Computing multifractal spectra

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Abstract

The famous Birkhoff Ergodic Theorem shows that given an ergodic measure the averages of an integrable function along typical orbits converges to the integral of the function. The multifractal spectral describes the sets of points for which the averages converge to another limit. In this note we will consider the specific setting of conformal repellors and show how to estimate the Hausdorff Dimension of such sets via approximations to their alternative characterizations as zeros of appropriate determinant functions.

1 Introduction

Given a measurable transformation $T : X \rightarrow X$ and an ergodic probability measure $\mu$ the Birkhoff ergodic theorem tells us that for almost every point the Cesàro averages (or Birkhoff averages) along an orbit converge to the integral. We summarise this as follows.

Theorem 1.1 (Birkhoff, 1931). Let $f \in L^1(X, \mu)$ then for a.e. $(\mu)$ we have that

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \rightarrow \int f \, d\mu \text{ as } N \rightarrow +\infty.$$ 

However, there is a still a set of zero measure about which Birkhoff’s theorem gives no information. It is a natural question to ask what is the “size” of the set of points of zero measure which don’t converge to $\int f \, d\mu$, but to some other given limit, as $N \rightarrow +\infty$. We will be interested in a particularly well known family of maps and invariant probability measures.

Definition 1.2. We say that a $C^2$ conformal map $T : X \rightarrow X$ ($X \subset \mathbb{R}^d$) is a conformal repeller if

1. $T$ is expanding, i.e., there exists $c > 0$, $\lambda > 1$ such that $\|DT^n v\| \geq c\lambda^n \|v\|$, for all $v \neq 0$ and $n \geq 1$;
2. $X$ is a repeller, i.e., there exists an open set $U \supset X$ with $X = \cap_{n=0}^{\infty} T^{-n}U$.

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1 INTRODUCTION

The conformality automatically holds for interval maps and hyperbolic rational maps restricted to their Julia sets, for example. Let $f : X \to \mathbb{R}$ be a $C^\omega$ function. Upper and lower bounds for the range of values of the accumulation points of the Cesàro averages $\left\{ \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) : n \geq 1 \right\}$ of this function come from the following quantities. We denote

\[
\alpha_+ = \sup \left\{ \int f d\mu : \mu = T - \text{invariant probability} \right\} \quad \text{and} \quad \alpha_- = \inf \left\{ \int f d\mu : \mu = T - \text{invariant probability} \right\}.
\]

In particular, for any $x \in [0, 1]$ we have that

\[
\alpha_- \leq \liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^nx) \leq \limsup_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^nx) \leq \alpha_+.
\]

It is natural to ask about the size of the set of points for which the limit exists for $\alpha$ in the range $(\alpha_-, \alpha_+)$. This leads to the following definition.

**Definition 1.3.** Given $\alpha_- < \alpha < \alpha_+$ we let

\[
\Lambda^{(f)}_\alpha = \left\{ x \in X : \lim_{n \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^nx) = \alpha \right\}.
\]

We can now state our main theorem on approximating the values of the multifractal spectra. In the context of $C^\omega$ conformal expanding repellers it provides a very efficient algorithm for the numerical computation of the multifractal spectrum.

**Theorem 1.4.** Let $T : X \to X$ be a $C^\omega$ conformal expanding repellor and let $f : X \to \mathbb{R}$ be a $C^\omega$ function. There exists $0 < \theta < 1$ such that given $\alpha \in (\alpha_-, \alpha_+)$ we can associate to the set of values $D_N = \{ f(x) : T^n x = x, n \leq N \}$ an approximation $d_N = d_N(D_N)$ such that

\[
\dim_H \left( \Lambda^{(f)}_\alpha \right) = d_N + O \left( \theta N^{(1 + \frac{1}{\omega})} \right),
\]

(1.1)

The significance of the super exponential error term in (1.1) is that it dominates the number of values to be computed in $D_N$, which grows exponentially with order $O(e^{hN})$ (where $h > 0$ denotes the topological entropy of the map). In the particular case of expanding Markov interval maps we will have $d = 1$ and we have the following corollary.

**Corollary 1.5.** Let $T : X \to X$ be a $C^\omega$ expanding Markov interval map and let $f : X \to \mathbb{R}$ be a $C^\omega$ function. There exists $0 < \theta < 1$ such that given $\alpha \in (\alpha_-, \alpha_+)$ we can associate to the set of values $D_N = \{ f(x) : T^n x = x, n \leq N \}$ an approximation $d_N = d_N(D_N)$ such that

\[
\dim_H \left( \Lambda^{(f)}_\alpha \right) = d_N + O \left( \theta N^2 \right).
\]

**Remark 1.6.** The $C^\omega$ hypothesis is crucial in proving these results. If we only assumed that $T$ is $C^\infty$ then we could only establish an exponential error term in the approximation in Theorem 1.4 (i.e., $O(\theta N)$ for some $0 < \theta < 1$).
We will describe the precise algorithm(s) later. In practise, there are two different approaches:

1. We can solve for $\alpha$ and $\mathcal{F}(f)(\alpha)$ independently in terms of a third variable $t$;
2. We can solve for $\mathcal{F}(f)(\alpha)$ in terms of $\alpha$.

The value $d_N$ comes from approximating an exact implicit expression for $\dim H \left( \Lambda^{(f)}_\alpha \right)$. In the specific setting of expanding interval maps, we define a determinant function of two variables:

$$d_2(s, t) = \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{T^n x = x} \exp \left( -t \sum_{i=0}^{n-1} f(T^i x) \right) \left| (T^n)'(x) \right|^{-s} \right).$$

This converges for $s$ and $t$ sufficiently large and extends to all values (as an analytic function). Assume without loss of generality that that $d_2(1, 1) = 0$ (otherwise we can add a constant to $f$ such that this hypothesis then holds, as we explain in detail later in §3, and then the multifractal spectrum is merely translated by this constant). The following gives an exact implicit characterization of the multifractal spectrum $\dim H \left( \Lambda^{(f)}_\alpha \right)$ for $\alpha_- < \alpha < \alpha_+$.

**Theorem 1.7.** Given $\alpha$ there is $s_\alpha$ and $t_\alpha$ such that:

1. $d_2(s_\alpha, t_\alpha) = 0$; and
2. $\frac{\partial d_2(s_\alpha, t_\alpha)}{\partial t} \big|_{t=t_\alpha} = \alpha \frac{\partial d_2(s_\alpha, t_\alpha)}{\partial s} \big|_{s=s_\alpha},$

and then we can write that $\dim H \left( \Lambda^{(f)}_\alpha \right) = s_\alpha + \alpha t_\alpha$.

Explicit estimates on $d_2(s, t)$ allow us to deduce the approximation result in the previous theorem.

