LIMIT POINTS FOR TETRAHEDRA AND INSCRIBED SPHERES

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Abstract. In this note we consider the limiting point for a nested sequence of inspheres and tetrahedra, starting from some initial choice. In two dimensional Euclidean space we give a proof that the dependence is analytic which extends to the hyperbolic plane. In three dimensions we show that it is a rational function.

In this note we consider a simple geometric construction which associates to a given triangle or tetrahedron a unique limit point. We shall study the dependence of this limit point on the initial tetrahedron. To explain the construction, we begin by recalling a simple geometric algorithm for triangles suggested by Synge [2].

The triangle algorithm. Let T_0 be a triangle in the Euclidean plane $\mathbb{R}^2$ with vertices a_0, b_0, c_0. We denote by S_0 the inscribed circle, i.e., the unique largest circle contained in T_0. We can then associate a new triangle T_1 whose vertices a_1, b_1, c_1 are the three points of intersection $S_0 \cap T_0$. (In particular, S_0 is the outscribed circle for T_1.) We proceed inductively. For each $n \geq 1$, we associate to the triangle T_n the inscribed circle S_{n+1}; and then we associate to S_{n+1} a new triangle T_{n+1} whose vertices are the three points of intersection $S_n \cap T_n$. The nested sequence of triangles $\{T_n\}_{n=0}^{\infty}$ (or, equivalently, the nested sequence of circles $\{S_n\}_{n=0}^{\infty}$) shrinks to a single limit point, which we can denote by $z = z(T_0) \in \mathbb{R}^2$.

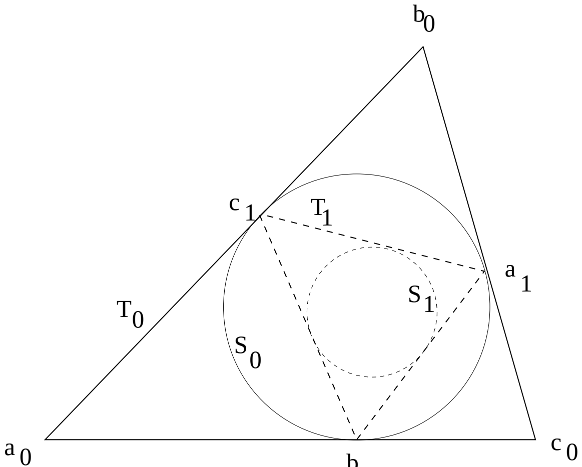

Figure 1. The first steps in the triangle algorithm in $\mathbb{R}^2$.

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There is an analogous construction in hyperbolic space, represented by the upper half plane $\mathbb{H}^2$ with the Poincaré metric $ds^2 = (dx^2 + dy^2)/y^2$. In this case, T_n is a hyperbolic triangle, whose sides are geodesic arcs between pairs of vertices. The inscribed circle S_n is again the largest circle (consisting of points at a fixed hyperbolic distance from a centre) contained in T_n [1, §7.14]. We again denote by $z(T_0)$ the limit point of the analogous iterated construction.

The first result describes the nature of the dependence of $z(T_0)$ on T_0, where the vertices of the triangle T_0 can be used to give natural coordinates.

Theorem 1. *In the case of either $\mathbb{R}^2$ or $\mathbb{H}^2$ the limit point $z = z(T_0)$ has a real analytic dependence on the initial triangle T_0.*

In particular, real analyticity of z means that each of its coefficients has a power series expansion (in terms of the coordinates of the three vertices of T_0) which has a non-zero radius of convergence.

In the case of $\mathbb{R}^2$ this, and more besides, was proved by Fried, using a novel argument involving transfer operators [5]. We shall present an alternative proof of this result, which easily extends to $\mathbb{H}^2$. In point of fact, the same method shows the corresponding result for any real analytic metric of non-positive curvature.

It is interesting to compare this with other well known construction of nested triangles. For example, from T_n ones could instead construct the medial triangle T_{n+1} whose vertices are the midpoints of the sides of T_n. However, here it is easy to see that this nested sequence of triangles share a common barycentre, which is necessarily the limit point. Alternatively, one could define the vertices of T_{n+1} to be the pedal points of the triangle T_n, i.e., the orthogonal projections of the vertices onto the opposite sides. However, here it is easy to see that the nested sequence of typical triangles converge to the barycentre of the initial triangle. However, there is a complication that if T_n is a right angled triangle, then T_{n+1} isn't defined [6].

