Markov operators + Ergodic theorems

Let $T_1, \dots, T_n : (X,\mu) \to (X,\mu)$ be measure preserving transformations on a probability space (ie, $A \in \mathcal{B}$, $\mu_i(T_i^{-1}A) = \mu_i(A)$ for $i = 1, \dots, n$).

Let $(p_1, \dots, p_n)$ be a probability vector (ie, $p_1 + \dots + p_n = 1$).

We define a linear operator:

$$\begin{cases} T : L^1(X,\mu) \longrightarrow L^1(X,\mu) \\ Tf(x) = \sum_{i=1}^{n} p_i f(T_i x) , \quad \forall f \in L^1(X,\mu) \\ \qquad\qquad\qquad\qquad\qquad\qquad a.e.(\mu)\ x \in X \end{cases},$$

More generally, we call $T : L^1(X,\mu) \to L^1(X,\mu)$ a Dunford-Schwartz operator if:

$$\|Tf\|_1 \leq \|f\|_1 \text{ and } \|Tf\|_\infty \leq \|f\|_\infty .$$

In addition, we say

(i) T is positive if $f_1(x) \leq f_2(x) \Rightarrow Tf_1(x) \leq Tf_2(x)$ a.e.

(ii) T is Markov if $T1 = 1$ (= constant f^n)

Finally, we say that T is ergodic if

$$T\chi_A = \chi_A, A \in \mathcal{B} \Rightarrow \mu(A) = 0 \text{ or } \mu(A) = 1$$

Example: We say that $\{T_1, \dots, T_n\}$ are ergodic if whenever $A \in \mathcal{B}$ satisfies $T_i^{-1}A = A$, $i = 1, \dots, n$ then $\mu(A) = 0$ or $\mu(A) = 1$.

If $T\chi_A(x) = \chi_A(x)$ then by convexity $T_i^{-1}A = A$ for $i=1,\dots,n$ and thus $\mu(A) = 0$ or 1, ie, the operator T is ergodic.

Theorem (Hopf-Dunford-Schwartz)

If $T : L^1(X,\mu) \to L^1(X,\mu)$ is ergodic then

$$\frac{1}{n}\sum_{k=0}^{n-1} T^k\phi(x) \xrightarrow[n\to\infty]{} \int \phi \, d\mu \quad \text{a.e.}(\mu)\ x\in X$$

and $\phi \in L^1(X,\mu)$.

Remark: This subsumes the Birkhoff ergodic theorem by letting $n=1$ and $p_1 = 1$

We begin with the following easy lemma:

Lemma 1: Let $\mathrm{Fix}(T) = \{ g \in L^\infty(X,\mu) \mid Tg = g \}$ then $B := \mathrm{Fix}(T) \oplus (I-T)L^\infty(X,\mu) \subseteq L^1(X,\mu)$ is dense.

(Proof postponed).

Definition: Let $S_n\phi(x) := \sum_{k=0}^{n-1} (T^k\phi)(x)$, $n \geq 0$

Lemma 2: If $\phi \in B$ then $\frac{1}{n}S_n\phi$ converges in $\|\cdot\|_\infty$

Proof. Let $f = g + (1-T)h$; $g \in \mathrm{Fix}(T)$, $h \in L^\infty(X,\mu)$

Then $\frac{1}{n}S_n\phi = g + \frac{1}{n}(h - T^n h)$ and

$$\left\| \frac{1}{n}S_n\phi - g \right\|_\infty \leq \frac{\|h\|_\infty + \|T^n h\|_\infty}{n} \leq \frac{2\|h\|_\infty}{n} \to 0$$

as $n\to\infty$ $\square$

Moreover, since T is ergodic we see that $\mathrm{Fix}(T)$ consists of constant functions (ie, $\phi \in B : \frac{1}{n} S_n f \to \int f \, d\mu$).

<u>Aim</u>: We want to extend the convergence to the L^1-closure of B.
(Then the ergodic theorem follows by Lemma 1).