In section 2 we illustrate the numerical application with two concrete examples, the details of which will be given in a later section. In sections 2-8 we will restrict to the simpler case that $T$ is the doubling map. In the subsequent sections we will generalise to where $T$ is a $C^\omega$ Markov expanding map.

## 2 Two examples

For the moment let us consider two specific examples to help illustrate this result. In both examples we take $X$ to be the unit interval and let $T : [0, 1] \to [0, 1]$ be the doubling map defined by $T x = 2x \pmod{1}$. This preserves the Lebesgue measure $\mu$. The two examples will correspond to the specific choices of functions $f(x) = \cos(2\pi x)$ or $f(x) = \sin(2\pi x)$.

**Remark 2.1.** In these particular cases, it is very easy to see the pointwise convergence to zero for the Cesàro averages without resorting to the use of the full weight of the Birkhoff theorem. More precisely, we can explicitly compute

$$\int_0^1 \left( \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \right)^4 \, dx = \frac{1}{N^4} \sum_{n_1, n_2, n_3, n_4=0}^{N-1} \int_0^1 f(T^{n_1+n_2+n_3+n_4} x) \, dx = \frac{2N^2 + N}{8N^4}.$$

In particular, we can deduce that $\int_0^1 \sum_{N=1}^{\infty} \left( \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \right)^4 \, dx < +\infty$ and thus we conclude that for a.e. $(\mu) x$ we have that $\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \to 0$ as $N \to +\infty$. 


Example 2.2 ($f(x) = \cos 2\pi x$). Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \cos(2\pi x)$. Clearly $\int f(x) \, d\mu(x) = 0$ and thus for almost all points (with respect to Lebesgue measure) we have the average converges to zero. In this case it is easy to check that $\alpha_+ = 1$ and $\alpha_- = -\frac{1}{2}$ For any other value $\alpha \neq 0$ in this range the lebesgue measure of the set of points converging to $\alpha \neq 0$ with be zero, and in fact it will have Hausdorff Dimension strictly less than 1. We can the estimate the Hausdorff Dimension of the set of points for which the Birkhoff averages converge to $\frac{1}{2}$ is:

$$\dim_H \left\{ x \in [0, 1] : \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \cos(2^{n+1}\pi x) = \frac{1}{2} \right\} = 0.73988277232849810681377573856\ldots$$

![Figure 1: Plots of the Cesàro averages $\frac{1}{N} \sum_{n=0}^{N-1} \cos(2^{n+1}\pi x)$ for (i) $N = 1$; (ii) $N = 6$; and (iii) $N = 12$.](image)

Example 2.3 ($f(x) = \sin(2\pi x)$). Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \sin 2\pi x$. Similarly, $\int f(x) \, d\mu(x) = 0$ and thus for almost all points (with respect to Lebesgue measure) we have the average converges to zero. In this case $\alpha_+ = \sqrt{15}/8 = 0.4841\ldots$ and $\alpha_- = -\sqrt{15}/8 = -0.4841\ldots$ (by a result of Bousch [2]). For any value $\alpha \neq 0$ in this range the lebesgue measure of the set of points converging to $\alpha$ with be zero, and in fact it will have Hausdorff Dimension strictly less than 1. We can the estimate the Hausdorff Dimension of the set of points for which the Birkhoff averages converge to $\frac{1}{4}$ is:

$$\dim_H \left\{ x \in [0, 1] : \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \sin(2^{n+1}\pi x) = \frac{1}{4} \right\} = 0.9014347510431874982161389164\ldots$$

![Figure 2: Plots of the Cesàro averages $\frac{1}{N} \sum_{n=0}^{N-1} \sin(2^{n+1}\pi x)$ for (i) $N = 1$; (ii) $N = 6$; and (iii) $N = 12$.](image)

In both of these simple examples we have considered the specific case of the doubling map. If $T^n x = x$ is a periodic point of period $n$ then we trivially see that $|(T^n)'(x)| = 2^n$.
and then we have the simplification
\[ d_2(s,t) = \exp \left( - \sum_{n=1}^{\infty} \frac{2^{-ns}}{n} \sum_{T^n x = x} \frac{\exp \left( -t \sum_{i=0}^{n-1} f(T^i x) \right)}{1 - (T^n)'(x)^{-1}} \right) \] (2.1)
in the complex function used in Theorem 1.7. We will return to this point in §6.

3 Hausdorff dimension

We want to begin by describing a standard approach using thermodynamic formalism. For the purposes of exposition, we will first consider the simplified case that \( T \) is the doubling map. We will then explain the modifications needed for the general case in a later section.

3.1 The pressure

We now introduce some notation and recall some standard results. Let \( g : [0,1] \rightarrow \mathbb{R} \) be a continuous function.

**Definition 3.1.** We define the pressure function of \( g \) by:
\[
P(g) = \lim_{n \to +\infty} \frac{1}{n} \log \left( \sum_{j=0}^{2^n-1} \exp \left( \sum_{k=0}^{n-1} g \left( \frac{2^k j}{2^n - 1} \right) \right) \right)
\]

**Remark 3.2.** The definition is in terms of the periodic points for the doubling map. It is well known that the \( 2^n \) periodic points of period \( n \) are of the form
\[
\frac{j}{2^n - 1}, \quad j = 0, 1, \ldots, 2^n - 1.
\]
The summation of the function \( g \) around the points of the orbit
\[
\left\{ \frac{2^k j}{2^n - 1} \right\} \text{ for } k = 0, \ldots, n - 1
\]
where \( \{ \cdot \} \) is the fractional part contributes to the pressure function the weight
\[
\sum_{k=0}^{n-1} g \left( \frac{2^k}{2^n - 1} j \right).
\]

There is an alternative formulation of the pressure using the variational principle:

**Lemma 3.3.** We can write
\[
P(g) = \sup_m \left\{ h(m) + \int g \, dm \right\}
\]
where the supremum is over all \( T \)-invariant probability measures \( m \). Moreover, providing \( g \) is Hölder continuous there is a unique \( T \)-invariant probability measure realising the supremum, and called the equilibrium state for \( g \).
Remark 3.4. We observe that when $\mu$ is Lebesgue measure we have that $h(\mu) = \log 2$. Moreover, for either $f(x) = \sin(2\pi x)$ or $f(x) = \cos(2\pi x)$ we have that $\int f d\mu = 0$. Thus by the variational principle we have that $P(-f) \geq \log 2$.

Remark 3.5. If we replace $g$ by $g + C$ then we see that $P(g + C) = P(g) + C$.

In principle, we would like to restrict to functions $f$ for which $P(-f) = 0$ and therefore like to “normalise” the function $f$ by adding a constant so as to obtain a new function $\tilde{f}$ which indeed has this property. This is achieved in the next lemma.

Lemma 3.6. If define $\bar{f} := f + P(-f)$ then $P(-\bar{f}) = 0$.

