Shift maps

Let Σ be the space of sequences $\omega = (\omega_i)_{i=0}^{\infty}$ taking values in the finite set $\{1, 2, \dots, N\}$

Define a metric by

$$d_\lambda(\omega, \omega') = \sum_{i=0}^{\infty} \frac{\delta(\omega_i, \omega_i')}{\lambda^i} \quad \text{for } \omega = (\omega_i)_{i=0}^{\infty}, \ \omega' = (\omega_i')_{i=0}^{\infty}$$

where $\delta(k, l) = \begin{cases} 1 & \text{if } k \neq l \\ 0 & \text{if } k = l \end{cases}$, for $\lambda > 2$.

Thus, two sequences are close if they agree on the first entries

Consider the cylinder defined by

$$C(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) = \{\omega \in \Sigma \mid \omega_i = \alpha_i, 0 \le i \le n-1\}$$

Fix a sequence $\alpha = (\alpha_i)_{i=0}^{\infty}$.

Lemma: $C(\alpha_0, \dots, \alpha_{n-1}) = \underbrace{\{\omega \in \Sigma \mid d(\alpha, \omega) < \lambda^{1-n}\}}_{B(\alpha, \lambda^{1-n})}$

Proof: If $\omega \in C(\alpha_0, \dots, \alpha_{n-1})$ then

$$d(\alpha, \omega) = \sum_{i=0}^{\infty} \frac{\delta(\alpha_i, \omega_i)}{\lambda^i} = \sum_{i=n}^{\infty} \frac{\delta(\alpha_i, \omega_i)}{\lambda^i} \le \frac{1}{\lambda^{n-1}} \cdot \frac{1}{\lambda - 1} < 1/\lambda^{n-1}$$

Conversely, if $\omega \notin C(\alpha_0, \dots, \alpha_{n-1})$ then

$$d(\alpha, \omega) = \sum_{i=0}^{\infty} \frac{\delta(\alpha_i, \omega_i)}{\lambda^i} \ge \lambda^{1-n},$$

since $\exists\, 0 \le i < n$ such that $\alpha_i \neq \omega_i$, ie, $\omega \notin B(\alpha, \lambda^{1-n})$ $\square$

Proof. For part (i), the periodic points are periodic sequences, ie, $\sigma^m x = x$ if $x_{n+m} = x_n$ for all $n \geq 0$.

To show that periodic orbits are dense we need to show there is a periodic point in every open set (or in every cylinder $\mathcal{C}(\alpha_0, \dots, \alpha_{m-1})$)

We then let
$$\begin{cases} \omega_i = \alpha_i & \text{for } 0 \leq i \leq m-1 \\ \omega_{i+m} = \alpha_i & \text{"} \\ \omega_{i+2m} = \alpha_i & \text{"} \end{cases}, \text{etc}$$

Thus $\omega \in \mathcal{C}(\alpha_0, -, \alpha_{m-1})$ and $\sigma^m \omega = \omega$.

For part (ii), to show σ is mixing let $\mathcal{C}(\alpha_0, -, \alpha_{m-1})$ and $\mathcal{C}(\beta_0, -, \beta_{m-1})$ be two cylinders then we need to find $n \geq 1$ such that

$$\mathcal{C}(\alpha_0, \dots, \alpha_{m-1}) \cap \sigma^{-n} \mathcal{C}(\beta_0, \dots, \beta_{m-1}).$$

Let $n = m$.

Choose any $\omega \in \Sigma$ such that

$$\begin{cases} \omega_i = \alpha_i & \text{for } i = 0, \dots, m-1 \\ \cancel{\omega_{i+m} = \beta_i \text{ for } i = 0, \dots, m-1} \end{cases}$$

Moreover $\sigma^{m}\mathcal{C}(\beta_0, \dots, \beta_{m-1}) = \Sigma_m$

Thus $\omega \in \mathcal{C}(\alpha_0, -, \alpha_{m-1}) \cap \sigma^{-m} \mathcal{C}(\beta_0, -, \beta_{m-1})$

Theorem 4.3. A subshift of finite type $\sigma : X_A \to X_A$ is transitive iff A is irreducible.

Proof ($\Rightarrow$) Assume that σ is transitive.

Let $[i]_0 := \{ (x_n)_{n\in\mathbb{Z}} \in X_A : x_0 = i \}$ $(i = 1, \dots, k)$

These sets are open

Transitivity implies: $\begin{cases} \forall\ 1 \le i,j \le k,\ \exists N > 0, \\ \sigma^{-N}[j]_0 \cap [i]_0 \neq \phi. \end{cases}$

Choose $(x_n)_{n\in\mathbb{Z}} \in \sigma^{-N}[j]_0 \cap [i_0]$

$\Rightarrow$ $x_0 = i$ and $x_N = j$

Thus, $A^N(i,j) = \sum_{r_1=1}^{k} \cdots \sum_{r_{N-1}}^{k} A(i,r_1)A(r_1,r_2) \dots A(r_{N-1},j)$
$\geq 1.$

($\Leftarrow$) Assume A = irreducible.
(i.e. for $1 \le i,j \le k$ we have that $A^N(i,j) \geq 1$).