<u>Definition</u>: We associate a maximal "operator" for $f \in L^1(X,\mu)$ by:
$$Mf(x) = \sup_n \left| \tfrac{1}{n} S_n f(x) \right|, \quad f \in L^1(X,\mu)$$

<u>Lemma 3</u>. (i) $Mf \geq 0$, $\forall f \in L^1(X,\mu)$
(ii) $M(\alpha f) = |\alpha| \cdot A(f)$, $\forall f \in L^1(X,\mu), \alpha \in \mathbb{C}$
(iii) $M(f+g) \leq Af + Ag$, $\forall f, g \in L^1(X,\mu)$.

(These follows from the definitions).

<u>Definition</u>: We say $(\frac{1}{n} S_n)_{n=1}^{\infty}$ satisfies a maximal inequality if
$$\mu\{x \mid Mf(x) > \lambda\} \leq \frac{\|f\|_1}{\lambda} \qquad (M).$$
for all $\lambda > 0$, $f \in L^1(X,\mu)$.

To achieve our "aim" we use the following:

<u>Lemma 4</u>. (Banach's Principle). Assuming (M) holds we have that
$$C := \{ f \in L^1(X,\mu) \mid (\tfrac{S_n f}{n})_{n=1}^{\infty} \text{ converges a.e} \} \subseteq L^1(X,\mu)$$
is a $\|\cdot\|_1$-closed subspace.

Proof. To see that $\mathcal{C}$ is closed: Let $f \in \overline{\mathcal{C}} \subseteq L^1(X,\mu)$ and $\varepsilon > 0$. Choose $g \in \mathcal{C}$ with $\|g-f\|_1 < \varepsilon$.

By the triangle inequality:

For a.e. (μ) $x \in X$:

$$\left|\frac{1}{k}S_k f(x) - \frac{1}{\ell}S_\ell f(x)\right| \leq \underbrace{\left|\frac{1}{k}S_k f(x) - \frac{1}{k}S_k g(x)\right|}_{\leq M(f-g)(x)} + \underbrace{\left|\frac{1}{k}S_k g(x) - \frac{1}{\ell}S_\ell g(x)\right|}_{\to 0 \text{ as } k,\ell\to\infty} \; (g \in \mathcal{C}) + \underbrace{\left|\frac{1}{\ell}S_\ell g(x) - \frac{1}{\ell}S_\ell f(x)\right|}_{\leq M(f-g)(x)}$$

Thus: $h(x) = \overline{\lim_{k,\ell\to\infty}} \left|\frac{1}{k}S_k f(x) - \frac{1}{\ell}S_\ell f(x)\right| \leq 2M(f-g)(x)$

Fix $\lambda > 0$, then:

$$\mu\{x \mid h(x) > 2\lambda\} \leq \mu\{x \mid M(f-g) > \lambda\}$$
$$\leq \frac{\|f-g\|_1}{\lambda} \qquad (\text{by } (M))$$
$$\leq \varepsilon/\lambda.$$

Since $\varepsilon > 0$ was arbitrary: $\mu\{x \mid h(x) > \lambda\} = 0$

Since $\lambda > 0$ was arbitrary: $\mu\{x \mid h(x) > 0\} = 0$, i.e, $h \equiv 0$ a.e.(μ)

In particular, $f \in \mathcal{C}$ (i.e, we deduce $\mathcal{C}$ is closed) $\square$

Recall: The Theorem follows from Lemma 1, Lemma 2 and Lemma 4.