Proof. This follows easily from the definition of pressure, or from the variational principle. In particular, $P(-\bar{f}) = P(-(f + P(-f))) = P(-f) - P(-f) = 0$.

We now trivially see that points for which the Cesaro averages of $f$ converge to $\alpha$ are precisely those points for which the Cesaro averages of $\tilde{f}$ converge to $\bar{\alpha} = \alpha + P(-f)$ for $\bar{f}$, i.e., $\mathcal{F}(f)(\alpha) = \mathcal{F}^{\tilde{f}}(\bar{\alpha})$. We are particularly interested in the value of the pressure for the following two examples.

Example 3.7 (Cosine function). In the particular case $f(x) = \cos(2\pi x) \in [-1, 1]$ we can explicitly compute $P(-f) = 0.8575307\cdots$ and thus we can replace $f(x)$ by the normalised function 
$$\tilde{f} = f + 0.8575307\cdots \in [-0.1424693\cdots, 1.8575307\cdots].$$

Example 3.8 (Sine function). In the particular case $f(x) = \sin(2\pi x) \in [-1, 1]$ we can explicitly compute $P(-f) = 8933924\cdots$ and thus we can replace $f(x)$ by the normalised function 
$$\tilde{f} = f + 8933924\cdots \in [-0.1066076\cdots, 1.8933924\cdots].$$

However, obtaining estimates on the Hausdorff Dimension for $\tilde{f}$ is equivalent to obtaining estimates on $f$ since we see from the definitions that 
$$\dim_H(\Lambda^\alpha(f)) = \dim_H(\Lambda^{\tilde{f}}_{\alpha + P(-f)}).$$

We complete this section by recalling an important property of the pressure function.

Lemma 3.9. There is an analytic dependence of the pressure function $P(g)$ where $g$ is an element of the Banach space of Hölder continuous functions (with a fixed Hölder exponent).

### 3.2 Pressure and the Hausdorff dimension

The pressure is an important ingredient in the general theory of thermodynamic formalism. Moreover, it plays an important role in the computation of Hausdorff dimension of certain sets, as is known from the work of Ruelle [7] (cf. [9] and [3]).

We begin by recasting the pressure function in a more convenient form. Let us assume that $f$ is not cohomologous to a constant.

Definition 3.10. We can consider the function $P : \mathbb{R} \to \mathbb{R}$ defined by 
$$P(t) := P(-t\tilde{f}) (= P(-tf) - tP(-f))$$
We can consider the unique equilibrium measure $\mu_t$ for the potential $-tf$.

**Lemma 3.11.** The function $P(t)$ is analytic with $P(0) = \log 2$ and $P(1) = 0$. Moreover,

1. $P'(t) = -\int \tilde{f} d\mu_t$;
2. $-P'(t)$ obtains all the values in $(\alpha_{\min}, \alpha_{\max})$; and
3. $P(t)$ is strictly convex and $P''(t) > 0$

**Proof.** These properties follow easily from those of the pressure. For part (1) we recall that $\frac{\partial P(-tf)}{\partial t} = -\int \tilde{f} d\mu_t$ from [4], for example. For part (2) we refer to [1]. For part (3), the convexity is well known since $f$ is not cohomologous to a constant [4].

The connection between the function $P_2(t)$ and the Hausdorff Dimension $\mathcal{F}(\tilde{f})(\alpha) := \dim_H(\Lambda_{\alpha}^{(\tilde{f})})$ of the level set is given by the following:

**Lemma 3.12.** Given $\alpha$ let us choose the unique $t = t_\alpha$ such that $P'(t_\alpha) = -\alpha$. We then have that

$$\mathcal{F}(\tilde{f})(\alpha) = \frac{P(t_\alpha) + t_\alpha \alpha}{\log 2}.$$

**Proof.** This is well explained in the article of Pesin and Weiss [5] and the book of Pesin [6]. The starting point is that we have $P(-tf - P(t)) = 0$ for any $t$. The dimension of the measure $\mu_t$ satisfies

$$\dim_H(\mu_t) := \frac{h(\mu_t)}{\log 2}$$

$$= \frac{\int (tf + P(t)) d\mu_t}{\log 2}$$

$$= \frac{P(t) - tP'(t)}{\log 2} \quad (3.1)$$

using the variational principle and part (1) of Lemma 3.11. Moreover, for almost all points with respect to $\mu_t$ we have that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \tilde{f}(T^k x) = \int \tilde{f} d\mu_t = -P'(t),$$

by using part (1) of Lemma 3.11 again. In particular, setting $t = t_\alpha$ and then substituting $\alpha = -P'(t_\alpha)$ into (3.1) gives the required result.

**4 Determinants and spectra**

To address the problem of computing the we need to compute the pressure and the derivative of the pressure. We can characterize the pressure using the zeta function and determinant, which in the context that $T$ is the doubling map takes a simple form.
4.1 Determinant of a single variable

We begin with a complex function of one variable which is useful in estimating $P(-f)$.

**Definition 4.1.** We formally define a function (for the doubling map) by

$$d_0(z) = \exp \left( -\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{2^n}{(2^n - 1)} \sum_{k=0}^{2^n-1} \exp \left( -\sum_{m=0}^{n-1} f \left( \frac{2^m k}{2^n - 1} \right) \right) \right)$$

where $z \in \mathbb{C}$.

In particular, we have the following properties for $d_0(z)$ which are useful in estimating $P(-f)$.

**Lemma 4.2.** We have the following properties:

1. The function $d_0(z)$ converges to a non-zero analytic function for $|z| < e^{-P(-f)}$;
2. The value $e^{-P(-f)}$ is a simple zero for $d_0(z)$;
3. For any $\epsilon > 0$, there exists $C > 0$ such that we can expand

$$d_0(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$$

where $|a_n| \leq C \left( \frac{1}{2} + \epsilon \right)^{n^2}$, for $n \geq 1$ and $a_n$ depends on the values

$$\bigcup_{m=1}^{n} \left\{ f \left( \frac{2^k}{2^m - 1} \right) : 0 \leq k \leq m - 1 \right\} ;$$

and

4. The function $d_0(z)$ has an analytic extension to $\mathbb{C}$ as an entire function.

**Proof.** These results can be deduced from Ruelle’s original article [8]. We briefly explain the construction in the present setting.

Part 1 follows easily from the definitions of $P(-f)$ and $d_0(z)$.

