Sub-exponential mixing rates for open systems with interacting particles

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Limit theorems for dynamical systems
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Steady states for mechanical particle systems
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- Equilibrium: explicit formulas for steady states
Steady states for mechanical particle systems

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- Nonequilibrium: pure existence of steady states is a nontrivial question due to non-compactness of the phase space
Steady states for mechanical particle systems

- Equilibrium: explicit formulas for steady states
- Nonequilibrium: pure existence of steady states is a nontrivial question due to non-compactness of the phase space
- The system may freeze or heat up
  - Freezing example: 1D model [Eckmann and Young 2011]
  - Problem: slow particles
  - Among many particles only ONE drives the system for extended periods of time
Sub-exponential mixing for mechanical particle systems

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- Similar effect for natural discretizations of systems with more than one particle:
  - slow particle(s) do not collide with anything for a while, while other particle(s) experience collisions and drive the system
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- Gibbs heat bath distributions: may emit arbitrarily slow particles
- Slow particles may prevent exponential mixing
- Similar effect for natural discretizations of systems with more than one particle:
  - slow particle(s) do not collide with anything for a while, while other particle(s) experience collisions and drive the system
- Present a system with particles interacting through ‘energy tanks’; rigorous results: existence, uniqueness, absolute continuity (w.r.t. Lebesgue measure), and sub-exponential mixing of (non-equilibrium) steady state.
\( \Gamma \) - a disk of radius \( R + d \).
Domain

- $\Gamma$ - a disk of radius $R + d$.
- $D$ - a disk of radius $R$ pinned at the center of $\Gamma$, rotating freely.
  - angular velocity $\omega$
  - marked position $\zeta$
Energy exchange with the disk

- $\Gamma$ - a disk of radius $R + d$.
- $D$ - a disk of radius $R$ pinned at the center of $\Gamma$, rotating freely.
  - angular velocity $\omega$
  - marked position $\zeta$
- particles exchange energy with the disk $D$
  - $v_\perp' = -v_\perp$
  - $v_t' = \omega$
  - $\omega' = v_t$
Heat Baths

- Split $\Gamma$ in two parts by vertical walls.
  - specular reflections off the walls
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- Left and Right heat baths $\partial \Gamma_L$ and $\partial \Gamma_R$
  - inverse temperatures $\beta_L = \frac{1}{T_L}$ and $\beta_R = \frac{1}{T_R}$
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  - upon collision of a particle with $\partial \Gamma_L$, new particle is emitted at the position of collision with speed $s$ and angle $\varphi$ distributed

$$\frac{2\beta_L^{3/2}}{\pi} s^2 e^{-\beta_L s^2} \cos(\varphi) ds d\varphi$$
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- $\Rightarrow$ particle’s velocity distributed as $\frac{\beta_L}{\pi} e^{-\beta_L |v|^2} dv$ in $\Gamma$
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- same for $\partial\Gamma_R$
The Markov process

The Phase Space

\[ \Omega = \left[ \{(x_i, v_i) : 1 \leq i \leq k\} \times \{(\zeta, \omega)\} \right]/\sim \]
The Markov process

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  \[ \Omega = \left\{ (x_i, v_i) : 1 \leq i \leq k \right\} \times \{ (\zeta, \omega) \} / \sim \]

- Markov Process \( \Phi_t \)
  - deterministic between collisions with heat baths
  - \((s, \varphi)\) at collision replaced by randomly drawn \((s, \varphi)\) from
  \[
  \frac{2\beta^{3/2}}{\pi} s^2 e^{-\beta L/R s^2} \cos(\varphi) ds d\varphi
  \]
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\[
\frac{2\beta^{3/2}_{L/R}}{\pi} s^2 \, e^{-\beta_{L/R}s^2} \cos(\varphi) \, dsd\varphi
\]

- Denote the transition probabilities for \( \Phi_t \) by \( \mathcal{P}^t \), i.e.

\[
\mathcal{P}^t((z, A) = \mathbb{P}(\Phi_t \in A | \Phi_0 = z)
\]
Main Theorem

Theorem

$\exists ! \text{ invariant measure } \mu \text{ for the continuous Markov process } \Phi_t.$
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\[ \mu \text{ is mixing and almost all initial probability distributions converge to } \mu \text{ (exceptions: distributions giving positive measures to the set of states with stopped particles, etc.)} \]
Main Theorem

