Ergodic Theory (MA-427)

Lectures $\begin{cases} \text{Tuesday 4-5 (MS.04)} \\ \text{Wednesday 12-1 (B1.01)} \to \text{Tomorrow D1.07} \\ \text{Thursday 2-3 (B3.03)} \end{cases}$

(Support class: Monday, 1-2 (B3.01)

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Suggested reference books

W. Parry, "Topics in Ergodic Theory" CUP, 1982

K. Peterson, "Ergodic Theory" CUP, 1983

Y. Sinai, "Introduction to Ergodic Theory", PUP, 1993

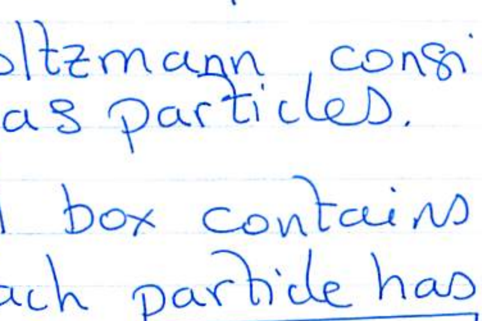
P. Walters, "Introduction to Ergodic Theory" Springer 2000

A historical perspective

Boltzmann considered the dynamics of gas particles.

A box contains $\sim 10^{20}$ particles

Each particle has 6 coordinates $\begin{cases} 3 = \text{position} \\ 3 = \text{velocity vector} \end{cases}$



Let $X = 6.10^{20}$ space of configurations

Let $\begin{cases} T^t : X \to X & \text{flow (evolution of system)} \\ T : X \to X & \text{time-one discrete map} \end{cases}$

preserves volume. (μ = normalized volume)

Boltzmann ergodic hypothesis:

"Spatial averages = time averages"

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \le k \le n \mid T^k x \in B\} = \mu(B).$$

Not always true, but introduced terminology

$$\text{"Ergodic"} = \underbrace{\text{"work" + "path"}}_{\text{(in Greek)}}.$$

Recurrence and Zermelo Paradox

Consider gas particles in a closed room

Let $\begin{cases} X = \text{phase space (all configurations)} \\ \mu = \text{normalized volume.} \end{cases}$

How frequently do all of the particles move to one half of the room (and suffocate the rest of the audience)?

$\sim 10^{20}$ particles $\Rightarrow$ $\sim 10^{18}$ seconds ($\gtrsim$ age universe)

Mathematical model.

Consider a polygon in the plane: P

Let T = (time one) flow for a particle moving at unit speed with elastic collisions on the boundary ∂P

m = Area × direction $(dx\, dy\, d\theta)$

Claim: For generic polygons T is "ergodic"

Main themes of course:

1) Understand the long term behaviour of typical orbits (Ergodic Theorems)
2) Consider subtler behaviour exhibited by some transformations (mixing, etc)
3) Classification of different transformations with similar properties (isomorphism + entropy)
4) Applications to problems in other parts of mathematics.

Example (Normal numbers). Let $b \geq 2$ and assume $0 \leq x \leq 1$ has an expansion:

$$x = \frac{i_1}{b} + \frac{i_2}{b^2} + \frac{i_3}{b^3} + \dots + \frac{i_n}{b^n} + \dots$$

where $i_n \in \{0, 1, \dots, b-1\}$

We say x is normal (base b) if each digit $j \in \{0, \dots, b-1\}$ occurs with frequency $1/b$:
i.e., $\lim_{N \to \infty} \frac{1}{N} \# \{1 \leq n \leq N \mid i_n = j\} = 1/b$

(eg. $b = 10$: Decimal expansion)

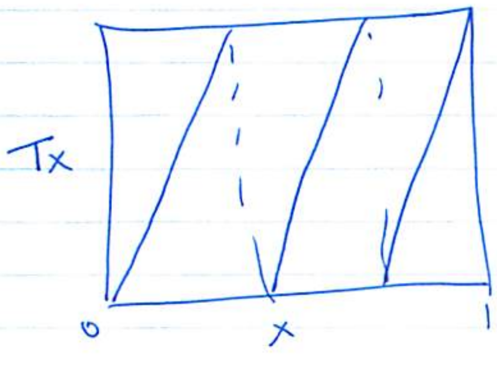
We say x is normal if it is normal (base b) for every $b \geq 2$.

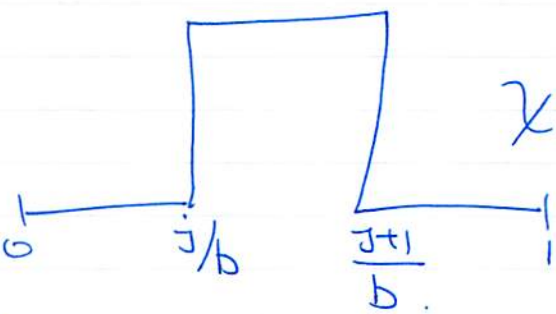
Theorem Almost every $x \in (0,1)$ is normal (i.e., the set of x which are not normal has zero Lebesgue measure)

Ergodic Theory formulation

$$\frac{1}{N} \# \{ 1 \le n \le N \mid i_n = j \}$$
$$= \frac{1}{N} \sum_{n=1}^{N} \chi(T^n x)$$

where
$$\begin{cases} T : [0,1) \to [0,1) \\ Tx = dx \quad (\mathrm{mod}\ 1) \\ \chi(x) = \begin{cases} 1 & \text{if } j/b \le x < \frac{j+1}{b}. \\ 0 & \text{otherwise}. \end{cases} \end{cases}$$





Birkhoff ergodic theorem. For a.e (Lebesgue) $x \in [0,1]$ we have:

$$\frac{1}{N} \sum_{n=1}^{N} \chi(T^n x) \longrightarrow \int \chi \, dx = 1/b$$

as $N \to \infty$.

<u>Sketch proof</u> (More general case later)

1) <u>Claim</u> : $\int \chi(T^n x)\,dx = \int \chi\,dx \ (= 1/b)$ (for $n \geq 0$).

Denote $\underline{\chi} = \underline{\lim}_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi(T^n x) \in L^2$

<u>Claim</u> $\underline{\chi}$ is independent of x

(Hint: Show $\underline{\chi} \circ T = \underline{\chi}$ and use Fourier series)

<u>Aim</u> : We want to show $\overline{f} = 1/b$.

2) Let $\varepsilon > 0$, $N(x) = \inf\left\{ N > 0 \,\middle|\, \sum_{n=1}^{N} \chi(T^n x) < \underline{\chi} + \varepsilon \right\}$

Let $M > 0$ and $A = \{ x \mid N(x) > M \}$

<u>Claim</u> $\mu(A) \to 0$ as $M \to +\infty$

We can bound:

$$\sum_{n=1}^{N} \chi(T^n x) \leq N(\underline{\chi} + \varepsilon) + \sum_{n=1}^{N} \chi_A(T^n x) + M\varepsilon.$$

By integrating:

$$\frac{1}{b} = \int \chi\,dx \leq (\underline{\chi} + \varepsilon) + \mu(A) + \frac{M}{N}\varepsilon$$

Let $N \to \infty$, $M \to \infty$ ($\Rightarrow \mu(A) \to 0$), $\varepsilon \to 0$

$\Rightarrow 1/b \leq \underline{\chi}$

Replacing χ by $-\chi$ above : $\overline{\chi} = \overline{\lim}_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi(T^n x) \leq 1/b$

Thus $\overline{\chi} = \underline{\chi} = 1/b$.