For $r > 0$ the disk $D(r) = \{ z \in \mathbb{C} : |z - \frac{1}{2}| < r \}$ and then the inverse branches $T_0(z) = \frac{1}{2}$ and $T_1(z) = \frac{z + 1}{2}$ satisfy $T_0(D(r)) \cup T_1(D(r)) \subset D(\frac{r}{2} + \frac{1}{2})$. Given $\epsilon > 0$, we choose $r$ sufficiently large that $\frac{r}{2} + \frac{1}{2} > \left( \frac{1}{2} + \frac{1}{2} \right) r$. Let $B$ be the Banach space of bounded analytic functions on $D$ with the supremum norm. The operator $L : B \to B$ defined by $Lw(z) = e^{-f(T_0z)} w(T_0z) + e^{-f(T_1z)} w(T_1z)$ is a nuclear operator since we can expand

$$Lw(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{Lw(\xi)}{z - \xi} d\xi = \sum_{n=0}^{\infty} z^n \left( \frac{1}{2\pi i} \int_{\Gamma} \frac{Lw(\xi)}{\xi^{n+1}} d\xi \right)$$

where $\Gamma = \{ z \in \mathbb{C} : |z - \frac{1}{2}| = r - \epsilon \}$. In particular, we can write $Lw = \sum_{n=0}^{\infty} \lambda_n w_n l_n(w)$ where $w_n \in B$, $l_n \in B^*$ with $\|w_n\|_B = \|l_n\|_{B^*} = 1$ and $|\lambda_n| \leq C \left( \frac{1}{2} + \epsilon \right)^n$, for $n \geq 0$. It then follows that for $z \in \mathbb{C}$ we can write

$$\det(I - zL) = 1 + \sum_{n=1}^{\infty} z^n \sum_{k_1 < \cdots < k_n} \lambda_{k_1} \cdots \lambda_{k_n} \det(l_{n_i}(w_{n_j}))_{i,j=1}^{n}$$
Moreover, \( \det(I - zL) = \exp\left( -\sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr}(L^n) \right) \) can be identified with \( d_0(z) \) by explicitly computing

\[
\text{tr}(L^n) = \sum_{k=0}^{2^n-1} \text{tr}(L_k^{(n)}) = \frac{2^n}{2^n - 1} \sum_{k=0}^{2^n-1} \exp\left( -\sum_{m=0}^{n-1} f\left( \frac{2^m k}{2^n - 1} \right) \right)
\]

by using the Taylor expansion at the periodic point \( \frac{2^m k}{2^n - 1} \) to compute the eigenvalues of \( L_k^{(n)} \) and thus, by summing, the trace \( \text{tr}(L_k^{(n)}) \). This completes the sketch of parts 3 and 4.

Finally, \( L \) has a maximal eigenvalue \( e^{P(-f)} \) from which part 2 follows.

As an immediate consequence we have the following.

**Corollary 4.3.** If \( z_0 > 0 \) is the largest zero for \( d_0(z) \) then \( P(-f) = -\log z_0 \).

Revisiting our previous examples, we can estimate the following.

**Example 4.4 (Sine function).** If \( f(x) = \sin(2\pi x) \) then we can estimate

\[
\begin{align*}
z_0 &= 0.409264981980930309113375642482 \ldots \quad \text{and} \\
P &= -\log z_0 = 0.893392455017504971692687831819 \ldots
\end{align*}
\]

In particular, we generate a sequence \( p_m = -\log z_m, m \geq 2 \), converging to \( P(-\sin(2\pi \cdot)) \), where \( z_m \) is a zero of the polynomial

\[
d_0^{(m)}(z) = 1 + \sum_{n=1}^{m} a_n z^n
\]

given by truncating the expansion for \( d_0(z) \). These approximations are illustrated in Table 1 (a), where we can easily see the super exponential convergence coming from Lemma 4.2 (3). The implied level of accuracy is already achieved when \( m = 9 \).

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Table 1: (a) Approximations \( p_m \) to \( P(-\sin(\cdot)) \); and (b) Approximations \( p_m \) to \( P(-\cos(\cdot)) \)
Example 4.5 (Cosine function). If \( f(x) = \cos(2\pi x) \) then we can estimate
\[
\begin{align*}
\text{Example 4.5} & \quad (\text{Cosine function}) \\
\text{We again generate a sequence } p_m = -\log z_m, m \geq 2, \text{ converging to } P(\cos(2\pi \cdot)) \\
\text{We observe that when } t = 1 \text{ this reduces to the previous function, i.e., } d_0(z) = d_1(z, 1).
\end{align*}
\]

4.2 Determinant of two variables

Once we replace \( f \) by \( \tilde{f} := f + P(-f) = f - \log z_0 \) we can consider a second function depending on two variables. This is the function used in giving an expression for the Hausdorff dimension.

Definition 4.6. We formally define a second complex function (for the doubling map)

\[
d_1(z, t) = \exp \left( -\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{m=0}^{2^n-1} \sum_{k=0}^{2^n-1} \exp \left( -t \sum_{m=0}^{n-1} \tilde{f} \left( \frac{2^m k}{2^n - 1} \right) \right) \right)
\]

where \( z \in \mathbb{C} \) and \( t \in \mathbb{R} \).

This converges provided \( t \) is sufficiently large and \( |z| \) is sufficiently small. We observe that when \( t = 1 \) this reduces to the previous function, i.e., \( d_0(z) = d_1(z, 1) \).

Remark 4.7. In the particular case that \( T \) is the doubling map then one can also write \( z = 2^{-s} \) and then we will consider instead the complex function \( d_2(z, t) \) which satisfies \( d_2(s, t) = d_1(2^{-s}, t) \), as in (2.1).

In particular, we have the following properties for \( d_1(z, t) \) which are analogous to those for \( d_0(z) \).

Lemma 4.8. We have the following properties:

1. The function \( z \) converges to a non-zero analytic function for \( |z| < e^{-P(-t\tilde{f})} \);
2. The value \( e^{-P(-t\tilde{f})} \) is a simple zero for \( d_1(z, t) \);
3. For any \( \epsilon > 0 \), there exists \( C > 0 \) such that we can expand

\[
d_1(z, t) = 1 + \sum_{n=1}^{\infty} a_n(t) z^n
\]

where \( |a_n(t)| \leq C \left( \frac{1}{2} + \epsilon \right)^n \), for \( n \geq 1 \) and \( a_n(t) \) depends on the values

\[
\bigcup_{m=1}^{\infty} \left\{ \tilde{f} \left( \frac{2^k}{2^m - 1} \right) : 0 \leq k \leq m - 1 \right\}.
\]
The function \( d_1(z, t) \) has an analytic extension to \( \mathbb{C} \) as an entire function.

**Proof.** These results too can be deduced from Ruelle’s original article [8], as in the proof of Lemma 4.2. The only difference is that we need to replace the operator \( \mathcal{L} \) by the operators \( \mathcal{L}_t \) defined by \( \mathcal{L}_t w(z) = e^{-t f(T_0 z)} w(T_0 z) + e^{-t f(T_1 z)} w(T_1 z) \). This is again a nuclear operator on \( \mathcal{L} \) and we can identify \( d_1(z, t) = \det(1 - z \mathcal{L}_t) \). The estimates are very similar to those in Lemma 4.2. \( \square \)

## 5 The first algorithm

We can now recast for formulae for the dimension in Lemma 3.12 using Lemma 4.2. This leads to two different approaches to the dimension. In this section we consider the first algorithm, where given a parameter \( t \) we can associate \( \alpha = \alpha(t) \) and \( \mathcal{F}(\bar{f})(t) \).