Theorem

- $\exists !$ invariant measure $\mu$ for the continuous Markov process $\Phi_t$.
- $\mu$ is mixing and almost all initial probability distributions converge to $\mu$ (exceptions: distributions giving positive measures to the set of states with stopped particles, etc.)
- $\mu$ is not exponential mixing and, for a large class of initial distributions, the convergence rate to $\mu$ is at best polynomial.
Methods

- Most commonly used methods: spectral gap and Harris ergodic theorem
  - both ensure exponential convergence of initial distributions to the initial state
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- Uniqueness, absolute continuity, and convergence of initial distributions to the steady state results can be acquired incrementally.

- Mechanical systems with stochastic boundaries by S. Goldstein, J.L. Lebowitz, and E. Presutti gives a great overview of ideas and difficulties associated with this approach. [Colloquia Mathematica Societatis János Bolyai 27 (1979). Random Fields, Esztergom (Hungary), 403-419 (1981)]
Existence: Hitting times of ‘good’ sets

- A closed set $C \in \mathcal{B}(\Omega)$ is called ‘good’ if $\exists \eta > 0, \ T > 0$ s.t. $\mathcal{P}^T(z, \cdot) = \mathcal{P}^T_{\star} \delta_z \geq \eta m_C$ (minorization condition)
  
  ▷ $m_C$ is a probability measure on $C$
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**Theorem (Meyn and Tweedie 1993)**

$C$ ‘good’ \+ $\mathbb{P}_z \{ \tau_C < \infty \} = 1 \ \forall z \in \Omega \implies$

$\exists$ an invariant measure for $\Phi_t$ (not necessarily finite).

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Theorem (Meyn and Tweedie 1993)

- $C$ ‘good’ $\Rightarrow \mathbb{P}_z\{\tau_C < \infty\} = 1 \ \forall z \in \Omega \implies \exists$ an invariant measure for $\Phi_t$ (not necessarily finite).

- Given $\delta \geq 0$ and $C \in \mathcal{B}(\Omega)$ define
  $$\tau_C(\delta) = \text{first hitting time of } C \text{ after time } \delta.$$
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Theorem (Meyn and Tweedie 1993)

The invariant measure for $\Phi_t$ is finite if $\exists$ ‘good’ $C$ and $\delta > 0$ s.t.

$$\sup_{z \in C} \mathbb{E}_z[\tau_C(\delta)] < \infty$$
Idea of Proof: similarity to a regenerative process

- Break sample paths from $z$ to $C$ into similarly behaving pieces of bounded average length
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- Break sample paths from $z$ to $C$ into similarly behaving pieces of bounded average length, i.e.,
- Introduce a stopping time $\tau$ such that most of the initial data is forgotten by time $\tau$ due to randomness of the heat baths.
  - the system ‘renews enough’ by time $\tau$ such that $\Phi_\tau$ hits $C$ with geometric rates
    (if $\Phi_\tau \sim \nu$ iid, then $\mathbb{P}_z[\Phi_\tau \in C] = \nu(C)$)
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- Control on certain expected values of \( \tau \) ⇒
  - \( \mathbb{P}_z\{\tau_C < \infty\} = 1 \ \forall z \in \Omega \) and
  - \( \sup_{z \in C} \mathbb{E}_z[\tau_C(\delta)] < \infty \)
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    (if $\Phi_\tau \sim \nu$ iid, then $\mathbb{P}_z[\Phi_\tau \in C] = \nu(C)$)
- Control on certain expected values of $\tau$ \Rightarrow
  - $\mathbb{P}_z\{\tau_C < \infty\} = 1 \ \forall z \in \Omega$ and
  - $\sup_{z \in C} \mathbb{E}_z[\tau_C(\delta)] < \infty$
- Balance between
  - fast enough renewal time $\tau$ and
  - thorough enough renewals
Idea of Proof: stopping time

Let $\tau$ be the minimum time at which all particles and the disk randomize.
Idea of Proof: stopping time