We can also consider the natural generalization of the construction above to tetrahedrons in three dimensional space.

The tetrahedron algorithm. Let $\mathcal{T}_0$ be a tetrahedron in $\mathbb{R}^3$. We denote by $\mathcal{S}_0$ the inscribed sphere, i.e., the largest sphere contained in $\mathcal{T}_0$. We can associate a new tetrahedron $\mathcal{T}_1$ whose vertices are the four points of intersection $\mathcal{S}_0 \cap \mathcal{T}_0$. For each $n \geq 2$, we associate to the tetrahedron $\mathcal{T}_n$ the inscribed sphere $\mathcal{S}_{n+1}$; and then we associated to S_n a new tetrahedron $\mathcal{T}_{n+1}$ whose vertices are the four points of intersection $\mathcal{S}_n \cap \mathcal{T}_n$. Again, one can show that the nested tetrahedra $\{\mathcal{T}_n\}_{n=0}^\infty$ (equivalently, the nested spheres $\{S_n\}_{n=0}^\infty$) shrink to a single limit point, which we denote by $w = w(\mathcal{T}_0) \in \mathbb{R}^3$.

There is an analogous construction in three dimensional hyperbolic space $\mathbb{H}^3$, with the Poincaré metric $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$. The hyperbolic tetrahedra is the hyperbolic convex hull of four vertices. The four faces are hyperbolic triangles contained in totally geodesic planes and the six edges are geodesic arcs between pairs of vertices. The existence of an inscribed sphere (whose centre is equidistant from each of the four faces) can be easily seen from the hyperboloid model of hyperbolic space.

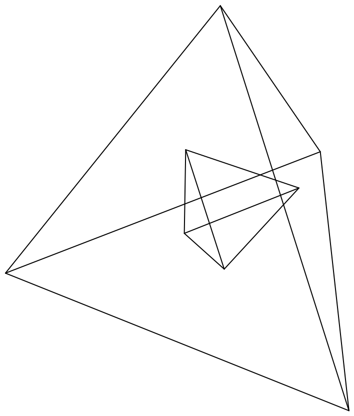

FIGURE 2. Regular tetrahedra $\mathcal{T}_0$ and $\mathcal{T}_1$ in $\mathbb{R}^3$.

Theorem 2. *In the case of either $\mathbb{R}^3$ or $\mathbb{H}^3$ the limit point $w = w(\mathcal{T}_0)$ has a real analytic dependence on the initial tetrahedron $\mathcal{T}_0$.*

In section 1 we present some preliminary observations and examples for the triangle algorithm in Euclidean space. In section 2 we shall give a proof of Theorem 1. In section 3 we give examples and results on the tetrahedron algorithm, and in section 4 we give a proof of Theorem 2.

1. The triangle algorithm in Euclidean space

In this section let us consider the triangle algorithm on Euclidean space $\mathbb{R}^2$. In Euclidean space, it is clear that similar triangles have their limit points described by the same similarity. It is therefore convenient to make a canonical choice for similar triangles. By translating, rotating and the scaling we can reduce the problem to considering those initial triangles $T_0 = T_0(w)$ whose vertices correspond to $\underline{0} = (0,0)$,$\underline{1} = (1,0)$ and $w \in \mathcal{H}$, say, where $\mathcal{H} = \{w = (w_1, w_2) \in \mathbb{R}^2 : w_2 > 0\}$ denotes the upper half plane. In this case we can define a map $f : \mathcal{H} \to \mathcal{H}$ from $\mathcal{H}$ to itself by $f(w) = z(T(w)) \in \mathbb{C}_+$. We plot part of the image of f in Figure 3.

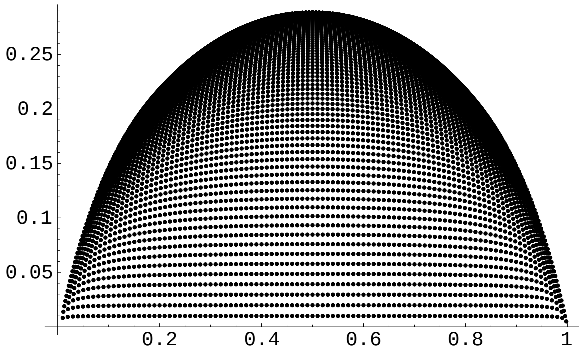

FIGURE 3. A plot of the images $f(w)$ of points $w \in [0,1] \times [0,1]$ chosen in a grid of size 0.01

Let us consider some special cases

Example 1.1 (Equilateral triangle). When $w = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ then T_0 is an equilateral triangle. Its barycentre (or "centre of gravity") is equal to $b = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and is also the centre of the inscribed circle S_1. One easily seems that b remains the barycentre for each T_n (and the centre for each C_n) for $n \geq 1$. In particular, in this case we can deduce that $f(w) = b$.