### 5.1 The algorithm

Given \( t \) we can solve implicitly for \( d_1(z(t), t) = 0 \). In particular, we can then differentiate this identity in \( t \) and write

\[
\frac{\partial d_1}{\partial z}(z(t), t) \frac{\partial z}{\partial t}(t) + \frac{\partial d_1}{\partial t}(z(t), t) = 0. \tag{5.1}
\]

Since by part (2) of Lemma 4.8 we have that \( z(t) = e^{-P(t)} \) we can differentiate in \( t \) and write

\[
\frac{\partial z}{\partial t}(t) = -z(t) P'(t) > 0. \tag{5.2}
\]

Comparing (5.1) and (5.2) we can now write

\[
P'(t) = \frac{1}{z(t)} \frac{\partial d_1}{\partial t}(z(t), t) < 0.
\]

In particular, can associate to \( t \) the value \( \alpha = \alpha(t) \) given by \( \alpha := -P'(t) \). We can then use Lemma 8.4 to write

\[
\mathcal{F}(\bar{f})(\alpha) = \frac{P(t) + t \alpha}{\log 2} = \frac{-\log z(t) + t \alpha}{\log 2}.
\]

We can now consider different choices of \( t \). For each value of \( t \) in a suitable range we can associate the corresponding value \( \alpha = \alpha(t) \) in the multi fractal analysis. We can similarly associate to \( t \) the value \( \mathcal{F} = \mathcal{F}(\bar{f})(\alpha(t)) \). In particular, given \( t \) we can consider the truncations

\[
d_1^{(m)}(z, t) = 1 + \sum_{n=1}^{m} a_n(t) z^n
\]

we can approximate the zero \( z(t) \) by the zero \( z^{(m)}(t) \) for \( d_1^{(m)}(z, t) \). We can then generate two approximating sequences:

1. A sequence

\[
\alpha_m := -\frac{1}{z^{(m)}(t)} \frac{\partial d_1^{(m)}}{\partial z}(z^{(m)}(t), t), \quad m \geq 2
\]

converging to \( \alpha \); and
2. A sequence
\[ F_m := -\log \frac{z^m(t) + t\alpha^m}{\log 2}, \quad m \geq 2 \]
converging to \( F^f \).

5.2 Examples

We can consider our two function \( f(x) = \cos(2\pi x) \) and \( f(x) = \sin(2\pi x) \) and examples for different \( t \). In particular, we will consider the cases \( t = 0.1, 0.5, 1.0 \) and \( 2.0 \) for \( m = 2, \cdots, 9 \). We begin with the function \( f(x) = \cos(2\pi x) \).

**Example 5.1** (\( f(x) = \cos(2\pi x) \) and \( t = 0.1 \)). When we choose \( t = 0.1 \) we can estimate \( \alpha = 0.0464374285064925289788106966 \cdots \) then \( F(\alpha) = 0.99673365723505025215451300157 \cdots \). This approximation can be seen in Table 2.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( F_m )</th>
<th>( \alpha_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.99921878521968777230398953543</td>
<td>-0.008312648150203493460440830</td>
</tr>
<tr>
<td>3</td>
<td>0.9967194485227907629706333834</td>
<td>-0.04657526394662772400556272</td>
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<td>0.996733667386981769809040088148</td>
<td>-0.0464373878923167366965004293</td>
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<td>0.99673365723529806312051914574</td>
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<td>0.996733657235051140332871000282</td>
<td>-0.0464374285064514507268995658</td>
</tr>
<tr>
<td>7</td>
<td>0.99673365723505025215451300157</td>
<td>-0.0464374285064936392018353217</td>
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<td>8</td>
<td>0.99673365723505025215451300157</td>
<td>-0.0464374285064925289788106966</td>
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<tr>
<td>9</td>
<td>0.99673365723505025215451300157</td>
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</tr>
</tbody>
</table>

Table 2: Approximations when \( t = 0.1 \)

**Example 5.2** (\( f(x) = \cos(2\pi x) \) and \( t = 1.0 \)). When we choose \( t = 1.0 \) we can estimate \( \alpha = -0.27261545269624043452694195366 \cdots \) then \( F(\alpha) = 0.843854384076045960227929754183 \cdots \). This approximation can be seen in Table 3.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( F_m )</th>
<th>( \alpha_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.784291089580260947222711820359</td>
<td>-0.2716355325028655417083182328</td>
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<td>0.820382435542054921917554111133</td>
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<td>4</td>
<td>0.845225458720281763724813117733</td>
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<tr>
<td>6</td>
<td>0.8438545587855474805375107971</td>
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<td>0.84385438379180432995045463875</td>
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<td>0.843854384076114016899339276279</td>
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<td>11</td>
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<td>0.843854384076045960227929754183</td>
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</tr>
</tbody>
</table>

Table 3: Approximations when \( t = 1.0 \)
We next turn to the function $f(x) = \sin(2\pi x)$.

**Example 5.3** ($f(x) = \sin(2\pi x)$ and $t = 0.1$). When we choose $t = 0.1$ we can estimate

$\alpha = -0.04969042542904811288195787711 \cdots$ and $\mathcal{F}(\alpha) = 0.996426724035268218671035356238 \cdots$.

This approximation can be seen in Table 4.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\mathcal{F}_m$</th>
<th>$\alpha_m$</th>
</tr>
</thead>
<tbody>
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<td>0.999999999999999999776977537480</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
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<td>0.996426724035268440715640281269</td>
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<td>0.996426724035268218671035356238</td>
<td>-0.04969042542904811288195787711</td>
</tr>
</tbody>
</table>

Table 4: Approximations when $t = 0.1$

**Example 5.4** ($f(x) = \sin(2\pi x)$ and $t = 1.0$). When we choose $t = 1.0$ we can estimate

$\alpha = -0.33451988790170 \cdots$ then $\mathcal{F}(\alpha) = 0.806282680953646 \cdots$.

This approximation can be seen in Table 5.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\mathcal{F}_m$</th>
<th>$\alpha_m$</th>
</tr>
</thead>
<tbody>
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<td>0.0</td>
</tr>
<tr>
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<td>0.851206720640423020185494351608</td>
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<td>4</td>
<td>0.80856384704673581076406208012</td>
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<td>0.806280457685890183938681730069</td>
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<td>0.806283011667093041374698714208</td>
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</tbody>
</table>

Table 5: Approximations when $t = 1.0$

6 The second algorithm

In the second algorithm we assume that we are given suitable $\alpha$ and then we want to solve for $\mathcal{F}(\alpha)$. 