It remains to show (M) holds. The following is key:

<u>Definition</u> Given $f \in L^1(X,\mu)$ we define

$$M_n f(x) = \max_{1\le k\le n}\{S_k f(x)\}, \quad n \geq 1, \ \text{a.e}(\mu)\ x \in X$$

<u>Lemma 5</u> (Hopf) $\displaystyle\int_{\{x \mid M_n f(x) \geq 0\}} f \, d\mu \geq 0$

<u>Proof</u> By definition, for $k = 2, 3, \ldots, n$:

$$S_{k-1} f(x) \leq M_n f(x) \leq \underbrace{\max\{M_n f(x), 0\}}_{=:(M_n f)_+(x)} \qquad (*)$$

In particular,

$$\begin{cases} S_k f(x) = f(x) + T S_{k-1} f(x) & k = 2, \ldots, n. \\ \qquad \leq f(x) + T M_n f(x) & \left(\text{Since } T \text{ is positive and } (*)\right) \\ S_1 f(x) := f(x) \leq f(x) + \underbrace{T M_n f(x)}_{\geq 0} & \left(\text{Since } (M_n f)_+ \geq 0 \text{ and } T \text{ positive}\right) \end{cases}$$

and thus:

$$M_n f(x) := \max_{1\le k\le n} S_k f(x) \leq f(x) + T(M_n f)_+(x). \qquad (**)$$

Therefore,

$$\int (M_n f)_+ \, d\mu = \int_{\{x \mid M_n f \geq 0\}} M_n f \, d\mu \qquad (\text{by def}^n \text{ of } (M_n f)_+)$$

$$\leq \int_{\{x \mid M_n f(x) \geq 0\}} f \, d\mu + \underbrace{\int_{\{x \mid M_n f(x) \geq 0\}} T(M_n f)_+ \, d\mu}_{\leq \int (M_n f)_+ \, d\mu} \qquad (\text{by } (**)) \quad \left(\text{Since } T \text{ is positive and contracts}\right)$$

Cancelling the first and last terms gives

$$\int_{\{x \mid M_n f(x) \geq 0\}} f \, d\mu \geq 0 \quad (\text{as required}) \qquad \square$$

Finally, this leads to the following

Corollary (Maximal Ergodic Theorem)

Let $A_n f = \max_{1\le k\le n} \left\{ \frac{S_k f(x)}{k} \right\}$, $n \ge 1$, $f \in L^1(X,\mu)$,

then for any $\lambda > 0$:

$$\mu\{x \mid A_n f(x) > \lambda\} \le \|f\|_1 / \lambda$$

(i.e., a "finite" version of (M)).

Proof: We can apply the Hopf Lemma (Lemma 5) to $\bar{f} = f - \lambda \mathbb{1}$ (constant fn value = 1) to get $\int_{\{x \mid M_n \bar{f}(x) \ge 0\}} \bar{f} \, d\mu \ge 0$, which gives

$$\int_{\{x \mid M_n g(x) > 0\}} (f - \lambda \mathbb{1}) \, d\mu \ge 0 \qquad (*)$$

(notice $\ge$ replaced by $>$) (since $\{x \mid M_n g = 0\} \supseteq \{x \mid g(x) \le 0\}$ by def^n of $M_n g$)

Let $1 \le k \le n$. Let $0 \le j \le k-1$:

$$T^j g = T^j f - \lambda T^j \mathbb{1}$$
$$= T^j f - \lambda \mathbb{1}. \qquad (T\mathbb{1} = \mathbb{1}, \text{ Markov property})$$

Averaging over $0 \le j \le k-1$:

$$\frac{1}{k} S_k g = \frac{1}{k} S_k f - \lambda \mathbb{1}$$

Taking supremum over $1 \le k \le n$:

$$A_n g = A_n f - \lambda \qquad (**)$$

and thus:

$$(***) \qquad \{x \mid A_n f(x) > \lambda\} \subseteq \{x \mid A_n g > 0\} \quad (\text{by } (**))$$
$$= \{x \mid M_n g > 0\} \quad (\text{compare def}^n\text{s of } A_n g, M_n g)$$

