^2 d\mu$. The following very elegant result has been announced by N.Anarathaman and S. Zelditch [AS].

Anarathaman-Zelditch Correspondence. *$L_f(s)$ has a meromorphic extension to $\mathbb{C}$. Let $s_n = \frac{1}{2} \pm ir_n$ denote the poles on the line $Re(s) = \frac{1}{2}$ with residue* $\mathrm{res}(L_f, s_n)$*. Then* $\mathrm{res}(L_f, s_n) \sim \int f dW_n$ *as $n \to +\infty$.*

In particular, if $\int f d(\mathrm{Vol}) = 0$ then Quantum Unique Ergodicity is equivalent to $\mathrm{res}(L_f, s_n) \to 0$ as $n \to +\infty$. It is known that that this is true for a density one sequence. However, this is not at all apparent from the dynamical viewpoint.

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