---

6 THE SECOND ALGORITHM
6.1 The algorithm

Given $\alpha$ we can define

$$P(z, t) = d_1(z, t) \quad \text{and}$$

$$Q(z, t) = \frac{\partial d_1}{\partial z}(z, t)z\alpha + \frac{\partial d_1}{\partial t}(z, t)$$

The following simple result gives the Hausdorff Dimension of the level set.

**Lemma 6.1.** Given the solution $(z_\alpha, t_\alpha) \in \mathbb{R}^2$ for

$$P(z_\alpha, t_\alpha) = Q(z_\alpha, t_\alpha) = 0$$

we can write

$$\mathcal{F}(f)(\alpha) = \frac{-\log z_\alpha + t_\alpha \alpha}{\log 2}.$$ 

**Proof.** By part (2) of Lemma 4.2, the identity $P(z_\alpha, t_\alpha) = 0 = d_1(z_\alpha, t_\alpha)$ ensures that $z_\alpha = e^{-P(t_\alpha)}$. In particular, by taking logarithms we can write

$$P(t_\alpha) = -\log z_\alpha. \quad (6.1)$$

Locally we have an implicit solution $z(t)$ for $d_1(z(t), t) = 0$ with $z(t_\alpha) = z_\alpha$. In particular, given $t$ we have by part (2) of Lemma 4.8 that $z(t) = e^{-P(t)}$. Differentiating this identity with respect to $t$ at $t_\alpha$ gives:

$$\frac{\partial z}{\partial t}(t_\alpha) = -z_\alpha P'(t_\alpha) > 0. \quad (6.2)$$

Differentiating the identity $d_1(z(t), t) = 0$ with respect to $t$ at $t_\alpha$ we can write:

$$\frac{\partial d_1}{\partial z}(z_\alpha, t_\alpha) \frac{\partial z}{\partial t}(t_\alpha) + \frac{\partial d_1}{\partial t}(z_\alpha, t_\alpha) = 0. \quad (6.3)$$

Comparing (6.2) and (6.3) gives:

$$\frac{\partial d_1}{\partial z}(z_\alpha, t_\alpha)z_\alpha + \frac{\partial d_1}{\partial t}(z_\alpha, t_\alpha) = Q(z_\alpha, t_\alpha) = 0, \quad (6.4)$$
providing $\alpha = -P'(t_0)$. Finally, by Lemma 3.12 and (6.1) we have that
\[
\mathcal{F}(f)(\alpha) = \frac{P(t_0) + t_0\alpha}{\log 2} = \frac{-\log z_0 + t_0\alpha}{\log 2}
\]
as required.

We can consider the truncations of the Taylor series in $z$ for $P(z, t)$ and $Q(z, t)$.

**Definition 6.2.** Given $\alpha$ and $m \geq 1$ we can define
\[
P^{(m)}(z, t) = \sum_{n=1}^{m} a_n(t)z^n \quad \text{and} \quad \frac{\partial}{\partial t}^{(m)}(z, t) = \sum_{n=1}^{m} b_n(t)z^n
\]
where $b_n(t) = a_n(t)\alpha + \frac{\partial}{\partial t}^{(m)}(t)$ (corresponding to the terms of the Taylor series expansion for (6.4)).

We can then approximate the solution $(z_0, t_0)$ for $P(z, t) = Q(z, t) = 0$ by a solution
\[
(z_0^{(m)}, t_0^{(m)}) \quad \text{for} \quad P^{(m)}(z, t) = Q^{(m)}(z, t) = 0.
\]
We can then generate an approximating sequence
\[
\mathcal{F}_m := \frac{-\log z^{(m)} + t^{(m)}\alpha}{\log 2}, \quad m \geq 2,
\]
converging to $\mathcal{F}(f)(\alpha)$.

### 6.2 Examples

We will again concentrate on the two basic functions $f(x) = \cos(2\pi x)$ and $f(x) = \sin(2\pi x)$

We will consider the cases $\alpha = 0.1$ and $0.25$ for $m = 2, \cdots, 12$.

We begin with the function $f(x) = \cos(2\pi x)$.

**Example 6.3** ($f(x) = \cos(2\pi x)$ and $\alpha = 0.1$). If we let $\alpha = 0.1$ then we see that $\mathcal{F}(0.1) = 0.986826533447210 \cdots$. The approximations can be seen in Table 6(a).

**Example 6.4** ($f(x) = \cos(2\pi x)$ and $\alpha = 0.25$). If we let $\alpha = 0.25$ then we see that $\mathcal{F}(0.25) = 0.92613854650709 \cdots$. The approximations can be seen in Table 6 (b).

**Example 6.5** ($f(x) = \cos(2\pi x)$ and $\alpha = 0.5$). If we let $\alpha = 0.5$ then we see that $\mathcal{F}(0.5) = 0.73988277232849 \cdots$. The approximations can be seen in Table 6 (c).
6.2 Examples

We now consider the function \( f(x) = \sin(2\pi x) \).

**Example 6.6** \((f(x) = \sin(2\pi x)\) and \(\alpha = 0.1\). If we let \(\alpha = 0.1\) then we see that \(\mathcal{F}(0.1) = 0.98538810432958044919196928737\). The approximations can be seen in Table 7 (a).

**Example 6.7** \((f(x) = \sin(2\pi x)\) and \(\alpha = 0.25\). If we let \(\alpha = 0.25\) then we see that \(\mathcal{F}(0.25) = 0.90143475104319124889630543175\). The approximations can be seen in Table 7 (b).

**Example 6.8** \((f(x) = \sin(2\pi x)\) and \(\alpha = 0.45\). If we let \(\alpha = 0.45\) then we see that \(\mathcal{F}(0.25) = 0.5128089471004681377573856\). The approximations can be seen in Table 7 (c).

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\mathcal{F}_m)</th>
<th>(m)</th>
<th>(\mathcal{F}_m)</th>
<th>(m)</th>
<th>(\mathcal{F}_m)</th>
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<td>0.7556304373136142018641163259</td>
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<td>12</td>
<td>0.739899137089877339005887469</td>
</tr>
</tbody>
</table>

Table 6: Approximations when: (a) \(\alpha = 0.1\); (b) \(\alpha = 0.25\); and \(\alpha = 0.5\)

Table 7: Approximations when: (a) \(\alpha = 0.1\); (b) \(\alpha = 0.25\); and \(\alpha = 0.45\)

![Figure 4: Plots of \(\mathcal{F}(\alpha)\) as a function of \(\alpha\) for: (a) \(\sin(2\pi x)\); and (b) \(\cos(2\pi x)\)](image_url)
7 Speed of approximation

We can consider the approximation using some simple estimates. We can approximate
\( d_1(z,t) \) by the complex function
\[
d_1^{(m)}(z,t) = 1 + \sum_{n=1}^{m} a_n(t)z^n,
\]
for \( m \geq 2 \), and observe that by the bounds in Lemma 4.2 (3), we have
\[
d_1(z,t) - d_1^{(N)}(z,t) = O\left(\left(\frac{1}{2} + \epsilon\right)^m\right)
\]
on any compact region. Moreover, by a simple application of Cauchy’s theorem we can bound the derivatives
\[
\frac{\partial d_1(z,t)}{\partial z} - \frac{\partial d_1^{(m)}(z,t)}{\partial z} = O\left(\left(\frac{1}{2} + \epsilon\right)^m\right)
\]and
\[
\frac{\partial d_1(z,t)}{\partial t} - \frac{\partial d_1^{(m)}(z,t)}{\partial t} = O\left(\left(\frac{1}{2} + \epsilon\right)^m\right).
\]

We now have the following result.