- Let $\tau$ be the minimum time at which all particles and the disk randomize, i.e.,
  $\tau = \min\{ t > 0 :$
  - all particles in $z$ have collided with $\partial \Gamma$ (all initial velocities are forgotten)
  - a particle originated from $\partial \Gamma$, hit the disk at some time $\tilde{t} > t_0(z)$, and collided with $\partial \Gamma$ again (disk’s angular velocity is forgotten) }
  - where $t_0(z)$ is a time at which all particles in $z$ heading for collision with the disk have collided with $\partial \Gamma$
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  \end{itemize}
  \* where $t_0(z)$ is a time at which all particles in $z$ heading for collision with the disk have collided with $\partial \Gamma$

- Let $\Phi_{\tau}$ be the discrete-time Markov chain obtained by stopping $\Phi_t$ at time $\tau$ and $P^\tau$ its transition probability kernel
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Proposition

For any initial distribution $\lambda$ and $\nu = P^\tau \ast \lambda$,

\[ \mathbb{E}_\nu[\tau] \leq D \]
The ‘good’ set $C$

Let $C$ be such that

- speeds of particles are bounded above and below
- $\omega$ is bounded above
- angles of incidence with disk are bounded away from $\pm \frac{\pi}{2}$
- $C$ is invariant between collisions with heat baths
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**Proposition**

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$0 < \gamma_{\text{min}} \leq \mathbb{P}[\Phi_\tau \in C] \leq \gamma_{\text{max}} < 1$ (variation depends on $\beta_L$ and $\beta_R$)
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### Proposition

- $\sup_{z \in C} \{\mathbb{E}_{z}[\tau(\delta)]\} \leq D'$
- $\mathbb{P}_z\{\tau(\delta) < \infty\} = 1$
Existence of unique invariant probability measure

Prop. $C$ is ‘good’

Prop. $\nu = \mathcal{P}^\tau \lambda \implies \mathbb{E}_\nu[\tau] \leq D$

Want: $\sup_{z \in C} \mathbb{E}_z[\tau_C(\delta)] < \infty$

Prop. $\gamma_{\min} \leq \mathbb{P}[\Phi \tau \in C] \leq \gamma_{\max}$

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Existence of unique invariant probability measure

- **Prop.**
  - $C$ is ‘good’

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  - $\sup_{z \in C} \{\mathbb{E}_z[\tau(\delta)]\} \leq D'$

- Let $\sigma_C$ be the hitting time of $C$ for $\Phi_{\tau}$: $\sigma_C = n \Rightarrow \Phi_{n\tau} \in C$
Existence of unique invariant probability measure

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- Let \( \sigma_C \) be the hitting time of \( C \) for \( \Phi \tau \): \( \sigma_C = n \Rightarrow \Phi_{n\tau} \in C \)
- For \( z \in C \)

\[
\mathbb{E}_z [\tau_C (\delta)] \leq \mathbb{P}(\sigma_C = 1) \mathbb{E}_z [\tau (\delta)] + \mathbb{P}(\sigma_C = 2) (\mathbb{E}_z [\tau (\delta)] + \mathbb{E}_{\mathbb{P}^\tau \delta z} [\tau])
\]

\[ + \mathbb{P}(\sigma_C = 3) (\mathbb{E}_z [\tau (\delta)] + \mathbb{E}_{\mathbb{P}^\tau \delta z} [\tau] + \mathbb{E}_{\mathbb{P}^2 \delta z} [\tau]) + \cdots \]
Existence of unique invariant probability measure

Prop. 

C is ‘good’

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Prop. 

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For \( z \in C \)

\[
\begin{align*}
\mathbb{E}_z[\tau_C(\delta)] & \leq \mathbb{P}(\sigma_C = 1)\mathbb{E}_z[\tau(\delta)] + \mathbb{P}(\sigma_C = 2)(\mathbb{E}_z[\tau(\delta)] + \mathbb{E}_{\mathcal{P}^\tau \delta z}[\tau]) \\
& \quad + \mathbb{P}(\sigma_C = 3)(\mathbb{E}_z[\tau(\delta)] + \mathbb{E}_{\mathcal{P}^\tau \delta z}[\tau] + \mathbb{E}_{\mathcal{P}^{2\tau} \delta z}[\tau]) + \cdots \\
& \leq \gamma_{\max} D' + (1 - \gamma_{\min}) \gamma_{\max}(D' + D) + (1 - \gamma_{\min})^2 \gamma_{\max}(D' + 2D) + \cdots
\end{align*}
\]
Existence of unique invariant probability measure

