(ii) there exists a solution of (#) together with (*) satisfying $\lambda(\xi_{0}$…
$m^{-n} > 0$ for any $\xi_0, \cdots, \xi_{n-1} \in \Sigma$.

Take a longest sequence $\xi^0, \xi^1, \cdots, \xi^{r-1}$ of elements in Σ^n such that

(1) $(\xi^i_1, \cdots, \xi^i_{n-1}) = (\xi^{i+1}_0, \cdots, \xi^{i+1}_{n-2})$ for any $i = 0, 1, \cdots, r-1$, and

(2) $\#\{i; 0 \leqq i < r, \xi^i = \xi\} \leqq N\lambda(\xi)$ for any $\xi \in \Sigma^n$,

where $\xi^i = (\xi^i_0, \cdots, \xi^i_{n-1})$ $(i = 0, 1, \cdots, r)$ and $\xi^r \equiv \xi^0$.

Then, by a graph-theoretical consideration, it is not difficult to prov
equality holds in the above inequality in (2) for any $\xi \in \Sigma^n$. Let β be th
element in $[0,1]^{\mathbb{N}}$ with period $n + r - 1$ defined by

$$(\beta(0), \beta(1), \cdots, \beta(n+r-2)) = (\xi^0_0, \xi^0_1, \cdots, \xi^0_{n-1}, \xi^1_{n-1}, \xi^2_{n-1}, \cdots, \xi$$

Then, it holds that $\mu_\beta \in U$.

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$$\frac{1}{n}\sum_{0}^{n-1} f(T^{i}Tx) = \frac{1}{n}\sum_{\phi}^{n} f(T^{i}x) = \frac{1}{n}\sum_{0}^{n} f(T^{i}x) - \frac{f(x)}{n}$$

$$= \left(\frac{n+1}{n}\right)\left(\frac{1}{n+1}\sum_{0}^{n} f(T^{i}x)\right) - \frac{f(x)}{n}$$

Step1

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A SIMPLE PROOF OF SOME ERGODIC THEOREMS

BY

YITZHAK KATZNELSON AND BENJAMIN WEISS

ABSTRACT

Some ideas of T. Kamae's proof using nonstandard analysis are employed to give a simple proof of Birkhoff's theorem in a classical setting as well as Kingman's subadditive ergodic theorem.

$(X, \mathcal{B}, \mu)$ be a probability measure space and let $T: X \to X$ be a
rable, measure preserving transformation, possibly noninvertible. Birk-
ergodic theorem states that for any integrable function f, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = f^*(x)$$

for μ-a.e. x, and f^* is a T-invariant function with the same integral as f.
dapt an idea of T. Kamae [1] to give a simple proof of this result. It is
ent to deal with nonnegative functions and defining

(i) Assume $f \geqslant 0$ since $f = f_1 - f_2$ $f_1, f_2 \geqslant 0$.

(ii) $$\bar{f}(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{0}^{n-1} f(T^j x), \quad \underline{f}(x) = \liminf_{n\to\infty} \frac{1}{n} \sum_{0}^{n-1} f(T^j x)$$

ces to show that

$$\int \bar{f}(x) d\mu(x) \leqq \int f(x) d\mu(x) \leqq \int \underline{f}(x) d\mu(x)$$

that gives equality a.e. $\bar{f}(x) = \underline{f}(x)$ and $\int f^* d\mu = \int f d\mu$ while the (iii) T-
ance of both $\bar{f}$ and $\underline{f}$ is clear. Fix some $M > 0$, $\varepsilon > 0$, denote

$$\bar{f}_M(x) = \min\{\bar{f}(x), M\}$$

define $n(x)$ to be the least integer $n \geqq 1$ for which

$$\bar{f}_M(x) \leqq \frac{1}{n} \sum_{0}^{n-1} f(T^j x) + \varepsilon. \qquad (*)$$

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Iterating n gives:

Step1 (4) $$\frac{1}{n} f_n(x) \leqq \frac{1}{n} \sum_{i=0}^{n-1} f_1(T^i x)$$

and thus by Birkhoff's ergodic theorem $\bar{f}(x) \leqq f_1^*(x)$.

