

Analysis in the 4th year at Warwick

¹Department of Mathematics, Warwick University,
<http://www.maths.warwick.ac.uk/~masdbl>

October 2, 2009

OverView

(Over) Simplifying things, there are three main themes in the 4th year analysis courses

Year	Dynamics	Stochastics	Classical Analysis
4	Dynamical Systems Ergodic Theory	Stochastic Analysis Brownian Motion	Advanced PDE Fourier Analysis
PhD TCC	Teichmuller Dynamics Homogeneous flows		Topics in Geom. Anal. Sobolev Spaces

TCC = Courses broadcasts between Bath, Bristol, Imperial, Oxford, Warwick

Connections to: *Riemann surfaces, Differential Geometry, Hyperbolic Geometry, Complexity, ...*

Geometric and Classical Analysis

Common themes:

- Understanding the regularity/differentiability of functions (Fourier Analysis);
- Analyzing the existence and uniqueness of solutions of PDEs on different domains (Advanced PDEs)

But requires useful classes of functions: Sobolev functions are Hilbert spaces of differentiable functions.

Example: An Elliptic PDE is the Laplacian $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Eigenfunctions/Eigenvalues on torus: $\nabla\psi_{n,m} = \lambda_{n,m}\psi_{n,m}$ where

$$\nabla\psi_{m,m} = e^{2\pi i(nx+my)} \text{ and } \lambda_{n,m} = -4\pi^2(n^2 + m^2)$$

Clearly, for some $C > 0$

$$\text{Card}\{|\lambda_{n,m}| \leq T\} \sim CT \text{ as } T \rightarrow +\infty.$$

This is a special case of **Weyl's Theorem** (One of the first Hilbert Problems to be solved)

Geometric and Classical Analysis

Common themes:

- Understanding the regularity/differentiability of functions (Fourier Analysis);
- Analyzing the existence and uniqueness of solutions of PDEs on different domains (Advanced PDEs)

But requires useful classes of functions: Sobolev functions are Hilbert spaces of differentiable functions.

Example: An Elliptic PDE is the Laplacian $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Eigenfunctions/Eigenvalues on torus: $\nabla \psi_{n,m} = \lambda_{n,m} \psi_{n,m}$ where

$$\nabla \psi_{m,m} = e^{2\pi i(nx+my)} \text{ and } \lambda_{n,m} = -4\pi^2(n^2 + m^2)$$

Clearly, for some $C > 0$

$$\text{Card}\{|\lambda_{n,m}| \leq T\} \sim CT \text{ as } T \rightarrow +\infty.$$

This is a special case of **Weyl's Theorem** (One of the first Hilbert Problems to be solved)

Geometric and Classical Analysis

Common themes:

- Understanding the regularity/differentiability of functions (Fourier Analysis);
- Analyzing the existence and uniqueness of solutions of PDEs on different domains (Advanced PDEs)

But requires useful classes of functions: Sobolev functions are Hilbert spaces of differentiable functions.

Example: An Elliptic PDE is the Laplacian $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Eigenfunctions/Eigenvalues on torus: $\nabla\psi_{n,m} = \lambda_{n,m}\psi_{n,m}$ where

$$\nabla\psi_{m,m} = e^{2\pi i(nx+my)} \text{ and } \lambda_{n,m} = -4\pi^2(n^2 + m^2)$$

Clearly, for some $C > 0$

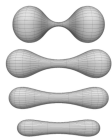
$$\text{Card}\{|\lambda_{n,m}| \leq T\} \sim CT \text{ as } T \rightarrow +\infty.$$

This is a special case of **Weyl's Theorem** (One of the first Hilbert Problems to be solved)

Application (Ricci flows: in Geometric Analysis): Solve differential equations for metrics g on manifolds:

$$\frac{\partial g}{\partial t} = -2R(g) \text{ where } R(g) = \text{Ricci curvature}$$

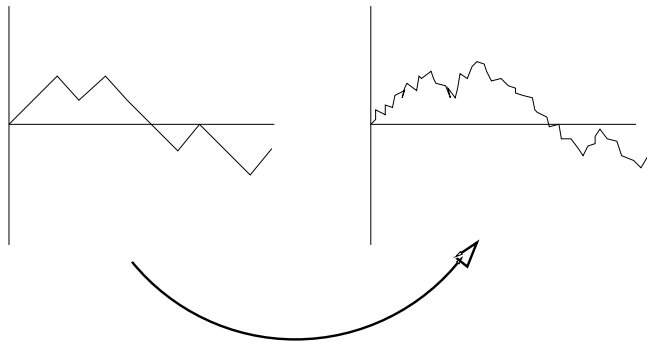
In three dimensions solves the Poincaré Conjecture (Solved by Perelman)



Stochastic Analysis and Brownian Motion

Stochastic Analysis deals with measures on finite dimensional spaces.

- Develop a theory of integration (Ito integral in Stochastic Analysis)
- Basic object is the space of paths (Brownian Motion)

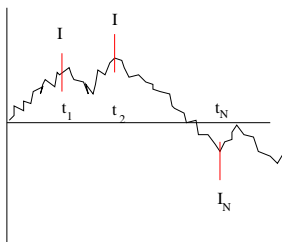


We need to define the measure (or “size”) of suitable sets (giving rise to the σ -algebra).

Let $I_1, \dots, I_N \subset \mathbb{R}$ be intervals. Let $t_1 < \dots < t_N$ be times.