**Prop.**

\( C \) is ‘good’

**Prop.**

\( \nu = P_\tau \lambda \implies E_\nu [\tau] \leq D \)

**Want:**

\[ \sup_{z \in C} E_z [\tau_C (\delta)] < \infty \]

**Prop.**

\[ \gamma_{\min} \leq P[\Phi \tau \in C] \leq \gamma_{\max} \]

\[ \sup_{z \in C} \{ E_z [\tau (\delta)] \} \leq D' \]

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For \( z \in C \)

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E_z [\tau_C (\delta)] \leq P(\sigma_C = 1) E_z [\tau (\delta)] + P(\sigma_C = 2) (E_z [\tau (\delta)] + E_{P_\tau \delta z} [\tau]) \\
+ P(\sigma_C = 3) (E_z [\tau (\delta)] + E_{P_\tau \delta z} [\tau] + E_{P_2 \tau \delta z} [\tau]) + \cdots
\]

\[ \leq \gamma_{\max} D' + (1 - \gamma_{\min}) \gamma_{\max} (D' + D) + (1 - \gamma_{\min})^2 \gamma_{\max} (D' + 2D) + \cdots \]

\[ = \frac{\gamma_{\max}}{\gamma_{\min}} D' + \frac{(1 - \gamma_{\min}) \gamma_{\max}}{\gamma_{\min}^2} D \]
Estimates for $\mathbb{E}_z(\tau)$

- $T^i$: (random) time it takes for particle $i$ to exit (hit $\partial \Gamma$)
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- $T^i$: (random) time it takes for particle $i$ to exit (hit $\partial \Gamma$)
- $T^{\text{flight}}$: (random) time to hit the disk originating from $\partial \Gamma$
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- however $\frac{1}{\sqrt{\omega^2 + s^2 \cos^2(\varphi')}} \leq \frac{1}{s \cos(\varphi')}$
Estimates for $\mathbb{E}_z(\tau)$

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however $\frac{1}{\sqrt{\omega^2 + s^2 \cos^2(\varphi')}} \leq \frac{1}{s \cos(\varphi')}$

Given $z \in \Omega$,

$$\tau \leq \max_{1 \leq i \leq k} \{ \sup_{\omega} \{ T^i \} \} + \sup_{\omega} \{ T^\text{flight} \} + \sup_{\omega} \{ T^\text{hit} \}$$
Estimates for $E_z(\tau)$

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however \[
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\]

Given $z \in \Omega$,

\[
\tau \leq \max_{1 \leq i \leq k} \{ \sup_\omega \{ T^i \} \} + \sup_\omega \{ T^{\text{flight}} \} + \sup_\omega \{ T^{\text{hit}} \}
\]

$\Rightarrow$

\[
E_z[\tau] \leq \max_{1 \leq i \leq k} E_z[\sup_\omega \{ T^i \}] + E_z[\sup_\omega \{ T^{\text{flight}} \}] + E[\sup_\omega \{ T^{\text{hit}} \}]
\]
Suspension flow coordinates

\((x_i, v_i) \rightarrow (r_i, \varphi_i, s_i, \xi_i)\)
Suspension flow coordinates

\[(x_i, v_i) \rightarrow (r_i, \varphi_i, s_i, \xi_i)\]

- \(r_i \in \partial \Gamma\): the point of the immediate past or future collision is with \(\partial \Gamma\)
- \(\varphi_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\): the angle with the normal to \(\partial \Gamma\)
- \(s_i \in [0, \infty)\): the speed of the particle
- \(\xi_i\) the distance from the collision point; \(\xi_i > 0 \leftrightarrow\) past or \(\xi_i < 0 \leftrightarrow\) future
Suspension flow coordinates

- \((x_i, v_i) \rightarrow (r_i, \varphi_i, s_i, \xi_i)\)

