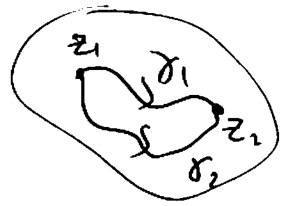
Applications of Cauchy's Theorem & Gousart's Theorem +

1. Independence of integrals

Let $f: U \to \mathbb{C}$ be analytic (U simply connected). Then for any piecewise smooth curve γ from z_1 to z_2 the integral $\int_\gamma f\,dz$ is independent of γ.

Proof: If γ_1, γ_2 are both curves from z_1 to z_2 then $\gamma = \gamma_1(-\gamma_2)$ is a closed curve and



$$0 \underset{\text{Gousart's Thm}}{=} \int_\gamma f\,dz = \int_{\gamma_1} f\,dz - \int_{\gamma_2} f\,dz \qquad \square$$

2. Derivatives

We have that for analytic fn's $f: U \to \mathbb{C}$ and simple closed curves $\gamma \subseteq U$:

Cor: We can write for z_0 inside γ:

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^2}\,dz$$

(& more generally, $f^{(k)}(z_0) = \dfrac{k!}{2\pi i} \displaystyle\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz$)

for $k \geq 2$.

Proof. We only need to justify differentiating under the integral sign.

Using Cauchy's theorem ($h \in \mathbb{C}$ small):

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_\gamma \frac{1}{h}\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right) f(z)\,dz$$

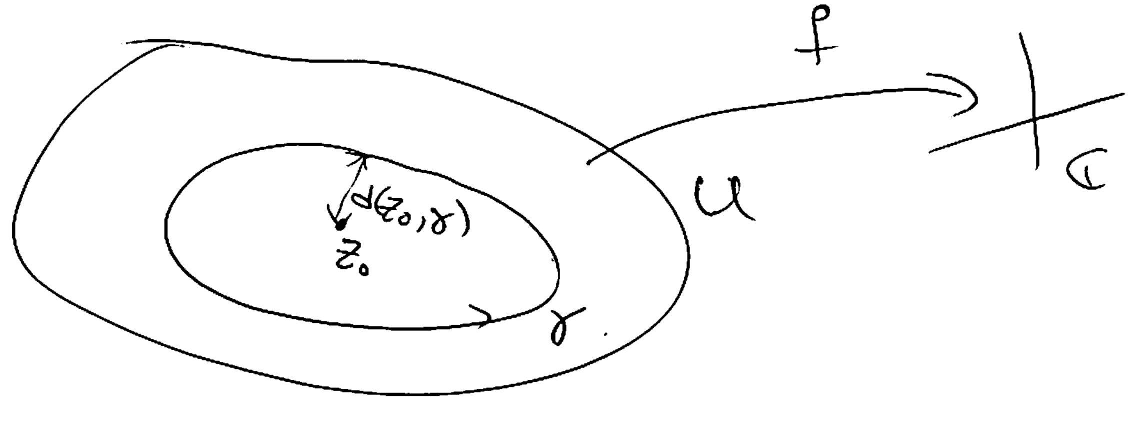
$$= \frac{1}{2\pi i}\int_\gamma \left(\frac{1}{(z-z_0-h)(z-z_0)}\right) f(z)\,dz$$

But then:

$$\left| \frac{f(z_0+h)-f(z_0)}{h} - \underbrace{f'(z_0)}_{:= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^2}\,dz} \right| \leq \left| \frac{h}{2\pi i}\int_\gamma \frac{1}{(z-z_0-h)(z-z_0)^2} f(z)\,dz \right|$$

$$\leq \underbrace{\frac{|h|}{2\pi}\left(\frac{\operatorname{length}(\gamma)}{(d(z_0,\gamma)-|h|)\,d(z_0,\gamma)^2}\right)\cdot \sup_{z\in\gamma}|f(z)|}_{\to 0 \text{ as } h \to 0.}$$

where $d(z_0,\gamma) = \inf_{z\in\gamma} |z_0 - z|$

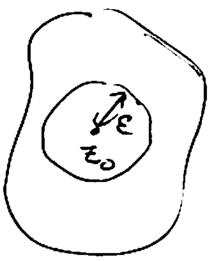


3. Completing equivalence of definitions of analyticity I:
(writing analytic functions as Taylor series)

We claim any complex differentiable function $f : U \to \mathbb{C}$ has a Taylor series at each $z_0 \in U$:

Choose $\varepsilon > 0$ such that

$$B(x, \varepsilon) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \} \subseteq U$$

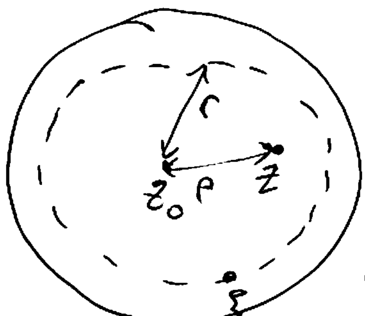


Choose $z \in B(z_0, \varepsilon)$ and then $\rho := |z_0 - z| < \varepsilon$

Choose $\rho < r < \varepsilon$ and by Cauchy's theorem:

$$f(z) = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\xi)}{\xi - z} \, d\xi \qquad (*)$$

where $C(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| = r \}$.



