Ergodic Theory MA427

Example Sheet 8

Entropy & Transfer operators

1. [Periodic automorphisms.] Let $T : X \to T$ be a periodic automorphism, i. e. there exists $m \in \mathbb{N}$ such that $T^m(x) = x$ for almost all $x \in X$. Show that T has zero entropy with respect to any T-invariant probability measure.

2. [Bernoulli shifts.] Complete the proof from the lectures that Bernoulli shifts are strong mixing (Recall: We showed the result when the associated sets are cylinders, but it remains to extend this results to all sets in the sigma algebra). Show that Bernoulli transformations have strictly positive entropy.

3. [Continuity of entropy.] Let $T : X \to X$ preserve a probability measure μ. Let Ξ be the set of all finite measurable partitions. Given $\alpha, \beta \in \Xi$ define

$$\rho(\alpha, \beta) = H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha)$$

Show that:

(a) ρ is a metric on Ξ;

(b) Given T, show that $h_\mu(\alpha, T)$ is continuous as a function of $\alpha \in \Xi$, i.e., for $\alpha, \beta \in Xi$ we have

$$|h_\mu(\alpha, T) - h_\mu(\beta, T)| \leq \rho(\alpha, \beta).$$

4. [Convexity.] Let $T : X \to X$ preserve a probability measure μ and let α be a measurable partition. Let ν be another measure and let $p \in [0, 1]$. Show that

$$ph_\mu(T, \alpha) + (1 - p)h_\nu(T, \alpha) \leq h_{p\mu+(1-p)\nu}(T, \alpha).$$

5. [One-sided generators.] Let T be an invertible measure-preserving transformation. Partition α is called a one-sided generator if partitions subordinate to partitions of the form $\bigvee_{i=0}^{k} T^{-1}(\alpha)$ are dense with respect to the metric ρ. Show that if a one-sided generator exists, then $h_\mu(T) = 0$.

6. [Bernoulli shifts are important.] Prove that if a measure preserving transformation $T\colon (X, \mu) \to (X, \mu)$ possesses a generator α with k elements, then T is metrically isomorphic to a full shift σ_k with uniform Bernoulli measure $\mu(1/k, 1/k, \ldots 1/k)$.

7. [Expanding maps.] Compute the entropy of expanding maps $T_b : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ of the circle defined by $T_b\colon x \mapsto bx$ for $b \in \mathbb{Z}$ with respect to usual Lebesgue measure.

8. [Torus translations are not complicated.] Prove that any topologically transitive translation of the torus (i.e., one for which there is a dense orbit) has a one-dimensional generator that consists of two elements.

9. [Markov transformations.] Let A be $n \times n$ matrix, $C \in \Omega_A$ be a cylinder, and let $\mathcal{N}(n, C)$ be the number of periodic points of period n in C. Show that

$$\lim_{n\to\infty} \frac{\mathcal{N}(n, C)}{\mathrm{tr}(A^n)} = \mu_\Pi(C)$$

where the matrix Π is given by

$$\pi_{i,j} = \frac{a_{ij} v_i}{\lambda v_j}.$$

Here a_{ij} are elements of A, $v = (v_1, \dots v_n)$ is a positive eigenvector of A^t and λ is corresponding eigenvalue.

[N.B. The measure μ_Π is called the Parry measure]

10. [Kac's Theorem for non-invertible maps.] Let $T : X \to X$ preserve an ergodic probability measure μ. Let $\mu(A) > 0$

(a) Show that if $A_n = \{x \in A : n_A(x) = n\}$ then $\sum_n \mu(A_n) = \mu(A)$.

(b) Show that if $B_n = \{x \in X : T^j x \notin A$ for $1 \leq j \leq n-1, T^n x \in A\}$ hen $\sum_n \mu(B_n) = 1$.

(c) Show that $\sum_{n=k}^{\infty} \mu(A_n) = \mu(B_k)$.

(d) Writing $\int_A n_A(x) d\mu(x) = \sum_{k=1}^{\infty} k\mu(A_k) = \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \mu(A_k)\right)$ complete the proof of Kac's Thereom.