We remark at this point that (4) implies that the sequence $\{(1/n)f$. equi-integrable, and combining this with the obvious inequality

$$\int \phi d\mu \leqq \int \frac{1}{n} f_n^* d\mu = \int \frac{1}{n} f_n d\mu, \quad \text{all } n, \quad \leqq \int f_1 d\mu.$$

$\frac{1}{n}\int f_n d\mu \leq \int f_1 d\mu$

we see that if $\int \phi d\mu > -\infty$, then the pointwise convergence a.e. of $(1/n)f$. implies convergence in L^1-norm. We have a similar, asymptotic, estimate wit instead of f_1 in (4). Fix $N > 1$ and let $n > N$. For each $i = 1, 2, \cdots, N$ wri $n = i + mN + k$ with $k < N$. Then by (3)

Step2

$$f_n(x) \leqq f_i(x) + \boxed{\sum_{l=0}^{m-1} f_N(T^{lN+i}x) + f_k(T^{mN+i}x)}$$

and summing over i,

$$Nf_n(x) \leqq \sum_{i=1}^{N-1} f_i(x) + \sum_{j=0}^{n-1} f_N(T^j x) + \sum_{i=1}^{N} f_{n-i-mN}(T^{mN+i}x)$$

hence

$$\frac{1}{n} f_n(x) \leqq \frac{1}{nN} \sum_{j=0}^{n-1} f_N(T^j x) + \overbrace{\boxed{\frac{1}{nN}\left(\sum_{i=1}^{N} f_i(x) + \sum_{i=1}^{N} f_{n-i-mN}(T^{mN+i}x)\right)}}^{O(1/n)}$$

As $n \to \infty$ the last two terms on the right converge to zero a.e. and, by th ergodic theorem, $\frac{1}{N}\left[\frac{1}{n}\sum_{j=0}^{n-1} f_N(T^j x) \to \int f_N d\mu\right]$

$$\bar{f}(x) \leqq \frac{1}{N} f_N^*(x) \quad \text{a.e.} \quad (\text{for all } N \geqslant 1)$$

which implies

$\inf\{\frac{1}{N} f_N^*\}$

(5) $$\bar{f}(x) \leqq \phi(x) \quad \text{a.e.}$$

For points x where $\phi(x) = -\infty$, (5) shows that the desired limit exists an equals $\phi(x)$. We restrict our attention to $X_M = \{x : \phi(x) \geqq -M\}$, which T-invariant, and proceed to show that

Step3

$M = N = \lambda$.

(6) $$\int_{X_M} f d\mu \geqq \int_{X_M} \phi d\mu.$$

This combined with (5) shows that the statement of the theorem is valid on \.

as $\bigcup_1^\infty X_M = \{x : \phi(x) > -\infty\}$ this will complete the proof. For case of lation we will simply assume $\phi(x) \geqq -M$ for all x. Set M=N

As in the proof of the Birkhoff theorem fix an $\varepsilon > 0$, set $\underline{f}_M = \max\{f, -N-1\}$, $\geq \underline{f}$. d put

const?

$$[redacted] \; n(x) = \min\left\{n \geqq 1 : \frac{1}{n} f_n(x) \leqq \underline{f}_M(x) + \varepsilon\right\}.$$

Step4

$A = \{x : n(x) > N\}$ where N is chosen so that

(7) $$\int_A (|f_1(x)| + N + 1) d\mu(x) < \varepsilon,$$

d define the modifications as before:

$$\bar{\underline{f}}_M(x) = \begin{cases} \underline{f}_M(x), & x \notin A, \\ f_1(x), & x \in A, \end{cases} \qquad \tilde{n}(x) = \begin{cases} n(x), & x \notin A, \\ 1, & x \in A. \end{cases}$$

ote that $\bar{\underline{f}}_M(x) \leqq \underline{f}_M(x)$ for all x, and by (7)

$f_1 > f$

$$\int \bar{\underline{f}}_M d\mu \leqq \int \underline{f}_M d\mu + \varepsilon.$$

Step5

sing the T-invariance of $\underline{f}_M$ we have for all x

$$f_{\tilde{n}(x)}(x) \leqq \sum_{j=0}^{\tilde{n}(x)-1} \bar{\underline{f}}_M(T^j x) + \tilde{n}(x) \cdot \varepsilon$$

(by (7), $x \notin A$.
trivial, $x \in A$.
$\to f_1(x) < f_1(x) + \varepsilon$

d can calculate for any $L > N$ as before:

$$f_L(x) \leqq \sum_{0}^{L-1} \bar{\underline{f}}_M(T^j x) + L \cdot \varepsilon + N(N+1) + \sum_{L-N}^{L-1} |f_1(T^j x)|.$$

tegrating and dividing by L we obtain

$$\int \phi(x) d\mu \leqq \int \frac{1}{L} f_L^* d\mu = \int \frac{1}{L} f_L d\mu$$
$$\leqq \int \bar{\underline{f}}_M d\mu + \varepsilon + \frac{N(M+1)}{L} + \frac{N}{L} \cdot \int |f_1| d\mu.$$

tting $L \to \infty$ and using (8), we see that

$$\int \phi(x) d\mu \leqq \int \underline{f}_M(x) d\mu.$$

ecall now that $\underline{f}_M(x) \leqq \phi(x)$ holds for all x and that, combined with (9), implies