This is a special case of the following more general class of examples.

Example 1.2 (Isoceles triangles). When $w = w(t) = (\frac{1}{2}, t)$, for $0 < t < \infty$, then T_0 is an isoceles triangle. It is easy to see by symmetry that the first coordinate of $f(w(t)) \in \mathcal{H}$ remains $\frac{1}{2}$. Thus we can write $f(w(t)) = (\frac{1}{2}, z_2(t))$, say. In Figure 4, we plot the second coordinate $z_2(t)$ of $f(w(t))$ against the initial height of the vertex t. The maximum corresponds to the equilateral triangle.

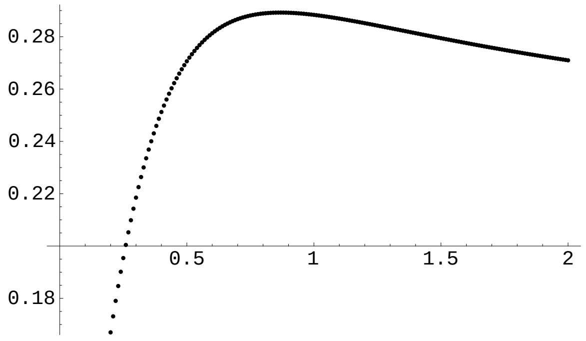

FIGURE 4. A plot of $z_2(t)$ where points $t \in [0, 2]$ are chosen with spacing by 0.01.

Finally, we shall need an elementary result in planar geometry. For each triangle T_n, $n \geq 1$, let $\alpha_n, \beta_n, \gamma_n$ be the internal angles.

Proposition 1. *$(\alpha_n, \beta_n, \gamma_n) \to (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ as $n \to +\infty$.*

Proof. By elementary trigonometry one sees that

$$\alpha_{n+1} = \frac{\beta_n + \gamma_n}{2}, \beta_{n+1} = \frac{\gamma_n + \alpha_n}{2} \text{ and } \gamma_{n+1} = \frac{\beta_n + \alpha_n}{2}.$$

The result then easily follows by considering the sequence of triples on the simplex $\{(\alpha, \beta, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ : \alpha + \beta + \gamma = \pi\}$. □

This is very different to what happens in the case of pedal triangles. In that case, the shape of the triangles $\{T_n\}_{n=0}^{\infty}$ typically will not converge [6], [7].

Let us denote by $\mathrm{diam}(T_n)$ the diameter of the triangle T_n. The following corollary will prove useful in the next section.

Corollary 1.1. *For each $\frac{1}{2} < \lambda < 1$ we have $diam(T_{n+1}) \leq \lambda diam(T_n)$ for all n sufficiently large.*

Proof. If T_n were an equilateral triangle then it is easy to see that so is T_{n+1} and $\mathrm{diam}(T_{n+1}) \leq \frac{1}{2}\mathrm{diam}(T_n)$. By continuity, we see that for T_n sufficiently close to equilateral we have $\mathrm{diam}(T_{n+1}) \leq \lambda\mathrm{diam}(T_n)$. Finally, for sufficiently large n this holds, by Proposition 1. □

2. PROOF OF THEOREM 1

We begin by presenting some explicit formulae. The space of triangles in $\mathbb{R}^2$ can be identified with the triples

$$X = \{(z_1, z_2, z_3) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 : z_1, z_2, z_3 \text{ are non-collinear}\}.$$

