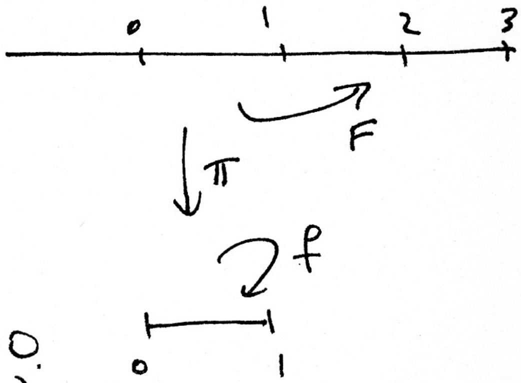
Exercise on lifts

Let $f: K \to K$ be an orientation preserving homeomorphism of the circle K. Let $F: \mathbb{R} \to \mathbb{R}$ be a lift, i.e., if $\pi: \mathbb{R} \to K$ is the usual projection ($\pi(x') = x' \pmod 1$) then $\pi(F(x')) = f(\pi(x'))$, $\forall x' \in \mathbb{R}$.



Claim: $F(x'+1) = F(x')+1$

Clearly $F(x'+1) - F(x') \in \mathbb{Z}$ since $\pi(x') = \pi(x'+1)$ implies:

$\pi(F(x'+1)) - \pi(F(x')) = f(\pi(x'+1)) - f(x') = 0$

(It is clear from the graphs that the difference is 1)

Claim: If $0 \leq x', y' \leq 1$ then $|F(x') - F(y')| \leq 1$.

Assume $x' \leq y' \leq x'+1$ then by monotonicity of F:

$$F(x') \leq F(y') \leq F(x') + 1 \quad \text{(using first claim)}$$

$\Rightarrow 0 \leq F(y') - F(x) \leq 1$.

Similarly, $y'-1 \leq x' \leq y'$ and by monotonicity of F

$$F(y') - 1 \leq F(x') \leq F(y') \quad \text{(using first claim)}$$

$\Rightarrow \quad -1 \leq F(x') - F(y') \leq 0$.

Thus $|F(x') - F(y')| \leq 1$.

(The case $y' \leq x' \leq y'+1$ is similar).