Since $\bar f$ is T-invariant so is $\bar f_M$ and thus averaging gives that for all

Make n bdd)

$$(1)\qquad n(x)\bar f_M(x) = \sum_0^{n(x)-1} \bar f_M(T^i x) \leqq \sum_0^{n(x)-1} f(T^i x) + n(x)\cdot\varepsilon.$$

by (x)

Step2 $(n \to \hat n)$

Now $n(x)$ is everywhere finite so that there is some N for which the se

$$A = \{x : n(x) > N\}$$

has measure less than ε/M. Define now

$$\hat f(x) = \begin{cases} f(x), & x \notin A, \\ \max\{f(x), M\}, & x \in A, \end{cases} \qquad \hat n(x) = \begin{cases} n(x), & x \notin A, \\ 1, & x \in A, \end{cases}$$

and observe that

$(f \le \hat f) \Rightarrow \hat f \geq 0$ (from (1))

$$(\bar 1)\qquad \sum_0^{\hat n(x)-1} \bar f_M(T^i x) \leqq \sum_0^{\hat n(x)-1} \hat f(T^i x) + \hat n(x)\cdot\varepsilon$$

$\hat n(x) \le N$

is also valid. The crucial improvement is that now $\hat n(x)$ is everywhere bou by N, while

$\le m(A)\cdot M \le \varepsilon/M \cdot M.$

$$(2)\qquad \int \hat f(x)\,d\mu(x) \leqq \int f(x)\,d\mu(x) + \int_A M\cdot d\mu(x) \leqq \int f(x)\,d\mu(x) + \varepsilon.$$

Step 3 $(\hat n \to \{n_k\})$

Choosing now L so that $NM/L < \varepsilon$ and defining inductively $n_0(x) = 0$ an

$$n_k(x) = n_{k-1}(x) + \hat n(T^{n_{k-1}(x)}x), \cdots$$

we have

$$\le \left[\sum_{n_{k-1}(x)}^{n_k(x)-1} f(T^i x) + \hat n(x)\varepsilon\right]$$

$$\sum_0^{L-1} \bar f_M(T^i x) = \sum_{k=1}^{k(x)} \underbrace{\sum_{n_{k-1}(x)}^{n_k(x)-1} \bar f_M(T^i x)} + \sum_{n_{k(x)}(x)}^{L-1} \bar f_M(T^i x)$$

$\to (L-1-n_{k(x)}(x)) \le N-1$

where $k(x)$ is the maximal k for which $n_k(x) \leqq L-1$. Applying $(\bar 1)$ to ea the $k(x)$ terms, and estimating by M the last $L - n_{k(x)}(x) \leqq N-1$ terms we h for all x

$(\hat n + \cdots + \hat n)$

$$\sum_0^{L-1} \bar f_M(T^i x) \leqq \sum_0^{L-1} \hat f(T^i x) + L\cdot\varepsilon + (N-1)M,$$

where the fact that $\boxed{\hat f \geqq 0}$ allows us to write $L-1$ as the upper limit of summation. Integrating both sides and dividing by L gives

$$\int \bar f_M d\mu \le \int \hat f d\mu + \varepsilon + \frac{(N-1)M}{} \le \int f d\mu + 3\varepsilon$$

ht of (2) and the choice of L. It is here that we use the fact that T is measure erving. Letting $\varepsilon \to 0$ and $M \to \infty$ gives half of what we wanted, namely

$$\int \bar f d\mu \leqq \int f d\mu.$$

$\int \bar f_M d\mu \to \int \bar f d\mu$ as $M \to +\infty$.

or the other half, fix $\varepsilon > 0$ and define now $n(x)$ as the least integer $n \geqq 1$ for

$$\frac{1}{n}\sum_0^{n-1} f(T^i x) \leqq \underline{f}(x) + \varepsilon.$$

efore $A = \{x : n(x) > N\}$ where now N is chosen so that $\int_A f(x)\,d\mu(x) < \varepsilon$. define now

$$\hat n(x) = \begin{cases} n(x), & x \notin A, \\ 1, & x \in A, \end{cases} \qquad \hat f(x) = \begin{cases} f(x), & x \notin A, \\ 0, & x \in A, \end{cases}$$

conclude the proof in the same way as before.