We introduce a map $S : X \to X$ defined by

$$\begin{aligned} &S(a,b,c) \\ &= \left(\frac{z(a,b,c)b + y(a,b,c)c}{|b-c|}, \frac{z(a,b,c)a + x(a,b,c)c}{|a-c|}, \frac{x(a,b,c)b + y(a,b,c)a}{|b-a|} \right) \end{aligned}$$

where

$$\begin{aligned} x(a,b,c) &= \frac{1}{2}(|a-b| + |a-c| - |b-c|), \\ y(a,b,c) &= \frac{1}{2}(|a-b| - |a-c| + |b-c|) \text{ and} \\ z(a,b,c) &= \frac{1}{2}(-|a-b| + |a-c| + |b-c|), \end{aligned}$$

and $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^2$. It is then easy to show that $S(a_n, b_n, c_n) = (a_{n+1}, b_{n+1}, c_{n+1})$, $n \geq 1$, using Heron's formula, cf. [3, pp.11-13],[4]. As we have already observed, S is equivariant under, in particular, rotations, translations and homothetic scaling, i.e.,

$$S\left((\lambda x, \lambda y, \lambda z) + (v_1, v_2, v_3)\right) = \lambda g z(x,y,z) + (v_1, v_2, v_3),$$

for any $(v_1, v_2, v_3) \in \mathbb{R}^3$, $g \in SO(2)$ and $\lambda \in \mathbb{R}$.

Let us recall an equivalent definition of real analyticity which is particularly useful for our purposes. We first identify the $\mathbb{R}^2$ with the real part $\mathbb{R}^2 \times \{0\}$ of $\mathbb{C}^2 = \mathbb{R}^2 + i\mathbb{R}^2$. In particular, to show that $f : \mathcal{H} \to \mathcal{H}$ is real analytic it suffices to find a neighbourhood $\mathcal{U}$ of $\mathcal{H}$ in $\mathbb{C}^2$ and an extension $f : \mathcal{U} \to \mathbb{C}^2$ which is analytic, in the usual sense, for two complex variables. More precisely, for every $(w_1, w_2) \in \mathcal{U}$ there should be a power series expansion

$$\begin{aligned} &f(z_1, z_2) = \\ &\left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_{n,m}^{(1)} (z_1 - w_1)^n (z_2 - w_2)^m, \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_{n,m}^{(2)} (z_1 - w_1)^n (z_2 - w_2)^m \right) \end{aligned}$$

which is uniformly convergent for (z_1, z_2) sufficiently close to (w_1, w_2).

Lemma 2. *The map* $s : w \mapsto S(\underline{0}, \underline{1}, w) \in \mathbb{R}^2$ *is real analytic for* $w \in \mathcal{H}$. *Moreover, there exists a neighbourhood* $\mathcal{U}_0$ *of* $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ *in* $\mathbb{C}^2$ *and a constant* $K > 0$ *such that the map* $s : \mathcal{U}_0 \to \mathbb{C}^2$ *is analytic and satisfies* $\sup_{w \in \mathcal{U}_0} |s(w)| \leq K$.

Proof. This follows from the previous formulae. In particular, we can write that $S(0,1,w) = (A(w), B(w), B(w))$, where

$$\begin{aligned} A(w) &= \frac{(-1 + |w| + |1-w|) + (1 - |w| + |1-w|)w}{2|1-w|} \\ B(w) &= \frac{(1 + |w| - |1-w|)w}{2|w|} \qquad (2.1) \\ C(w) &= \frac{(1 + |w| - |1-w|)}{2}, \end{aligned}$$

for $w \in \mathcal{H}$. In particular, provided w is bounded away from both 0 and 1 (corresponding to degenerate triangles) we see from the formulae that $(A(w), B(w), C(w))$ is real analytic, i.e., it has an analytic extension to a neighbourhood $\mathcal{U}$ of $\mathcal{H}$ in $\mathbb{C}^2$, say. It is easy to see that we can choose a neighbourhood $\mathcal{U}_0 \subset \mathcal{U}$ of $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ such that its closure is compact and contained in $\mathcal{U}$. Then the restriction $s : \mathcal{U}_0 \to \mathbb{C}^2$ is uniformly bounded, and we can set $K = \sup_{w \in \mathcal{U}_0} |s|$. $\square$

We shall now apply this lemma to give a direct proof of Theorem 1.

Proof of Theorem 1. We shall first concentrate on the case of $\mathbb{R}^2$, and then indicate the minor changes required for the case of $\mathbb{H}^2$. We denote the vertices of the triangle T_n by (a_n, b_n, c_n), for $n \geq 1$. When $n = 0$, we set $(a_0, b_0, c_0) = (\underline{0}, \underline{1}, w)$.