Given $U, V \neq \phi$ open sets chose $\begin{cases} (i_n)_{n\in\mathbb{Z}} \in U \\ (j_n)_{n\in\mathbb{Z}} \in V \end{cases}$

and $M > 0$ large, such that

$$\begin{cases} U \supseteq \{ (x_n)_{n\in\mathbb{Z}} \in X_A \mid x_k = i_k,\ |k| \le M \} \\ V \supseteq \{ (x_n)_{n\in\mathbb{Z}} \in X_A \mid x_k = j_k,\ |k| \le M \} \end{cases}$$

Choose $N > 0$: $A(\cancel{i_M}, j_{-M}) \geq 1$

In particular, $\exists\ x_1, \ldots, x_{N-1}$ with

$$A(i_m, x_1) = A(x_1, x_2) = \cdots = A(x_{N-1}, j_{-m}) = 1$$

and then define:

$$x_n = \begin{cases} i_n & \text{if } n \leq m \\ x_{n-m} & \text{if } m+1 \leq n \leq M+N-1 \\ j_{n-(2m+N)} & \text{if } M+N \leq n. \end{cases}$$

Thus, $x \in U \cap \sigma^N V$ i.e. $U \cap \sigma^N V \neq \phi$. $\boxtimes$

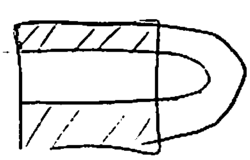
~~Smale Horse shoe~~:

Definition: A continuous bijective map $\pi : \Sigma_2 \to \Lambda$ is called a conjugacy if $\pi \circ \sigma = T \circ \pi$

Theorem 4.4. There exists a conjugacy $\pi : X_2 \to \Lambda$ from the shift space to the Smale horse-shoe.

Proof. We define

$$\pi\left((x_n)_{n\in\mathbb{Z}}\right) = \bigcap_{n\in\mathbb{Z}} T^{-n} R_{x_n}.$$



f is well-defined: For each $r \geq 1$ we consider the rectangle $C_r\left((x_n)_{n\in\mathbb{Z}}\right) = \bigcap_{|n|\leq r} C_r\left((x_n)_{n\in\mathbb{Z}}\right)$

where (i) $C_r((x_n)_{n\in\mathbb{Z}})$, $r \geq 1$ are closed

(ii) $\mathrm{diam}\left(C_r((x_n)_{n\in\mathbb{Z}})\right) \to 0$ as $r \to +\infty$

Thus, $\bigcap_{r\geq 1} C_r((x_n)_{n\in\mathbb{Z}}) \neq \emptyset$

and is a single point.

<u>π is continuous</u>: If $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$ are close in X_2 $\Rightarrow$ $\begin{cases}\exists\, N \text{ large, with} \\ x_n = y_n \text{ for } |n| \leq N.\end{cases}$

Thus $\pi((x_n)_{n\in\mathbb{Z}}), \pi((y_n)_{n\in\mathbb{Z}}) \in C_N((x_n)_{n\in\mathbb{Z}})$

and $\mathrm{diam}\left(C_N((x_n)_{n\in\mathbb{Z}})\right) \to 0$, as $N \to +\infty$

(i.e π is continuous).

<u>π is injective</u>: $\pi((x_n)_{n\in\mathbb{Z}}) = x = \pi((y_n)_{n\in\mathbb{Z}})$

$\Longleftrightarrow f^n x \in R_{x_n} = R_{y_n} \Longleftrightarrow x_n = y_n$.

<u>π is surjective</u>: Given $x \in \Lambda$ define $x_n \in \{1,2\}$ by $f^n x \in R_{x_n}$, $n \in \mathbb{Z}$. Then $\pi((x_n)_{n\in\mathbb{Z}}) = x$

<u>π is a conjugacy</u>.

$$f\left(\pi((x_n)_{n\in\mathbb{Z}})\right) = f\left(\bigcap_{n\in\mathbb{Z}} T^{-n} R_{x_n}\right)$$

$$= \bigcap_{n\in\mathbb{Z}} T^{-n} R_{x_{n+1}} = \pi((x_n)_{n\in\mathbb{Z}})$$

$\boxtimes$

Corollary 4.4.1

(1) The map $f:\Lambda \to \Lambda$ is transitive;

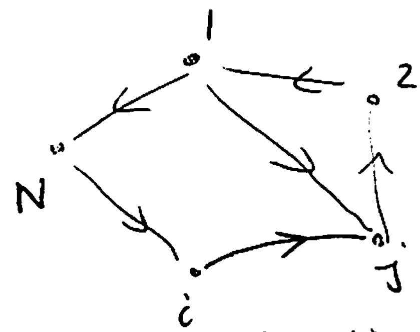
(2) The number of periodic points $f^n x = x \in \Lambda$ of period n is 2^n

Proof. These are true and easy for $\sigma : X_2 \to X_2$. They are preserved for $T : \Lambda \to \Lambda$ by the conjugacy.