Thus:

$$0 \leq \int_{\{x \mid M_n g(x) > 0\}} (f - \lambda \mathbb{1})\, d\mu \qquad \text{(by } (*)\text{)}$$

$$= \int_{\{x \mid A_n f(x) > \lambda\}} (f - \lambda \mathbb{1})\, d\mu + \underbrace{\int_{\left\{x \,\middle|\, \substack{M_n g(x) > 0, \\ A_n f(x) \leq \lambda}\right\}} (f - \lambda \mathbb{1})\, d\mu}_{\leq 0 \text{ (since } f \leq A_n f)} \qquad \text{(using } (***)\text{)}$$

Thus:

$$\mu \{x \mid A_n f(x) > \lambda\} \leq \frac{1}{\lambda} \int_{\{x \mid A_n f(x) > \lambda\}} f\, d\mu \leq \frac{\|f\|_1}{\lambda}.$$

as required $\square$

We can now prove that (m) holds.

For $f \in L^1(X, \mu)$ we have

$$\mu \left\{x \,\middle|\, \max_{1 \leq k \leq n} \left| \frac{1}{k} S_k f(x) \right| > \lambda \right\}$$

$$\leq \mu \left\{x \,\middle|\, \max_{1 \leq k \leq n} \left\{ \frac{S_k |f| (x)}{k} \right\} > \lambda \right\} \qquad \text{(since } T \text{ positive)}$$

$$\leq \|f\|_1 / \lambda \qquad \text{(by Corollary)}$$

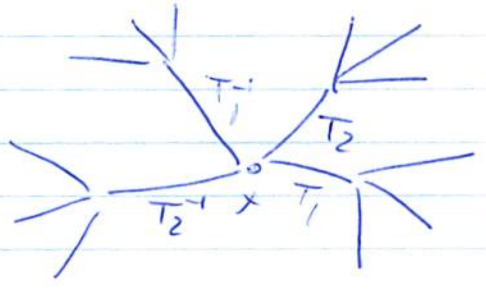
Let $n \longrightarrow +\infty$: $\quad \max_{1 \leq k \leq n} \left\{ \frac{S_k f(x)}{k} \right\} \nearrow \underbrace{\sup_{k \geq 1} \left\{ \frac{S_k f(x)}{k} \right\}}_{=: Mf(x)}.$

Thus: $\mu \{x \mid Mf(x) > \lambda\} \leq \frac{\|f\|_1}{\lambda}$.

(i.e, (m) holds, as required).

Example (Actions by free groups)

- Let $T_1, \ldots, T_k : (X,\mu) \to (X,\mu)$ be invertible measure preserving transformations. (Let $\underset{T_{k+1}}{\overset{\shortparallel}{T_1^{-1}}}, \ldots, \underset{T_{2k}}{\overset{\shortparallel}{T_k^{-1}}} : (X,\mu) \to (X,\mu)$ be the inverses)
- Assume they generate a free group



Let $\underline{i} = (i_1, \ldots, i_n) \in \{1, \ldots, k\}^n$ denote a string for where $T_{i_j} \neq T_{i_{j+1}}^{-1}$

$\#\{\underline{i} \mid T_{i_j} \neq T_{i_{j+1}}^{-1}\} = 2k(2k-1)^{n-1}$

- Consider the spherical averages:

$$\sigma_n \phi(x) = \sum_{|\underline{i}|=n} \frac{\phi(T_{i_1} \ldots T_{i_n} x)}{2k(2k-1)^{n-1}}, \quad n \geq 1$$

Theorem (Nevo-Stein, Bufetov):

$$\frac{1}{N} \sum_{n=1}^{N} \sigma_n \phi(x) \text{ converges as } N \to +\infty,$$
for a.e. (μ) x.

Proof Consider $\hat{T} : (L^1(X,\mu))^{2k} \to (L^1(X,\mu))^{2k}$

$$\left[\hat{T}(\phi_1, \ldots, \phi_{2k})(x)\right]_j \qquad (j = 1, \ldots, 2k)$$
$$= \frac{1}{(2k-1)} \sum_{\substack{i=1 \\ (T_i^{-1} \neq T_j)}}^{2k} \phi_i(T_j x)$$