Let us assume for induction that we have an analytic extension of (a_n, b_n, c_n) in some neighbourhood $V \subset \mathbb{C}^2$ of $(w, 0)$. We can assume without loss of generality that $V \subset \mathbb{C}^2$ is a polydisk, i.e., $V = V_1 \times V_2$ where $V_1, V_2 \subset \mathbb{C}$. By the equivariance descrsibed above, we can write

$$\begin{aligned}(a_{n+1}, b_{n+1}, c_{n+1}) &= S(a_n, b_n, c_n) \\ &= \lambda_n g f(w_n) + (a_n, a_n, a_n) \\ &= \lambda_n g(A(w_n), B(w_n), C(w_n)) + (a_n, a_n, a_n)\end{aligned} \tag{2.2}$$

where we denote $\lambda_n := |b_n - a_n|$, $g = \begin{pmatrix} \cos\theta & \sin\theta \\ -sin\theta & \cos\theta \end{pmatrix}$ (with θ the angle of $b_n - a_n$ with the horizontal) and $w_n := \frac{c_n - a_n}{b_n - a_n}$.

We see that the analyticity of (a_n, b_n, c_n) on V implies that $w \mapsto \lambda_n$ and $w \mapsto w_n$ are again analytic on V. Thus, by the inductive hypothesis, equation (2.1) gives us that $(a_{n+1}, b_{n+1}, c_{n+1})$ has an analytic dependence for w in V.

To complete the proof it suffices to show that $a_n(w)$ converges to an analytic function on V. We begin with the following simple lemma.

Lemma 3.

1. *The sequence* $w_n \to \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ *as* $n \to +\infty$. *Moreover, there exists* $N \geq 1$ *such that for any* $n \geq N$ *we have that* $w_n \in \mathcal{U}_0$.
2. *For any* $\frac{1}{2} < \lambda < 1$ *we can choose* N *sufficiently large that for* $n \geq N$ *we have* $\lambda_n \leq \lambda^n$.

Proof. The first part follows from Proposition 1, since the triangles T_n approach equilateral triangles. The second part follows from Corollary 1.1. We need to be slightly careful to remember that we are considering a neighbourhood in the complexification, rather than in $\mathcal{H}$, but the contraction implicit in the statement leads to the same conclusion. $\square$

By Lemma 3 (1) and Lemma 2, for $n \geq N$ we can bound $|a_{n+1} - a_n| \leq |\lambda_n| K$ and thus, Lemma 3 (2) , we see that $|a_n - a_{n+1}| \leq K\lambda^n$. In particular, for $N \leq m < n$ we can write

$$|a_n - a_m| \leq K \sum_{k=m+1}^{n} \lambda^k \leq \frac{K\lambda^m}{1 - \lambda}$$

and deduce that $a_n(\cdot) : V \to \mathbb{C}^2$, for $n \geq 0$, are a uniformly converging family of analytic functions. We can denote the uniform limit $f(w_1, w_2) = \lim_{n \to +\infty} a_n(w_1, w_2)$, which is again analytic. More precisely, using Cauchy's theorem (for functions of two complex variables) the function $f(w_1, w_2)$, can be written in the form:

$$\begin{aligned} f(w_1, w_2) &= \lim_{n\to+\infty} a_n(w_1, w_2) \\ &= \lim_{n\to+\infty} \frac{1}{2\pi i} \int_{\partial V_1} \int_{\partial V_2} \frac{a_n(\xi_1, \xi_2)}{(u-\xi_1)(v-\xi_2)} d\xi_1 d\xi_2 \\ &= \frac{1}{2\pi i} \int_{\partial V_1} \int_{\partial V_2} \left(\lim_{n\to+\infty} a_n(\xi_1, \xi_2) \right) \frac{1}{(u-\xi_1)(v-\xi_2)} d\xi_1 d\xi_2. \end{aligned}$$

However, this last expression is seen to be analytic as a function of w.

The proof of Theorem 1 for $\mathbb{H}^2$ is very similar to that of $\mathbb{R}^2$, although the notation is more complicated since a key difference is that the geodesic triangles are no longer similar under homothetic scaling. Therefore, in this case we can make such a convenient choice of coordinates and we need to consider $V^{(1)}, V^{(2)}, V^{(3)}$ to be sufficiently small neighbourhoods of each of the initial vertices a_0, b_0, c_0, respectively. One can see that $(a_{n+1}, b_{n+1}, c_{n+1})$ has an analytic dependence on (a_n, b_n, c_n). By induction, it has an analytic dependence on $(a_0, b_0, c_0) \in V^{(1)} \times V^{(2)} \times V^{(3)}$. As the hyperbolic triangles get smaller they tend to approximate Euclidean triangles. In particular, the analogue of Corollary 1.1 still holds and the basic iterative construction described above for the Euclidean case again applies. Thus we again we have that given $\frac{1}{2} < \lambda < 1$ we have that $\mathrm{diam}(T_n) \le \lambda \mathrm{diam}(T_{n+1})$, for n sufficiently large. □

