The subadditive ergodic theorem

Let μ be a T-invariant probability measure. Let $f_n(x) \in L^1(X,\mu)$, $n \geq 1$, be a subadditive sequence of functions i.e.:

$$f_{n+m}(x) \leq f_n(x) + f_m(T^n x)$$

Example: Let $M: X \longrightarrow SL(2,\mathbb{R})$ be a measurable function.

We let $\|A\| = \sup\{\|Ax\|_2 \mid \|x\|_2 \leq 1\}$

We then define:

$$f_n(x) = \log \|M(x)M(Tx)\cdots M(T^{n-1}x)\|_2$$

and assume: $\int \log\|M(x)\|\, d\mu < +\infty$.

Theorem (Subadditive ergodic theorem, Kingman '67).

Assume μ is ergodic. Then

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = \ell := \inf\left\{\int f_n\, d\mu / n \;\middle|\; n \geq 1\right\}$$

for a.e. (μ) $x \in X$.

(Assume that $\|f_1\|_\infty < +\infty$, for simplicity).

Proof: We define $\begin{cases} \overline{f}(x) = \overline{\lim}_{n\to\infty} f_n(x)/n & \text{a.e.}(\mu)\, x \in X. \\ \underline{f}(x) = \underline{\lim}_{n\to\infty} f_n(x)/n \end{cases}$

Since

$$\underbrace{\underline{\lim}_{n\to\infty} \frac{f_{n+1}(x)}{n+1}}_{=\underline{f}(x)} \leq \underbrace{\underline{\lim}_{n\to\infty} \frac{f_1(x)}{n+1}}_{=0} + \underbrace{\underline{\lim}_{n\to\infty} \frac{f_n(Tx)}{n+1}}_{\underline{f}(Tx)}$$

Thus $\underline{f}(x) \leq \underline{f}(Tx)$ and thus by ergodicity $\underline{f}$ is constant. Similarly, $\overline{f}(x) \leq \overline{f}(Tx)$ and $\overline{f}$ is constant.

Fix $N \geq 1$ and choose $1 \leq i \leq N$.

We can then write any $n = i + mN + r$ where $0 \leq r \leq N-1$ and $m \geq 0$.

By subadditivity,

$$f_n(x) \leq f_i(x) + \sum_{j=0}^{m-1} f_N(T^{jN+i}x) + f_r(T^{mN+i}x).$$

Summing over $1 \leq i \leq N$ gives:

$$Nf_n(x) \leq \sum_{i=1}^{N} f_i(x) + \underbrace{\sum_{i=1}^{N}\sum_{j=0}^{m-1} f_N(T^{jN+i}x)}_{\sum_{k=0}^{n-1} f_N(T^k x)}$$

$$+ \sum_{i=1}^{N} f_{\underbrace{n-i-mN}_{=k}}(T^{mN+i}x)$$

Dividing by nN and letting $n \to +\infty$ gives

$$\underbrace{\overline{\lim_{n\to\infty}} f_n(x)/n}_{=\overline{f}(x)} \leq \underbrace{\overline{\lim_{n\to\infty}} \frac{\|f_1\|_\infty}{n}}_{=0} + \underbrace{\frac{1}{N}\overline{\lim_{n\to\infty}} \frac{f_N^n(x)}{N}}_{=\frac{1}{N}\int f_N d\mu \text{ (Birkhoff ergodic Thm)}} + \underbrace{\overline{\lim_{n\to\infty}} \frac{\|f_1\|_\infty}{n}}_{=0},$$

for any $N \geq 1$.

Thus $\overline{f}(x) \leq \ell = \inf\left\{\frac{1}{N}\int f_N d\mu \mid N \geq 1\right\}$.

Next we turn to $\underline{f}$

Moreover, we may as well assume $l > -\infty$.

Fix $\varepsilon > 0$. Let us define $n : X \longrightarrow \mathbb{N}$

by $n(x) = \inf\{n \geq 1 \mid f_n(x) \leq n(\underline{f}+\varepsilon)\}$

Fix $M > 0$ Let us define $A = \{x \mid n(x) \geq M\}$.

<u>Claim</u>: For $n \geq 1$,

$$f_n(x) \leq n(\underline{f}+\varepsilon) + \sum_{i=0}^{n-1} \chi_A(T^i x)\|f_1\|_\infty + M\|f_1\|_\infty$$

for a.e. (μ) $x \in X$.

<u>Proof of claim</u> (Similar to the proof of the Birkhoff Ergodic Theorem).

We can cover the set $\{1, 2, \ldots, n-1\}$ by the sets:

1) $\{k : T^k x \in A\}$;
2) $\{l, l+1, \ldots, l+n(T^l x)-1\}$; or
3) $\{n-M, \ldots, n-1\}$

This leads to the claim $\square$

Thus:

$$\frac{f_n(x)}{n} \leq (\underline{f}+\varepsilon) + \frac{1}{n}\sum_{i=0}^{n-1} \chi_A(T^i x)\|f_1\|_\infty + \frac{M}{n}\|f_1\|_\infty$$

Integrating this inequality gives:

$$\frac{1}{n}\int f_n \, d\mu \leq (\underline{f}+\varepsilon) + \mu(A) \cdot \|f_1\|_\infty + \frac{M}{n}\|f_1\|_\infty$$

for any $n \geq 1$.

First we let $n \to \infty$.

Next we let $M \to \infty$ which in turn implies that $\mu(A) \to 0$

Finally, we let $\varepsilon \to 0$. We can then deduce

$$l \leq \underline{f} \quad \text{a.e.}(\mu)\ x \in X.$$

Therefore $\overline{f} = \underline{f} = l$, ie, $l = \lim_{n \to \infty} f_n(x)/n$.