We can consider the set of paths $B(t) \in C([0, \infty), \mathbb{R})$ such that

$$\{f(t) \in C([0, \infty), \mathbb{R}) : f(t_1) \in I_1, \dots, f(t_N) \in I_N\}$$



The Wiener measure gives an explicit formula for the measure of such sets.

Dynamical Systems and Ergodic Theory

Consider a map $T : X \rightarrow X$ on a space X . We are interested in the orbits

$$x, Tx, T^2x, \dots, T^n x, \dots$$

of points $x \in X$.

- For continuous maps on can consider the recurrence of orbits (Dynamical Systems)
- For maps preserving a measure μ we can consider the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \text{ for } f \in L^1(X, \mu)$$

which converges for almost all points $x \in X$ (Ergodic Theory)

Applications to Number Theory

(i) van der Waerden's Theorem (1926): Consider any finite partition $\mathbb{N} = B_1 \cup \dots \cup B_k$ (with $B_i \cap B_j = \emptyset$) for $i \neq j$ then at least one set B_i contains arithmetic progressions of arbitrary length (i.e., $\forall l \in \mathbb{N}, \exists a, b \in \mathbb{N}$ such that $a + kb \in B_i$, for $k = 0, \dots, l - 1$).

Dynamical proof by Furstenberg-Weiss (1975). Generalizations to "fat" subsets of \mathbb{N} : Szemerédi (1975), Furstenberg (1976). Generalizations to prime numbers: Green-Tao (2004)

(ii) Oppenheim Conjecture (1929): Let $Q(n_1, n_2, n_3) = n_1^2 + n_2^2 - \sqrt{2}n_3^2$ (irrational indefinite quadratic form). Then

$$\{Q(n_1, n_2, n_3) : n_1, n_2, n_3 \in \mathbb{N}\} \subset \mathbb{R}$$

is actually dense in \mathbb{R} .

Proved by Margulis in 1987. Proof uses ergodic theory of flows in spaces of matrices.

Applications to Number Theory

(i) van der Waerden's Theorem (1926): Consider any finite partition $\mathbb{N} = B_1 \cup \dots \cup B_k$ (with $B_i \cap B_j = \emptyset$) for $i \neq j$ then at least one set B_i contains arithmetic progressions of arbitrary length (i.e., $\forall l \in \mathbb{N}, \exists a, b \in \mathbb{N}$ such that $a + kb \in B_i$, for $k = 0, \dots, l - 1$).

Dynamical proof by Furstenberg-Weiss (1975). Generalizations to "fat" subsets of \mathbb{N} : Szemerédi (1975), Furstenberg (1976). Generalizations to prime numbers: Green-Tao (2004)

(ii) Oppenheim Conjecture (1929): Let $Q(n_1, n_2, n_3) = n_1^2 + n_2^2 - \sqrt{2}n_3^2$ (irrational indefinite quadratic form). Then

$$\{Q(n_1, n_2, n_3) : n_1, n_2, n_3 \in \mathbb{N}\} \subset \mathbb{R}$$

is actually dense in \mathbb{R} .

Proved by Margulis in 1987. Proof uses ergodic theory of flows in spaces of matrices.

Applications to Number Theory

(i) van der Waerden's Theorem (1926): Consider any finite partition $\mathbb{N} = B_1 \cup \dots \cup B_k$ (with $B_i \cap B_j = \emptyset$ for $i \neq j$) then at least one set B_i contains arithmetic progressions of arbitrary length (i.e., $\forall l \in \mathbb{N}, \exists a, b \in \mathbb{N}$ such that $a + kb \in B_i$, for $k = 0, \dots, l - 1$).

Dynamical proof by Furstenberg-Weiss (1975). Generalizations to "fat" subsets of \mathbb{N} : Szemerédi (1975), Furstenberg (1976). Generalizations to prime numbers: Green-Tao (2004)

(ii) Oppenheim Conjecture (1929): Let $Q(n_1, n_2, n_3) = n_1^2 + n_2^2 - \sqrt{2}n_3^2$ (irrational indefinite quadratic form). Then

$$\{Q(n_1, n_2, n_3) : n_1, n_2, n_3 \in \mathbb{N}\} \subset \mathbb{R}$$

is actually dense in \mathbb{R} .

Proved by Margulis in 1987. Proof uses ergodic theory of flows in spaces of matrices.

Applications to Number Theory

(i) van der Waerden's Theorem (1926): Consider any finite partition $\mathbb{N} = B_1 \cup \dots \cup B_k$ (with $B_i \cap B_j = \emptyset$) for $i \neq j$ then at least one set B_i contains arithmetic progressions of arbitrary length (i.e., $\forall l \in \mathbb{N}, \exists a, b \in \mathbb{N}$ such that $a + kb \in B_i$, for $k = 0, \dots, l - 1$).
Dynamical proof by Furstenberg-Weiss (1975). Generalizations to "fat" subsets of \mathbb{N} : Szemerédi (1975), Furstenberg (1976). Generalizations to prime numbers: Green-Tao (2004)

(ii) Oppenheim Conjecture (1929): Let $Q(n_1, n_2, n_3) = n_1^2 + n_2^2 - \sqrt{2}n_3^2$ (irrational indefinite quadratic form). Then

$$\{Q(n_1, n_2, n_3) : n_1, n_2, n_3 \in \mathbb{N}\} \subset \mathbb{R}$$

is actually dense in \mathbb{R} .

Proved by Margulis in 1987. Proof uses ergodic theory of flows in spaces of matrices.