bserve that we could have restricted the integration to any T-invariant set so we really have shown that $f^*(x)$ is a version of the conditional expectation with respect to the σ-algebra of invariant sets. The same basic idea, of ifying the function so that $n(x)$ becomes bounded, can be used to simplify ofs of other ergodic theorems as well. To illustrate the possibilities we give a proof of Kingman's subadditive ergodic theorem [2]:

EOREM. *If T is a measure preserving transformation of the probability sure space $(X, \mathscr{B}, \mu)$ and $\{f_n\}_1^\infty$ is a sequence of L^1-functions satisfying*

$$(3)\qquad f_{n+m}(x) \leqq f_n(x) + f_m(T^n x), \qquad \textit{all } n, m \geqq 1$$

$\lim_{n\to\infty}(1/n)f_n(x)$ *exists* a.e. *and may be identified as* $\phi(x) = \inf_n (1/n) f_n^*(x)$ *re f_n^* is the projection of f_n onto the space of T-invariant functions.*

or the proof, note first that (3) implies

Step'

$$f_{n+m}^*(x) \leqq f_n^*(x) + f_m^*(x)$$

(since f^* invariant),

If μ ergodic
then
$f_n^* = \int f_n d\mu$
a_n = sub. seq.

hence $(1/n)f_n^*(x)$ converges to $\phi(x)$. Next, denote

$$\bar f(x) = \limsup \frac{1}{n} f_n(x), \qquad \underline{f}(x) = \liminf \frac{1}{n} f_n(x),$$

$f_M(x) = \phi(x)$ a.e. Since whenever $f_M(x) \neq f(x)$ we have f_{M+1} $- M - 1 \neq \phi(x)$, this can happen only on a null set and $f(x) = \phi(x)$ a.e.

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Cor: $T: X \to X$, $A: X \to L(\mathbb{R}^k, \mathbb{R}^k)$,
let $f_n^a(x) = \log \|A(x).A(Tx) \cdots A(T^{n-1}x)\|$
then $\|A(x) \cdots A(T^{n-1}x)\|^{1/n} \to e^{\lambda}$, some $\lambda \neq 0$

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A COMBINATORIAL CONDITION FOR THE EXISTENCE OF POLYHEDRAL 2-MANIFOLDS

BY

U. BETKE AND P. GRITZMANN

ABSTRACT

Let $\mathcal{P}$ denote a polyhedral 2-manifold, i.e. a 2-dimensional cell-complex in $\mathbf{R}^d$ ($d \geqq 3$) having convex facets, such that set($\mathcal{P}$) is homeomorphic to a closed 2-dimensional manifold. Let E be any subset of odd valent vertices of $\mathcal{P}$, and c_E its cardinality. Then for the number $c_{P(E)}$ of facets containing a vertex of E the inequality $2c_{P(E)} \geqq c_E + 1$ is proved. This local combinatorial condition shows that several combinatorially possible types of polyhedral 2-manifolds cannot exist.

A polyhedral 2-manifold $\mathcal{P}$ is a 2-dimensional cell-complex in $\mathbf{R}^d$ ($d \geqq 3$), se facets are convex polygons, such that set($\mathcal{P}$) is homeomorphic to a closed imensional manifold.

Given an abstract 2-dimensional cell-complex which has the structure of a binatorial 2-manifold, the question arises, whether there exists a polyhedral anifold which is combinatorially equivalent to it.

For a cell-complex $\mathcal{P}$ let $F_0(\mathcal{P})$ denote the set of all vertices of $\mathcal{P}$. For every tex e of $\mathcal{P}$ the valence val(e, $\mathcal{P}$) is the number of polygons of $\mathcal{P}$ containing e.

It is easy to see that simple polyhedral 2-manifolds, i.e. where all the vertices 3-valent, do not exist apart from the case of genus 0.

In [1] it is shown that for orientable polyhedral 2-manifolds $\mathcal{P}$ the "valence-ance" $\Sigma_{e \in F_0(\mathcal{P})}(\text{val}(e, \mathcal{P}) - 3)$ is bounded from below by a constant determined the genus of set($\mathcal{P}$). Here we give a local combinatorial condition for the stence of polyhedral 2-manifolds.

HEOREM. *Let $\mathcal{P}$ be a polyhedral 2-manifold, E any subset of odd valent tices of $\mathcal{P}$, and* $c_E :=$ card(E) *its cardinality. Then for the number $c_{P(E)}$ of facets taining a vertex of E we have:*

$$2c_{P(E)} \geqq c_E + 1.$$

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