We next expand (for $\xi \in C(z_0, r)$)

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0)\left(1 - \left(\frac{z - z_0}{\xi - z_0}\right)\right)}$$

$$= \frac{1}{(\xi - z_0)} \left(1 + \left(\frac{z - z_0}{\xi - z_0}\right) + \left(\frac{z - z_0}{\xi - z_0}\right)^2 + \cdots \right)$$

where $\left|\frac{z - z_0}{\xi - z_0}\right| = \rho/r < 1$. Thus by $(*)$:

$$f(z) = \underbrace{\frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\xi)}{\xi - z_0} \, d\xi}_{a_0} + (z - z_0) \underbrace{\left(\frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\xi)}{(\xi - z_0)^2} \, d\xi \right)}_{a_1} + (z - z_0)^2 \underbrace{\left(\frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\xi)}{(\xi - z_0)^2} \, d\xi \right)}_{a_2} + \cdots$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ where } a_n = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

In particular, we see that the radius of convergence satisfies: $R = \overline{\lim_{n\to\infty}} |a_n|^{1/n} > r > 0$, as required.

4. <u>Completing the definitions of analyticity II: (Continuity of partial derivatives).</u>

By writing $f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^2} dz$

(for some "small" simple closed curve around γ)

we see that $z_0 \longmapsto f'(z_0)$ varies continuously

(where γ is fixed)

Since complex differentiability of $f(z) = u(x,y) + iv(x,y)$ [where $z = x+iy$] gives:

$$f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial y}(z_0) = \frac{\partial v}{\partial \gamma}(z_0) - i\frac{\partial u}{\partial y}(z_0)$$

[Cauchy-Riemann equations]

we can deduce continuity of the partial derivatives:

$$z_0 \longmapsto \frac{\partial u}{\partial x}(z_0), \frac{\partial v}{\partial y}(z_0), \frac{\partial v}{\partial x}(z_0), \frac{\partial u}{\partial y}(z_0)$$

<This completes all parts of proof of equivalence of defns of analyticity>.

Corollary: Let $f: U \to \mathbb{C}$ be analytic and let z_0 be the only zero (of multiplicity one) inside the simple closed curve γ. Then:

$$z_0 = \frac{1}{2\pi i} \int_\gamma z \cdot \frac{f'(z)}{f(z)}\, dz$$

Proof. We can write $f(z) = (z - z_0) g(z)$, where $g: U \to \mathbb{C}$ is non-zero and analytic (inside γ)

There $\frac{f'(z)}{f(z)} = \frac{1}{z - z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\psi(z)\text{ analytic on } U}$.

By Cauchy's theorem:

$$\frac{1}{2\pi i} \int_\gamma z \frac{f'(z)}{f(z)}\, dz = \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{z}{z - z_0}}_{= z_0} + \underbrace{\frac{1}{2\pi i} \int_\gamma z \psi(z)}_{= 0}$$

$$= z_0.$$

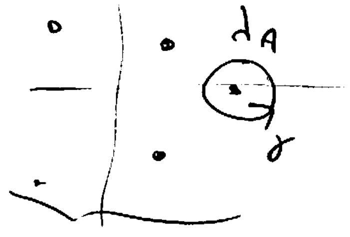
Application (Pert^n Theory for matrices). Let A be a $k \times k$ matrix with complex entries $A = (a_{ij})_{i,j=1}^{k}$.

Let λ_A be an eigenvalue of multiplicity one.

Then $z = \lambda_A$ is a zero of the characteristic polynomial $P_A(z) = \det(z \cdot I - A)$.

If we choose a (simple closed) curve γ around λ_A which doesn't contain any other eigenvalues then we can write:

$$\lambda_A = \frac{1}{2\pi i}\oint_\gamma z \, \frac{P_A'(z)}{P_A(z)} \, dz$$

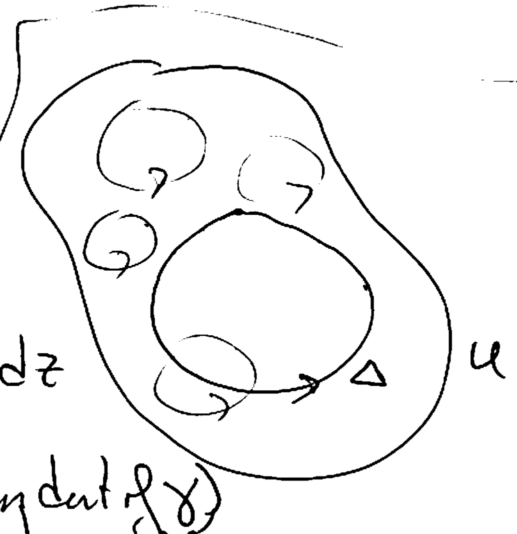


Eigenvalues of A

In particular, $\underbrace{A = (a_{ij})_{i,j=1}^{k}}_{\in \mathbb{R}^{k^2}} \longmapsto \lambda_A$ is continuous (even smooth)
as a function of $(a_{ij}) \in \mathbb{R}^{k^2}$.

Converse to Cauchy's Theorem

Theorem (Moreira's theorem)
Let $f : U \to \mathbb{C}$ be <u>continuous</u> and assume that $\oint_\Delta f(z)\,dz = 0$, for <u>all</u> simple closed curves $\Delta \subseteq U$.
Then $f : U \to \mathbb{C}$ is analytic.



<u>Proof</u>: Fix $z_0 \in U$.
For $z \in U$ associate $F(z) := \int_\gamma f(z)\,dz$
where $\left\{ \begin{array}{l} \gamma : [0,1] \to \mathbb{C} \\ \gamma(0) = z_0,\ \gamma(1) = z \end{array} \right\}$ (independent of γ)
– already seen this as Application 1.

Moreover, $F'(z) = f(z)$. Thus
- $F(z)$ is complex differentiable (with derivative $f(z)$)
- All derivatives exist for $F(z)$, including $F''(z) = f'(z)$.

Thus f is complex differentiable.