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- \(\varphi_i \in (-\frac{\pi}{2}, \frac{\pi}{2})\): the angle with the normal to \(\partial \Gamma\)
- \(s_i \in [0, \infty)\): the speed of the particle
- \(\xi_i\): the distance from the collision point;
  \(\xi_i > 0 \leftrightarrow \text{past}\) or \(\xi_i < 0 \leftrightarrow \text{future}\)

- Denote angles of collision with disk by \(\varphi'\); note that
  \[
  \sin(\varphi') = \frac{R+d}{R} \sin(\varphi).
  \]
Estimates for $\mathbb{E}_z[\sup_\omega \{ T^i \}]$

- $T^i$: (random) time it takes for particle $i$ to hit $\partial \Gamma$
Estimates for $E_z[\sup_\omega \{ T^i \}]$

- $T^i$: (random) time it takes for particle $i$ to hit $\partial \Gamma$
- if the particle is headed for collision, the disk has angular velocity
  - either $\omega$
  - or $s_j \sin(\varphi_j')$ (if $j^{th}$ particle collided with $\partial \Gamma$ right before the $i^{th}$)
  - or $\tilde{\omega} \sim \sqrt{\frac{\beta L}{R \pi}} e^{-\beta L/R \tilde{\omega}^2} d\tilde{\omega}$
Estimates for $\mathbb{E}_z[\sup_\omega \{ T^i \}]$

- $T^i$: (random) time it takes for particle $i$ to hit $\partial \Gamma$
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\[
\frac{l}{\sqrt{\omega^2 + s^2_i \cos^2(\phi'_i)}} \leq \frac{l}{s_i \cos(\phi'_i)} \quad \text{and} \quad \frac{l}{\sqrt{\tilde{\omega}^2 + s^2_i \cos^2(\phi'_i)}} \leq \frac{l}{\tilde{\omega}}
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- where $l$ is a maximal flight distance
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$\implies$

$$E_z[T^i] \leq \frac{2l}{s_i \cos(\varphi'_i)} + \frac{l}{\sqrt{\beta_{\min} \pi}}$$
Estimates for $\mathbb{E}[\sup_\omega \{ T^{\text{flight}} \}]$

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Estimates for $\mathbb{E}[\sup_\omega \{T^{\text{flight}}\}]$

- $T^{\text{flight}}$: (random) time of exit of a particle emitted from $\partial \Gamma$

$$
\mathbb{E}[\sup_\omega \{T^{\text{flight}}\}] \leq \int_0^\infty \int_{\sin(\varphi) \geq \alpha} \frac{2I}{s} \rho_{\beta L/R}(s, \varphi) ds d\varphi \\
+ \int_0^\infty \int_{\sin(\varphi) \leq \alpha} \left[ \frac{I}{s} + \frac{I}{s \cos(\varphi')} \right] \rho_{\beta L/R}(s, \varphi) ds d\varphi \\
\leq K
$$
Estimates for $\mathbb{E}[\sup_\omega \{ T^{hit} \}]$ and $\mathbb{E}_z[\tau]$

- $T^{hit}$: (random) time to hit the disk emitted from $\partial \Gamma$
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$$\mathbb{E}[\sup_\omega \{ T^{flight} \}] \leq K \Rightarrow$$

$$\mathbb{E}[\sup_\omega \{ T^{hit} \}] \leq \alpha K + (1 - \alpha)\alpha 2K + (1 - \alpha)^2\alpha 3K + \cdots$$
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- $\implies$

\[
\mathbb{E}_z[\tau] \leq \max_{1 \leq i \leq k} \left\{ \frac{2l}{s_i \cos(\varphi'_i)} \right\} + \frac{l}{\sqrt{\beta_{\min}}} + \frac{(1 + \alpha)K}{\alpha}
\]
Prop. $\nu = P^\tau_\star \lambda \implies \mathbb{E}_{\nu}[\tau] \leq D$

\[ \nu = P^{nt}_\star \lambda \implies \]

\[ d(s^T_i, \varphi^T_i) \sim \rho_{\beta L/R}(s_i, \varphi_i) ds_i d\varphi_i := \frac{2\beta^{3/2}}{\sqrt{\pi}} \frac{s_i^2 e^{-\beta L/R s_i^2}}{s_i^2} \cos(\varphi_i) ds_i d\varphi_i \]
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\( \nu = \mathcal{P}_*^{n\tau} \lambda \implies \)