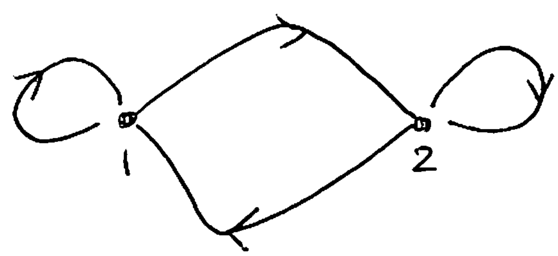
Note: Let $P_n(\sigma)$ be the number of periodic points of period n (i.e., $\sigma^n x = x$). Then we see from the proof that $P_n(\sigma) = N^n$.

Subshifts and graphs.

We can represent a subshift by a (directed) graph: $\begin{cases} \text{vertices } 1, 2, \ldots, N \\ \text{edges from } i \text{ to } j \text{ iff } A(i,j) = 1 \end{cases}$



Example: Σ = (full) shift on 2 symbols.



$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Lemma For every $i, j \in \{1, \ldots, N\}$ the number N_{ij}^m = # paths of <u>length</u> m from i to j (pass through m edges)

$$= A^m(i,j)$$

(i.e., (i,j)-th entry of matrix $A^m = \underbrace{A.A \ldots A}_{m \text{ matrix mult}^n}$).

Proof. When $m=1$, $N_{ij}^{1} = A(i,j)$.
We continue by induction.

Assume $N_{ij}^{m} = A^{m}(i,j)$.
For each $k \in \{1, \dots, N\}$ a path from i to k of length m can be extended to a path from i to j of length $m+1$ iff $A(k,j)=1$.

Thus
$$N_{ij}^{m+1} = \sum_{k=1}^{N} N_{ik}^{m} A(k,j)$$
$$= \sum_{k=1}^{N} A^{m}(i,k) A(k,j) = A^{m+1}(i,j) \quad \boxtimes$$

Corollary For a subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$ the number of periodic points $P_m(\sigma)$ of period m is $\mathrm{trace}(A^m)$

Proof Periodic points $\sigma^m x = x$ correspond to sequences with period m, and thus paths of length m. Thus
$$P_m(\sigma) = \sum_{i=1}^{N} N_{ii}^{m} = \sum_{i=1}^{n} A^{m}(i,i)$$
$$= \mathrm{trace}(A^m) \quad \boxtimes$$

Example: Consider the graph:

Fixed points
$$\begin{cases} (1,1,1,\dots) \\ (2,2,2,\dots) \\ (5,5,5,\dots) \end{cases}$$

Period 2 points
$$\begin{cases} (1,2,1,2,1,2,\dots) \\ (2,1,2,1,2,1,\dots) \\ (3,4,3,4,3,4,\dots) \\ (4,3,4,3,4,3,\dots) \end{cases}$$

$(1,2,1,2,1,2,\dots)$, $(2,1,2,1,2,1,\dots)$ ← same orbit

$(3,4,3,4,3,4,\dots)$, $(4,3,4,3,4,3,\dots)$ ← same orbit

Period 3 orbits
$$\begin{cases} (1,2,2,1,2,2,\dots) \\ (2,2,1,2,2,1,\dots) \\ (2,1,2,2,1,2,\dots) \\ (1,1,2,1,1,2,\dots) \\ // \\ // \\ (3,4,5,3,4,5,\dots) \\ // \\ // \end{cases}$$

$(1,2,2,1,2,2,\dots)$, $(2,2,1,2,2,1,\dots)$, $(2,1,2,2,1,2,\dots)$ ← same orbit

$(1,1,2,1,1,2,\dots)$, //, // ← same orbit.

$(3,4,5,3,4,5,\dots)$, //, // ← same orbit

> **Definition.** A matrix A is called positive if all of its entries are positive (i.e., $A(i,j) > 0$).

Lemma. If A^m is positive, then so is A^n for any $n \geq m$.

Proof. If $i, j \in \{1, 2, \ldots, N\}$ then

$$A^{m+1}(i,j) = \sum_{k=1}^{N} \underbrace{A^m(i,k)}_{>0} \cdot A(k,j)$$

But there must be some path ending at j of length one (else impossible that A^m is positive), i.e., $\exists k$ such that $A(k,j) = 1$.