**Proposition 7.1.** Given solutions \((z_\alpha^{(m)}, t^{(m)})\) for
\[
Q_\alpha^{(m)}(z,t) = P^{(m)}(z,t) = 0,
\]
we can then write
\[
F(\alpha) = \frac{\log z_\alpha^{(m)}}{\log 2} + t_\alpha^{(m)} + O\left(\left(\frac{1}{2} + \epsilon\right)^m\right)
\]

**Remark 7.2.** Although Proposition 7.1 gives that the error term tends to zero at the same super exponential rate, the inferred constant in the Landau \( O \) term may vary. From the proof of Lemma 4.2 we can get bounds of the form \(|a_n| \leq C_n(\frac{1}{2} + \frac{1}{4r})n^2/2\) and \(C_n \leq n^{n/2} \left(\sup_{|z| < r} |e^{-tf(z)}|\right)^n\), for any \( r > \frac{1}{2} \). Thus, for \(|t|\) larger the estimate on \(C\), and consequently for \(a_n\), may be worse. In particular, the approximation to \(d_1(z,t)\) by truncating to a given number of terms may give a worse estimate.

In the explicit case that \(f(z) = \cos(2\pi z)\) (or \(f(z) = \cos(2\pi z)\)) we can bound \(|\hat{f}(z)| \leq \frac{1}{2}(e^{2\pi r} + e^{-2\pi r})\). Thus we can bound the \(n\)th term in \(d_1(e^{-P(-t\hat{f})}, t)\) by
\[
n^{n/2} \left(e^{-P(-t\hat{f})}\exp(e^{2\pi r} + e^{-2\pi r})\right)^n \left(\frac{1}{2} + \frac{1}{4r}\right)^{n^2/2} \tag{7.1}
\]
This explains the observed variation in the accuracy in the numerical approximations in the examples and can be quantified with a little extra work (to optimise the choice of \(r > 0\) so as to minimise the contributions from the bound (7.1) for \(n \geq m\), for a given \(m\)).
8 Generalizations

In this final section, we will indicate how these results can be generalised and applied to other problems.

8.1 The case of general expanding interval maps

Thus far, we have concentrated on the simpler case that the underlying transformation is the doubling map. Let us now outline the modifications for the case of a general $C^\omega$ expanding Markov map $T$.

**Definition 8.1.** We can consider the function $Q : \mathbb{R} \to \mathbb{R}$ defined implicitly by

$$P(-tf - Q(t) \log |T'|) = 0 \left(= P(-tf - Q(t) \log |T'|) + tP(-f)\right)$$

**Remark 8.2.** It is well known that we always have $P(-\log |T'|) = 0$.

We can consider the unique equilibrium measure $\mu_t$ for the potential $-tf - Q(t) \log |T'|$.

**Lemma 8.3.** The function $Q(t)$ is analytic with $Q(0) = 1$ and $Q(1) = 0$. Moreover,

1. $Q'(t) = -\int \frac{f d\mu_t}{\log |T'|d\mu_t}$;
2. $-Q'(t)$ obtains all the values in $(\alpha_{\min}, \alpha_{\max})$; and
3. $Q''(t) > 0$

**Proof.** These properties are easily checked. In particular, Part (1) follows from the implicit function theorem.

The connection between the function $P_2(t)$ and the Hausdorff Dimension $F(\alpha)$ of the level set is given by the following:

**Theorem 8.4.** Given $\alpha$ let us choose the unique $t = t_\alpha$ such that $Q'(t_\alpha) = -\alpha$. We then have that

$$F(f)(\alpha) = Q(t_\alpha) + t_\alpha \alpha$$

**Proof.** This is well explained in the article of Pesin and Weiss [5] and the book of Pesin [6]. We can write

$$Q'(t_\alpha) = -\frac{\partial P(-t_\alpha f - u \log |T'|)}{\partial u}|_{u = Q(t_\alpha)} = -\int \frac{f d\mu_t}{\log |T'|d\mu_t} = -\alpha \quad (8.1)$$

for $t = t_\alpha$. The essential idea is that for any $t$ we have from the variational principle that

$$P(-tf - Q(t) \log |T'|) = 0 = h(\mu_t) + \int (-tf - Q(t) \log |T'|) d\mu_t \quad (8.2)$$

and that the dimension of the measure $\mu_t$ satisfies

$$\dim_H(\mu_t) := \frac{h(\mu_t)}{\int \log |T'| d\mu_t} = \frac{\int (tf + Q(t) \log |T'|) d\mu_t}{\int \log |T'| d\mu_t} = Q(t) + t \alpha$$
using (8.1) and (8.2) and Part (1) of Lemma 8.3.

We can define a complex function by:

\[ d_0(z) = \exp \left( -\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\exp\left( -\sum_{i=0}^{n-1} f(T^i x) \right)}{1 - (T^n)'(x)^{-1}} \right) \]

which converges for \(|z| \) sufficiently small. In fact, when \( T \) and \( f \) are \( C^\omega \) it follows by work of Ruelle (after Grothendieck) that \( d_0(z) \) is entire in \( \mathbb{C}^2 \). There is a zero at \( z_0 = e^{-P(-f)} \) and we can replace \( f \) by \( \bar{f} = f + P(-f) = f - \log z_0 \).

As explained in the introduction, we can define a complex function by:

\[ d_2(s, t) = \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{T^n x = x} \exp\left( -t \sum_{i=0}^{n-1} f(T^i x) \right) \left| (T^n)'(x)^{-1} \right|^{-t} \right) \]

which converges for \( s, t \) sufficiently large. In fact, when \( T \) and \( f \) are \( C^\omega \) it follows by work of Ruelle (after Grothendieck) that \( d_2(s, t) \) is entire in \( \mathbb{C}^2 \).