3. The Tetrahedron algorithm in Euclidean space

Consider a tetrahedron $\mathcal{T}$ with vertices $\{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^3$ and let $\mathcal{S}$ be the associated insphere. If we write $\mathcal{S} = \{v \in \mathbb{R}^3 : ||v - p|| = r\}$ then p is called the *incentre* and r is called the *inradius*. We recall some simple results.

Lemma 4. *We let $v_{ij} = v_i - v_j$, with $1 \le i \ne j \le 4$.*

1. *The volume is given by $V(\mathcal{T}) = \frac{1}{6} v_{il}.(v_{jl} \times v_{kl})$.*
2. *The area A_{ijk} of the face with vertices $\{v_i, v_j, v_k\}$ is $\frac{1}{2}||v_{ik} \times v_{jk}||$.*
3. *The inradius is given by $r = 3\frac{\mathrm{V}(\mathcal{T})}{\mathrm{A}(\mathcal{T})}$.*
4. *The incentre takes the form $p = \frac{r}{V}\left(v_{14}A_{234} + v_{24}A_{124} + v_{34}A_{124}\right)$.*

Proof. Parts (1) and (2) are standard. Part (3) is easily seen by decomposing the tetrahedron into four subtetrahedra, each sharing one of the original faces but having the incentre as a new vertex. Part (4) comes from observing that the incentre lies on a line equidistant from three given faces [6] □

We can identify the four points $C \cap \mathcal{T}$ where the insphere touches the tetrahedron by the following lemma.

Lemma 5. *We can write $C \cap \mathcal{T} = p + \{rv_1', rv_2', rv_3', rv_4'\}$ where*

$$v_1' = \mp \frac{v_{24} \times v_{21}}{||v_{24} \times v_{21}||}, v_2' = \pm \frac{v_{14} \times v_{34}}{||v_{14} \times v_{34}||}, v_3' = \pm \frac{v_{34} \times v_{24}}{||v_{33} \times v_{24}||}, v_4' = \mp \frac{v_{21} \times v_{32}}{||v_{21} \times v_{32}||}.$$

Proof. The vectors joining the incentre p to the points in $C \cap \mathcal{T}$ correspond to radii for the incircle. In particular, the vectors have length one and are perpendicular to the faces of the tetrahedron. There are two cases, depending on the orientation of the choice of labelling of the vertices (e.g., the sign of $(v_{24} \times v_{14}).v_{34}$). □

There is an even simpler inductive method to construct the sequence $\{\mathcal{T}_n\}$.

Proposition 6. *The tetrahedra $\mathcal{T}_{n+2}$ is similar to the tetrahedron whose vertices are given by the projection of the vertices $\mathcal{T}_n$ onto its own insphere.*

Proof. Let us denote by $\{v_1, v_2, v_3, v_4\}$ the vertices of $\mathcal{T}_0$. By Lemma 4, the vertices of $\mathcal{T}_1$ are $p + r\{v_1', v_2', v_3', v_4'\}$, where p and r are the incentre and the inradius of $\mathcal{T}_1$. Thus, by applying the lemma a second time, we see that the vertices of $\mathcal{T}_2$ are

$$p' + \{r'v_1'', r'v_2'', r'v_3'', r'v_4''\}\,.$$

For sufficiently regular tetrahedra that the the centre of the sphere containing the vertices of $\mathcal{T}_0$ (the outcentre $\mathcal{T}_0$) lies inside $\mathcal{T}_1$. Without loss of generality, let us assume for convenience that $v_4 = 0$, otherwise we can translate the tetrahedron (since the formulae in Lemma 5 are invariant under translation). We can use Lemma 5 to write, for example,

$$v_4'' = \left(\frac{v_1 \times v_3}{||v_1 \times v_3||} - \frac{v_2 \times v_1}{||v_2 \times v_1||}\right) \times \left(\frac{v_3 \times v_2}{||v_3 \times v_2||} - \frac{v_1 \times v_3}{||v_1 \times v_3||}\right).$$