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\]

\[
\mathbb{E}_\nu[\tau] \leq \int_{\Omega} \left[ \max_{1 \leq i \leq k} \left\{ \frac{2l}{s_i \cos(\varphi_i')} \right\} \right] + \frac{l}{\sqrt{\beta_{\min}}} + \frac{(1 + \alpha)K}{\alpha} \right] d\nu
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- \( \nu = \mathcal{P}^{n\tau} \lambda \Rightarrow \)

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\]

\[
\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\infty} \frac{2l}{s \cos(\varphi')} \rho_{\beta L/R}(s, \varphi') dr d\varphi + \frac{l}{\sqrt{\beta_{\min} \pi}} + \frac{(1 + \alpha)K}{\alpha} \leq D
\]
Prop. $\sup_{z \in C} \{ \mathbb{E}_z[\tau(\delta)] \} \leq D'$

If $z \in C$

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\mathbb{E}_z[\tau] \leq \max_{1 \leq i \leq k} \left\{ \frac{2l}{s_i \cos(\varphi_i')} \right\} + \frac{l}{\sqrt{\beta_{\text{min}}}} + \frac{(1 + \alpha)K}{\alpha}
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- Wait time $\delta \Rightarrow$ some of the particles experience collisions with $\partial \Gamma$ and redistribute their speeds and angles according to $\rho_{\beta_L/R}(s, \varphi)drd\varphi$. 

Tatiana Yarmola (University of Geneva)  
Open interacting particle systems  
Lausanne, June 6, 2013  
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Let $\Phi_1$ be time-1 Markov chain sampled from $\Phi_t$
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**Theorem (Meyn and Tweedie 1993)**

$\Phi_1$ is irreducible and $\Phi_t$ has invariant probability measure $\mu \implies$

$\Phi_t$ is ergodic, i.e.

$$\lim_{t \to \infty} ||P^t(x, \cdot) - \mu|| = 0$$
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- Ergodicity $\Rightarrow$ any distribution converges to the invariant measure.

$$\lim_{t \to \infty} \| P^t \lambda - \mu \| = \lim_{t \to \infty} \sup_{t \to \infty} \left| \int_{\Omega} (P^t(x, A) - \mu(A)) d\lambda \right|$$

$$\leq \lim_{t \to \infty} \int_{\Omega} \| P^t(x, \cdot) - \mu(\cdot) \| d\lambda = 0$$
Rates of mixing

Let $B_T$ be the set of configurations such that one of the particles would not collide with $\partial \Gamma \cup \partial D$ in time $T$. 
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- Equilibrium invariant measure: $\beta = \beta_L = \beta_R$

$$d\mu = ce^{-\beta(\omega^2 + \sum_{i=1}^{k} |v_i|^2)} d\omega d\zeta dv_1 \cdots dv_k dx_1 \cdots dx_k$$

[Eckmann and Young 2006]
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[Eckmann and Young 2006]

- in $(r, \varphi, s, t)$-coordinates the part that corresponds to particle $i$:

$$d\mu_i = cs_i^2 e^{-\beta s_i^2} \cos(\varphi_i) dr_i d\varphi_i ds_i dt_i$$

- integrate over distance of flight $< d$, $s_i < d/T$, and $t_i < \frac{d}{s_i} - T$

$$\nu(B_T) \approx \int_0^{\frac{d}{s_i}} s_i^2 \left( \frac{d}{s_i} - T \right) ds_i \approx \frac{1}{T^2}$$
Rates of mixing are not exponential [Young 1999]

\[ z \in B_{k+n} \Rightarrow \mathcal{P}^n(z, B^c_k) = 0 \]
Rates of mixing are not exponential [Young 1999]

- $z \in B_{k+n} \Rightarrow P^n(z, B_k^c) = 0$
- Let $\lambda \ll \mu$ be such that $d\lambda = \varphi d\mu$ with $\varphi \geq 1 + c$ on $B_{k_0}$
Rates of mixing are not exponential [Young 1999]

