Complex Dynamics

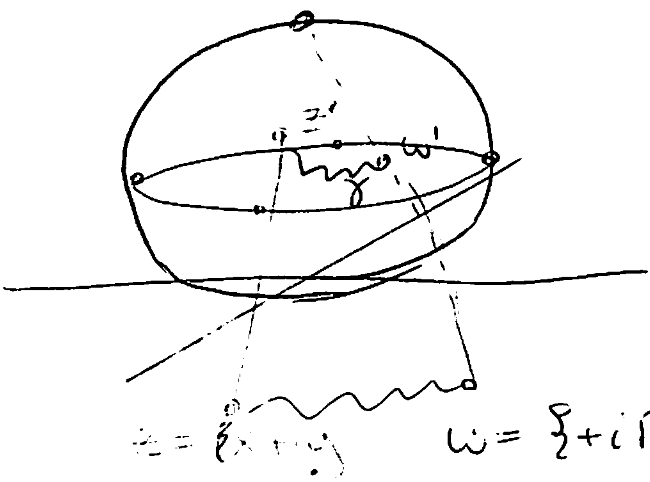
We considered the linear expanding map $f : K \to K$, by $f(z) = z^n$ $(n \geq 2)$ (where $K = \{z \in \mathbb{C} \mid |z| = 1\}$).

However, we can also replace K by the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

(This is compact: Using stereographic projections

$d(z', w')$ in place of $|z-w|$)

$$\text{length}(\gamma) = \int_\gamma \frac{2|dz|}{1+|z|^2}$$



we can consider $\begin{cases} f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \\ f : z \longmapsto z^n \end{cases}$

More generally, we can consider polynomial maps:

$$\left.\begin{aligned} f &: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \\ f(z) &= P(z) \end{aligned}\right\} \text{ where } P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \in \mathbb{C}[z]$$

Examples (a) Let $\begin{cases} f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \\ f_c(z) = z^2 + c \end{cases}$ where $c \neq 0$

(b) For $c = 0$ this reduces to
$f_0(z) = z^2$.

Even more generally, we can consider rational maps

$$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$
$$f : z \longmapsto P(z)/Q(z)$$

where $P(z), Q(z) \in \mathbb{C}[z]$ are (coprime) polynomials.

Remark Rational maps are holomorphic maps on $\hat{\mathbb{C}}$.

We want to consider rational maps such that

$$d = \max(\deg P, \deg Q) \geqslant 2,$$

where $\deg P, \deg Q$ are largest powers of z.
This is related to the previous notion of degree:

z is called a critical point if $f'(z)=0$
z is called a regular point if $f'(z)\neq 0$

If z is a regular point then:

$$\#\{w : \underbrace{f(w) = z}_{\equiv P(w)-zQ(w)=0}\} = d$$

(since f is locally one-one).

Let $U \subseteq \hat{\mathbb{C}}$ be an open set.
We say that a family of functions $\{f_n : U \to \hat{\mathbb{C}}\}$ is normal or equicontinuous if

$\forall \varepsilon > 0, \exists \delta > 0$ such that for $z, w \in U$ with $|z-w| < \delta \Rightarrow |f_n(z) - f_n(w)| < \varepsilon$, $\forall n \geq 1$.

Definition. The Fatou set $F \subseteq \hat{\mathbb{C}}$ of a rational map $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the set of $z \in \hat{\mathbb{C}}$ for which there is a neighbourhood V of z such that $f_n := f^n, n \geq 1$, is normal

The Julia set $J \subseteq \hat{\mathbb{C}}$ is the complement of the Fatou set (i.e., $J = \hat{\mathbb{C}} - F$)

Proposition (i) J is a closed set.
(ii) $J \neq \phi$ (if $d \geq 2$).