We can expand out these cross products to write

$$\begin{aligned} v_4'' &= \frac{v_1.(v_2 \times v_3)}{||v_1 \times v_3||.||v_1 \times v_3||.||v_1 \times v_3||}(||v_2 \times v_3||v_1 + ||v_1 \times v_3||v_2 + ||v_1 \times v_2||v_3) \\ &= \frac{v_1.(v_2 \times v_3)}{||v_1 \times v_3||.||v_1 \times v_3||.||v_1 \times v_3||}\frac{V(\mathcal{T})}{r}p, \end{aligned}$$

by Lemma 4. In order to compare all four new vertices, it is convenient to make a second translation of the tetrahedron so that p is now at the origin, and we see that v_4'' corresponds to v_4 scaled by a factor of

$$\lambda = \frac{v_1.(v_2 \times v_3)}{||v_1 \times v_3||.||v_1 \times v_3||.||v_1 \times v_3||}\frac{V(\mathcal{T})}{r}.$$

A similar calculation applies to v_2'', v_3'', v_4''. $\square$

It is easy to deduce the following weak analogue of Corollary 1.1.

Corollary 6.1. *There exists $0 < \lambda < 1$ such that $diam(\mathcal{T}_{n+1}) \leq \lambda diam(\mathcal{T}_n)$ for all n sufficiently large.*

Example 3.1. Let us consider a tetrahedron $\mathcal{T}_0$ whose vertices are $v_1 = (1, 0, 0)$, $v_2 = (0.87, 0.48, 0)$, $v_3 = (-0.98, 0.20, 0)$ and $v_4 = (0, 0, 1)$. The easiest way to compare the tetrahedra $\{\mathcal{T}_n\}_{n=0}^{\infty}$ experimentally is to compute the six edge lengths for the first few tetrahedra.

n	Lengths of edges of $\mathcal{T}_n$
0	{0.49, 1.99, 1.41, 1.87, 1.41, 1.41 }
1	{1.97, 1.18, 1.62, 1.62, 1.34, 1.84 }
2	{1.00, 1.91, 1.82, 1.82, 1.89, 1.05 }
3	{1.92, 1.27, 1.63, 1.63, 1.28, 1.91 }
4	{1.02, 1.90, 1.82, 1.82, 1.90, 1.02 }

Table . The lengths of the six edges for each of the tetrahedra $\mathcal{T}_n$, for $n = 0, 1, \ldots, 4$

Example 3.2. The regular tetrahedron has vertices

$$\left\{(0,0,1)\,,\left(0,\frac{2\sqrt{2}}{3},-\frac{1}{3}\right),\left(\sqrt{\frac{2}{3}},-\frac{\sqrt{2}}{3},-\frac{1}{3}\right),\left(-\sqrt{\frac{2}{3}},-\frac{\sqrt{2}}{3},-\frac{1}{3}\right)\right\}$$

If we take $\mathcal{T}_1$ to be this tetrahedron then each tetrahedron $\mathcal{T}_n$ is regular. Moreover, the each incentre for each insphere $\mathcal{S}_n$ is the origin $(0,0,0) \in \mathbb{R}^3$. Thus we have that $w(\mathcal{T}_1) = (0,0,0)$

Example 3.3. Consider the family of tetrahedrons $\mathcal{T}_t$ corresponding to replacing the vertex $v_1 = (0,0,1)$ in the regular tetrahedron in Example 3.2 by $(0,0,t)$. By symmetry, on can see that $w(\mathcal{T}_t)$ must lie on the line $x = 0$, $y = 0$. In Figure 3 we plot the z-cordinate for a range of values.

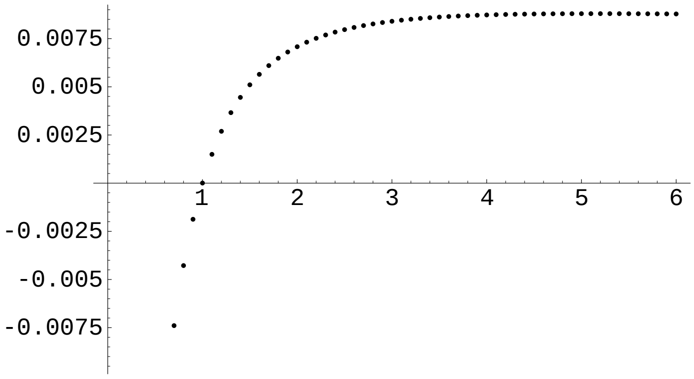

FIGURE 5. A plot of the height of $w(\mathcal{T}_t)$ where $0 < t < 6$ are chosen with a spacing size 0.1

Example 3.4. Consider the family of tetrahedrons $\mathcal{T}_{x,y}$ corresponding to replacing the vertex $(0,0,1)$ in the regular tetrahedron in Example 3.2 by $(x,y,1)$. In Figure 5, we plot the z coordinate for a range of values of x and y.