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- Let $\lambda \ll \mu$ be such that $d\lambda = \varphi d\mu$ with $\varphi \geq 1 + c$ on $B_{k_0}$

\[ \|\mathcal{P}_\ast^n \lambda - \mu\| \geq \mathcal{P}_\ast^n \lambda(B_k) - \mu(B_k) \geq \lambda(B_{k+n}) - \mu(B_k) \]
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$$\|P^n_* \lambda - \mu\| \geq P^n_* \lambda(B_k) - \mu(B_k) \geq \lambda(B_{k+n}) - \mu(B_k)$$

$$\geq (1 + c)\mu(B_{k+n}) - \mu(B_k) \approx [(1 + c)\frac{k^2}{(k+n)^2} - 1] \frac{1}{k^2}$$
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  \[
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- \( \exists N \) s.t. \( [(1 + c)\frac{N^2}{(N+1)^2} - 1] > \frac{c}{2} \).
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\| \mathcal{P}_*^n \lambda - \mu \| \geq \mathcal{P}_*^n \lambda(B_k) - \mu(B_k) \geq \lambda(B_{k+n}) - \mu(B_k)
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- \( \exists N \) s.t. \( [(1 + c)\frac{N^2}{(N+1)^2} - 1] > \frac{c}{2} \). Let \( k = Nn \). Then

\[
\| \mathcal{P}_*^n \lambda - \mu \| \geq \frac{c}{2N^2} \frac{1}{n^2}
\]
Rates of mixing: non-equilibrium

- The information we need: \( C \) is a ‘good’ set, i.e. \( \exists \eta > 0, \ T_0 > 0 \) s.t.
  \[
  \mathcal{P}^{T_0}(z, \cdot) = \mathcal{P}^{T_0}_* \delta_z \geq \eta m_C \quad \forall z \in C
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  - where \( m_C \) is the uniform probability measure on \( C \)
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- The probability that a randomly emitted particle will not collide in time $T + \delta$ is $\geq \frac{\varsigma}{T^3}$
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For $z \in C_\delta$, $P^\delta(z, B_T) \geq \frac{\varsigma}{T^3} \Rightarrow$

$$\mu(B_T) - \mu(B_{T+\delta}) \geq \alpha \eta \gamma \times \frac{\varsigma}{T^3}$$
Prop. $\mu(B_T) \approx \frac{1}{T^2}$

- $B_T^c \to B_T$: emit slow particle from $\partial \Gamma$
Prop. \( \mu(B_T) \approx \frac{1}{T^2} \)

- \( B^c_T \to B_T \): emit slow particle from \( \partial \Gamma \)
- \( \Rightarrow \) for \( z \in B^c_T \), \( P^\delta(z, B_T) \leq \frac{\varsigma'}{T^3} \)
Prop. $\mu(B_T) \approx \frac{1}{T^2}$

- $B^c_T \rightarrow B_T$: emit slow particle from $\partial \Gamma$
- $\Rightarrow$ for $z \in B^c_T$, $\mathcal{P}^\delta(z, B_T) \leq \frac{\varsigma'}{T^3}$
- $\Rightarrow \mu(B_T) = \mu(B_{T+\delta}) + \Theta\left(\frac{1}{T^3}\right)$
Prop. $\mu(B_T) \approx \frac{1}{T^2}$

- $B^c_T \to B_T$: emit slow particle from $\partial \Gamma$
- $\Rightarrow$ for $z \in B^c_T$, $P^\delta(z, B_T) \leq \frac{\zeta'}{T^3}$
- $\Rightarrow \mu(B_T) = \mu(B_{T+\delta}) + \Theta\left(\frac{1}{T^3}\right)$
- $\Rightarrow \mu(B_T) \approx \frac{1}{T^2}$
Remarks

‘Almost renewing’ stopping time idea applicable to more general classes of systems

▷ underlying deterministic dynamics may create ‘conspiracies’ (invariant sets or traps)
▷ minorization condition on $C$ is harder to prove for more complex geometries
Remarks

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- Main geometric assumption: a particle hits the disk at most once before randomizing its velocity at $\partial \Gamma$
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  - used in estimating $\mathbb{E}_z[\tau]$
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- Mechanical particle systems coupled to Gibbs-like heat reservoirs do not mix exponentially
- Coupling arguments may be developed to get the polynomial upper bounds on the rates of mixing
Thank You!