Thus A^{m+1} is positive. Proceeding inductively, A^n is positive $\square$

Definition: We call A aperiodic if there exists $m \in \mathbb{N}$ such that A^m is positive.

Proposition. If A is aperiodic then $\sigma : \Sigma_A \to \Sigma_A$ is (topologically) mixing and the periodic points are dense in Σ_A (i.e., σ is chaotic and hence has sensitive dependence on initial conditions)

Proof. The proof is very similar to that for $\sigma : \Sigma \to \Sigma$. For example, to show periodic points are dense it suffices to show every cylinder $C_A(\alpha_0, \ldots, \alpha_{k-1}) := C(\alpha_0, \ldots, \alpha_{k-1}) \cap \Sigma_A$ contains a periodic orbit.

However, let $x = (\underbrace{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \beta_1, \ldots, \beta_m}, \underbrace{\alpha_0, \alpha_1, \ldots}$

repeat string of length $k+m-1$

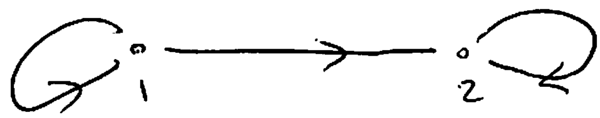
where $\begin{cases} A(\alpha_{k-1}, \beta_1) = 1 \\ A(\beta_i, \beta_{i+1}) = 1, & i = 1, \ldots, m-1. \\ A(\beta_m, \alpha_0) = 1 \end{cases}$

Then $\sigma^{k+m-1} x = x \in \mathcal{C}_A(\alpha_0, \ldots, \alpha_{k-1})$.

We know such a $\beta_1, \ldots, \beta_m$ exists since $A^m(\alpha_{k-1}, \alpha_0) > 0$. $\square$

<u>Example</u>. Neither $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ nor $A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

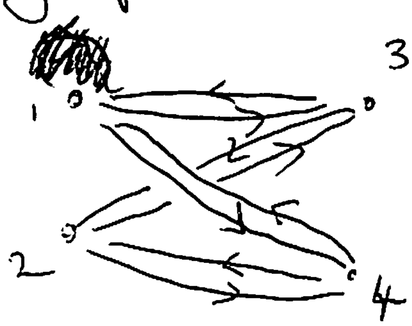
are aperiodic. In the first case $A^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and the graph is



In the second case

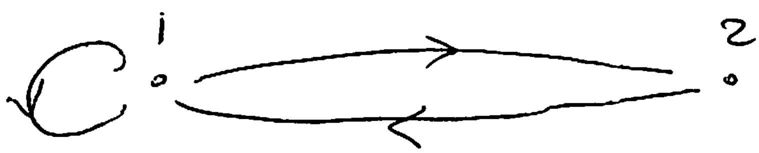
$$A^{2m} = \begin{pmatrix} 2^{2m-1} & 2^{2m-1} & 0 & 0 \\ 2^{2m-1} & 2^{2m-1} & 0 & 0 \\ 0 & 0 & 2^{2m-1} & 2^{2m-1} \\ 0 & 0 & 2^{2m-1} & 2^{2m-1} \end{pmatrix} \text{ \& } A^{2m+1} = \begin{pmatrix} 0 & 0 & 2^{2m} & 2^{2m} \\ 0 & 0 & 2^{2m} & 2^{2m} \\ 2^{2m} & 2^{2m} & 0 & 0 \\ 2^{2m} & 2^{2m} & 0 & 0 \end{pmatrix}$$

and the graph is:



Example Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ then A is aperiodic since $A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is positive.

Associated graph:



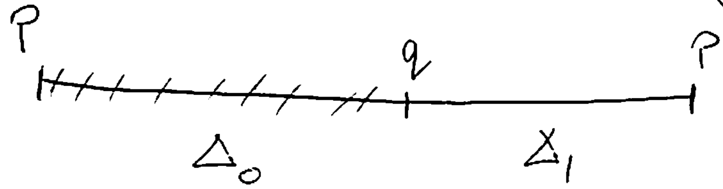
Coding of nonlinear expanding maps

Let $f: K \to K$ be an expanding map.
(For simplicity assume $\deg(f) = 2$).

- f has a fixed point $f(p) = p$, say.
- $f^{-1}(p) = \{p, q\}$ say.

(eg when $f(x) = 2x \pmod 1$ then $p = 0, q = \frac{1}{2}$)

- Partition $K = \Delta_0 \cup \Delta_1$ where Δ_0, Δ_1 are arcs with end points p, q



NB. $\Delta_0 \cap \Delta_1 = \{p, q\}$ so not quite a partition in usual sense

Given $x \in K$ we code the orbit $(f^n x)_{n=0}^{\infty}$ such that $f^n(x) \in \Delta_{\omega_n}$, where $\omega_n \in \{1, 2\}$