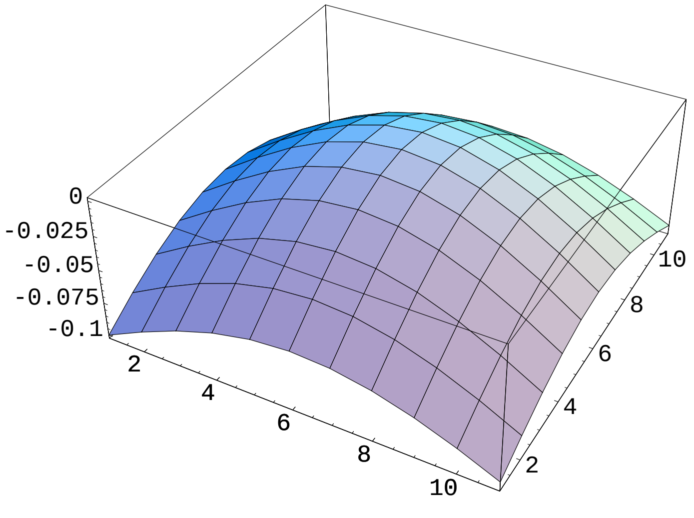

FIGURE 6. A plot of the z-coordinate of $w(\mathcal{T}_{x,y})$ where $-1 < x, y < 1$ are chosen in a grid of size 0.2.

4. PROOF OF THEOREM 2

Proof of Theorem 2. For the case of $\mathbb{R}^3$ this is very similar to the proof of Theorem 1. Let $X = \{(v_1, v_2, v_3, v_4) : v_1, v_2, v_3, v_4$ are non-coplanar$\}$ then we have a well defined map $S : X \to X$ defined by $S(a_n, b_n, c_n, d_n) = (a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1})$. From the construction we see that S is equivariant under similarities. In particular, if we identify $\mathbb{R}^3 = \mathbb{C} \oplus j\mathbb{R}$ then we can consider a canonical choice for the initial tetrahedra of the form $\mathcal{T}_1 = \{0, 1, z, w + jt\}$ there $z \in \mathbb{C} \ominus \mathbb{R}$, $w \in \mathbb{C}$ and $t > 0$.

Whereas in the proof of Theorem 1 the triangles $\{T_n\}_{n=0}^{\infty}$ approach equilateral triangles, in the case of Euclidean tetrahedrons $\{\mathcal{T}_n\}_{n=0}^{\infty}$ the tetrahedra $\mathcal{T}_n$ may not similarly converge. However, in the method of proof of Theorem 1 what was really used was that the triangles have diameters which contract exponentially quickly (Corollary 1.1). The corresponding property holds for tetrahedron (Corollary 6.1), the same method of proof applies.

For the case of $\mathbb{H}^3$ and hyperbolic tetrahedrons $\{\mathcal{T}_n\}_{n=0}^{\infty}$, the tetrahedra $\mathcal{T}_n$ approximate Euclidean tetrahedra as they get smaller. In particular, they will be bounded away from the degenerate cases and the same method or proof applies yet again. ☐

REFERENCES

1. A. Beardon, *The Geometry of Discrete Groups*, Springer, Berlin, 1983.
2. H. Croft, K. Falconer and R. Guy, *Unsolved Problems in Geometry*, Springer, Berlin, 1991.
3. H. Coxeter, *Geoemtry revisited*, MAA, New York, 1996.
4. W. Dunham, *Journey through Genius*, Penguin, London, 1991.
5. D. Fried, *Poncelet Points*, Unpublished manuscript.
6. J. Kingston and J. Synge, *The sequence of pedal triangles*, Am. Math. Mon. **95** (1998), 609-620.
7. P. Lax, *The ergodic character of sequences of pedal triangles*, Am. Math. Mon. **97** (1990), 377-381.
8. E. C. Zeeman, *Appendix 11: 3dimensional geometry*, Teaching and learning geometry, Science Policy - Reports and Statements, Royal Society, London, 2001.

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