Proof (i) By definition, the Fatou set F is open, thus $J = \hat{\mathbb{C}} - F$ is closed;

(ii) Assume for a contradiction $F = \hat{\mathbb{C}}$. Then $f^n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is normal. By the Arzela-Ascoli theorem there is a uniformly convergent subsequence f^{n_k}. Moreover, by Montel's theorem the limit g is analytic. However,

$$\deg(g) = \lim_{k \to \infty} \deg(f^{n_k}).$$

But $\deg(f^{n_k}) \nearrow +\infty$, giving a contradiction □

We say that $E \subseteq \hat{\mathbb{C}}$ is completely invariant if $f(E) \subseteq E$ and $f(\hat{\mathbb{C}} - E) \subseteq \hat{\mathbb{C}} - E$ (i.e., $f^{-1}E = E$).

$$\boxed{\text{Proposition } J \text{ is completely invariant}}$$

Proof From the definitions, $f^{-1}(J) \subseteq J$.

Let $z_0 \in F$ and $f^{n_j + 1}$ converges uniformly on a neighbourhood $V \ni z_0$.

Since f maps open neighbourhoods of z_0 to open neighbourhoods of $f(z_0)$, then f^{n_j} converges uniformly on a neighbourhood of $f(z_0)$.

Thus $f(z_0) \in F$. Thus $f(F) \subseteq F$.

In particular F (and so $J = \bar{\mathbb{C}} - F$) is completely invariant. $\square$

$\boxed{\text{Proposition. For any } N \geq 1, \text{ the Julia set of } f \text{ coincides with that of } f^N}$

Proof. Since (f^n) is normal iff (f^{nN}) is normal, f and f^N have the same Fatou sets (thus the same Julia sets).

Other properties of Julia sets

- J = closure of repelling periodic points

  i.e., $J = \overline{\{w \mid f^n(w) = w, |(f^n)'(w)| > 1\}}$

- J is an infinite set.

Other properties of Fatou sets

- If U is a component of F then $\underbrace{f^n U = U}_{\text{periodic}}$ (some $n \geq 1$) or $\underbrace{f^n U = f^{n_0} U}_{\text{preperiodic}}$ (some $n \geq n_0$)

(Sullivan '85)

Quadratic maps & Mandelbrot set

Consider the family

$$\begin{cases} f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \\ f_c : z = z^2 + c \end{cases}$$

Def^n The Mandelbrot set $\mathcal{M} \subseteq \mathbb{C}$

is a ~~closed subset~~

$$\mathcal{M} = \{ c \in \mathbb{C} \mid f_c^n(0) \not\to +\infty \}$$

Proposition. $\mathcal{M}$ is a closed bounded subset of $\mathbb{C}$

Proof. • If $|c| > 2$ then one shows by induction that

$$|(f_c^n)'(0)| \geq |c|.(|c|-1)^{2^{n-1}}, n \geq 1$$

so $|f_c^n(0)| \to \infty$ and thus $c \notin \mathcal{M}$

i.e. $\mathcal{M} \subseteq \{c : |c| \leq 2\}$

Suppose $|f_c^m(0)| = 2+\delta > 2$, for some $m \geq 1$

If $|c|$ $= |f_c(0)| > 2 \Rightarrow c \notin \mathcal{M}$.

If $|c| \le 2$

$$\Rightarrow |f_c^{n+1}(0)| \geq (2+\delta)^2 - 2 \geq 2 + 4\delta$$

Proceeding by induction:

$$|f_c^{n+k}(0)| \geq 2 + 4^k \delta \to \infty \text{ as } k \to \infty$$

$$\Rightarrow c \notin M$$

Thus $M = \underbrace{\{ c \in \mathbb{C} \mid |f_c^n(0)| \le 2, \forall n \ge 1 \}}$

$$= \underbrace{\bigcap_n \{ c \in \mathbb{C} \mid |f_c^n(0)| \le 2 \}}_{\text{closed set}} \quad \blacksquare$$

M has a complicated structure.

<u>Known</u>: M is connected

<u>Unknown</u>: Whether M is locally connected

(i.e. $\forall x \in M$ and $x_n \in M$, $x_n \to x$
$\exists$ connected sets $x, x_n \in L_n$ with $\mathrm{diam}(L_n) \to 0$)

Entropy and rational maps

We can consider a general result which applies to rational maps.