Given \( \alpha \) there is \( t \) such that there is zero at \((s(t), t)\), i.e., \( d_2(s(t), t) = 0 \). Using the Implicit Function Theorem we can write

\[ \frac{\partial d_2(s_\alpha, t)}{\partial t} \bigg|_{t=t_\alpha} + \frac{\partial d_2(s, t_\alpha)}{\partial t} \bigg|_{s=s_\alpha} \frac{\partial s}{\partial t} = 0 \]

and we want to solve \((s_\alpha, t_\alpha)\) such that

\[ \frac{\partial d_2(s_\alpha, t)}{\partial t} \bigg|_{t=t_\alpha} - \alpha \frac{\partial d_2(s, t_\alpha)}{\partial s} \bigg|_{s=s_\alpha} = 0 \]

In particular, we can write that

\[ \mathcal{F}(\bar{f})(\alpha) = s_\alpha + \alpha t_\alpha. \]

**Example 8.5.** We can consider the simple example \( T : [0, 1] \to [0, 1] \) given by

\[ T(x) = 2x + \frac{1}{4\pi} \sin(2\pi x) \]

and the function \( f(x) = \sin(2\pi x) \).

### 8.2 Pointwise dimension of measures

There is a natural generalisation of this to the case that we look at measures and pairs of functions, following the full version of the analysis of Pesin and Weiss in [5], [6].

The previous ideas are based on the different possible limits of the Birkhoff averages

\[ \frac{1}{N} \sum_{n=1}^{\infty} \bar{f}(T^n x) \] \( \to +\infty \). The distinction now is that we fix a a reference measure \( \mu \) and consider instead the limits of

\[ \frac{\log \mu(B(x, r))}{\log r} \]

as \( r \to +\infty \). Replacing the Birkhoff ergodic theorem we want the following limiting result associated to measures.
**Definition 8.6.** We say that \(\mu\) has pointwise dimension \(\alpha\) if for almost all \((\mu) x \in X\) we have that

\[
\alpha = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha
\]

Let us assume that \(\mu\) is a Gibbs measure for a Hölder continuous function \(\psi\). We can assume without loss of generality that \(P(-\psi) = 0\).

**Proposition 8.7.** If \(\mu\) is a Gibbs measure for a Hölder continuous function \(\psi\) then there exists \(\alpha = \alpha_\mu\) such that for almost all \((\mu) x \in X\) we have that

\[
d_\mu(x) = \alpha
\]

Assume that we have a real value \(\alpha\) and that we want to choose a value \(q\) such that

\[
\alpha = \frac{\int \psi d\mu_q}{\int \log |T'| d\mu_q}
\]

where \(t(q)\) satisfies \(P(-t(q) \log |T'| - q\psi) = 0\) and \(\mu_q\) is the unique equilibrium state for \(-t(q) \log |T'| - q\psi\). In particular, if \(\Lambda_\alpha = \{x : d_\mu(z) = \alpha\}\) then we see that for a.e.\((\mu_q), x \in \Lambda_\alpha\) we have that

\[
d_\mu(x) = \lim_{n \to +\infty} \frac{\log \mu(I_n(x))}{\log |(T^n)'(x)|} = \frac{\int \psi d\mu_q}{\int \log |T'| d\mu_q} = \alpha
\]

where \(I_n(x)\) is the dyadic interval containing \(x\), by using the Birkhoff ergodic theorem for \(\mu_q\).

In particular, we see that \(\mu_q(\Lambda_\alpha) = 1\) and then we deduce that \(\dim_H(\Lambda_\alpha) \geq \dim_H(\mu_q)\). In fact, a simple estimate shows that there is an equality: \(\dim_H(\Lambda_\alpha) = \dim_H(\mu_q)\). Moreover, we know that:

**Lemma 8.8.** \(\dim_H(\mu_q) = t(q) + q\alpha\).

**Proof.** We know that

\[
P(-t(q) \log |T'| - q\psi) = 0 = h(\mu_q) + \int (-t(q) \log |T'| - q\psi) d\mu_q
\]

which allows us to rewrite

\[
\dim_H(\mu_q) = \frac{h(\mu_q)}{\int \log |T'| d\mu_q} = t(q) + q\alpha.
\]

\(\square\)

**Example 8.9.** A trivial example would be where we took the measure \(\mu\) to be the \((p, 1 - p)\)-Bernoulli measure. In this case we let \(T(x) = 2x (\mod 1)\) and see that

\[
\phi(x) = \begin{cases} 
\log p & \text{if } 0 \leq x \leq \frac{1}{2} \\
\log(1 - p) & \text{if } \frac{1}{2} \leq x \leq 1 
\end{cases}
\]

---

8.3 Other examples of conformal repellers

Finally, we briefly mention other familiar examples of repellers to which our results apply.
8.3.1 Hyperbolic Julia sets

We let $T : \hat{C} \to \hat{C}$ be a rational map. We define the Julia set to be $J$ to be the closure of the the periodic points. We say that $T : J \to J$ is a hyperbolic rational map if there exists $c > 1$ such that $|T'(z)| > c$, for all $z \in J$ [9]. We can apply the algorithm(s) to $T : J \to J$ and any real analytic map $f : J \to \mathbb{R}$.

8.3.2 Schottky Groups

Let $C_1, \cdots, C_k, C_{k+1}, \cdots, C_{2k}$ be circles in $\mathbb{C}$ with disjoint interiors $D_1, \cdots, D_{2k}$. We can let $\gamma_i : \hat{C} \to \hat{C}$ be a linear fractional transformation which maps $C_i$ to $C_{k+i}$, for $i = 1, \cdots, k$. A Schottky group $\Gamma$ is generated by $\gamma_1^\pm, \cdots, \gamma_k^\pm$ and we denote by $\Lambda$ the associated Limit set (i.e., the set of accumulation points for the orbits $\gamma z_0$, $g \in \Gamma$, for any fixed reference point $z_0$).

We can consider the transformation $T : \Lambda \to \Lambda$ defined by

$$ T(z) = \begin{cases} 
\gamma_i(z) & \text{if } z \in D_i \\
\gamma_i^{-1}(z) & \text{if } z \in D_{k+i}
\end{cases} $$

for $i = 1, \cdots, k$. There is also associated to this a natural conformal measure $\mu$ such that $\mu \circ T = |T'|^\delta \mu$, where $\delta$ is the Hausdorff Dimension of $\Lambda$. It is possible to apply the algorithm(s) for $T : \Lambda \to \Lambda$ and any real analytic function $f : \Lambda \to \mathbb{R}$. It is also natural to apply the results on point wise dimension multifractal spectrum to the measure $\mu$.

Remark 8.10. In the case of non-conformal expanding maps it is possible to recover many of these results by replacing the Hausdorff dimension of the sets $\Lambda^f$ by their entropy (which is defined in terms of covers by dynamical Bowen-balls, rather than the standard definition). We have also considered only discrete transformations. However, for real analytic (semi-)flows many of the results can be modified by using Markov sections.

References