Let $f: M \to M$ be a C^1 map
Let $d = \deg(f)$ be the number of preimages of a regular point

$$\boxed{\textbf{Theorem} \quad h(f) \geq \log|\deg(f)|.}$$

Proof Let ω denote the area on M

Let $0 < \alpha < 1$ be arbitrary.

Let $L = \sup_{x \in M} |Jac_x(f)|$

eg. $Jac_x(f) = |f'(x)|$.

Let $\varepsilon = L^{-\alpha/(1-\alpha)}$

Let $B = \{x \mid |Jac_x(f)| \geq \varepsilon\}$

- Cover $B \subseteq M$ by open sets U_i such that $f: U_i \to fU_i$ is injective.
- Let δ be the Lebesgue number of $\{U_i\}$.
  (If $x \neq y$ and $d(x,y) < \delta \Rightarrow f(x) \neq f(y)$). $\circledast$

Given $n \geq 1$, let

$$A = \{x \in M \mid \#(B \cap \{x, fx, \dots, f^{n-1}x\}) < \alpha n\}$$

(ie, those x which are "rarely" in B)

If $x \in A$ then:

$$|\mathrm{Jac}_x(f^n)| = \prod_{j=0}^{n-1} |\mathrm{Jac}_{f^j_x}(f)| \quad \text{(Chain rule)}$$

$$< \varepsilon^{(1-\alpha)n} \cdot L^{\alpha n}$$

$$= \left(\varepsilon^{(1-\alpha)} L^{\alpha}\right)^n = 1$$

Thus $\mathrm{Area}(f^n A) < \mathrm{Area}(M)$

$\Rightarrow$ $\boxed{\text{There exists a regular value } x \in M - f^n(A) \text{ (by Sard's Theorem) for } f^n \text{, i.e., } x \text{ has } d^n \text{ preimages (wrt } f)}$

- If all d preimages of x are in B, let $Q_1 = \{f^{-1}x\}$
- Otherwise, let Q_1 be a single preimage which is outside B

$\boxed{\underline{\text{Note}} \quad y \in Q_1 \Rightarrow y \text{ is regular (wrt } f) \text{ since } x \text{ is regular (wrt } f^n) \text{ - chain rule}}$

For each $y \in Q_1$ continue the construction to get $Q_2 \subseteq f^{-2}x$ etc.

Finally, construct $Q_n \subseteq f^{-n}(x)$.

$\boxed{\underline{\text{Claim}} \quad Q_n \text{ is } (n, \delta)\text{-separated}}$

Proof. If $y_1 \neq y_2$ in Q_n and $d(f^i y_1, f^i y_2) < \delta$ for $i = 0, \dots, n-1$ then $f^{n-1}(y_1) = f^{n-1}(y_2)$ or $f^{n-1}(y_1) \neq f^{n-1}(y_2)$ both in B.

But in last case, $d(f^{n-1}y_1, f^{n-1}y_2) \leq \delta$

$$\Rightarrow f^n y_1 \neq f^n y_2 \quad (\text{by } (*))$$

contradicting that y_1, y_2 are f^n-preimages of x.

Thus $f^{n-1}y_1 = f^{n-1}y_2$.

Proceeding inductively, $f^{n-2}y_1 = f^{n-2}y_2$ and, eventually, $y_1 = y_2$. This proves the claim

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Since $Q_n \subseteq f^{-n}(x) \subseteq f^{-n}(M - f^n(A))$
$$\subseteq M - A$$

$$\Rightarrow Q_n \cap A = \phi.$$

(i.e., if $y \in Q_n$ then $\#\{0 \leq i \leq n-1 : f^i y \in B\} \geq \alpha n$)

In particular, $\# Q_n \geq d^{\alpha n}$

Thus $N(n, \delta) \geq d^{\alpha n}$ and so $h(f) \geq \alpha \log d$

Since $0 < \alpha < 1$ was arbitrary we deduce $h(f) \geq \log d$. $\square$

Remark: For rational maps we have also $h(f) = \log d$. However this needs some more machinery.