A NOTE ON THE ARCSINE LAW FOR GEODESICS

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0. Introduction

The classical arcsine theorem was proved by Levy in 1939, and generalized to independent identically distributed random real valued variables X_i with mean zero by Erdös and Kac in 1946. This described the behavior of the summations $X_1 + \cdots + X_n$, and the distribution of the proportion of summations which are positive. In this note we want to describe a simple geometric analogue of this result for geodesic flows on negatively curved surfaces. There are a number of interesting dynamical analogues for discrete dynamical systems cf. [Zweimuller].

Let us now consider a geometric result for geodesic flows in terms of homology. Let T_1V denote the unit tangent bundle of a compact manifold V with negative sectional curvatures. Given a unit tangent vector $v \in T_1V$ we can associate the geodesic $\gamma_v : \mathbb{R} \to V$. Let $\phi_t : T_1V \to T_1V$ denote the usual geodesic flow, which is given by $\phi_t(v) = \dot{\gamma}_v(t)$ and let X be the associated vector field on T_1V. Let μ denote the normalized Liouville measure.

Given any non-trivial non-zero closed 1-form w (i.e., not of the form df, where f is a function) and $T > 0$ can denote $c(v,T) := \int_0^T \langle \omega, X\rangle(\phi_t v)dt \in \mathbb{R}$. This gives some measurement of the displacement of the orbit in a direction in homology determined by ω.

Notation. For any $T > 0$, we can consider the proportion $\rho_v^+(T)$ of time $0 \leq t \leq T$ that $c(v,T) \geq 0$. Similarly, we can consider the proportion $\rho_v^-(T)$ of time $0 \leq t \leq T$ that $c(v,T) \leq 0$.

Theorem 1. *For any* $0 < \alpha < 1$ *we have that*

$$\lim_{T\to+\infty} \mu \left\{ v \in T_1V \,:\, \frac{\rho_v^+(T)}{T} \geq \alpha \right\} = \frac{2}{\pi} \sin^{-1}(\sqrt{\alpha}).$$

This gives counter intuitive results. For example, the probability that a typical geodesic spends a proportion of its time greater than $\frac{3}{4}$ with $c(v,t) \geq 0$ is $\frac{1}{3}$.

More generally, the result also holds for any Gibbs measure μ with a vanishing winding cycle.

Remark. There is a corresponding result for $\rho_v^+(T)$:

$$\lim_{T\to+\infty} \mu \left\{ v \in T_1V \,:\, \frac{\rho_v^+(T)}{T} \geq \alpha \right\} = \frac{2}{\pi} \sin^{-1}(\sqrt{\alpha}).$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

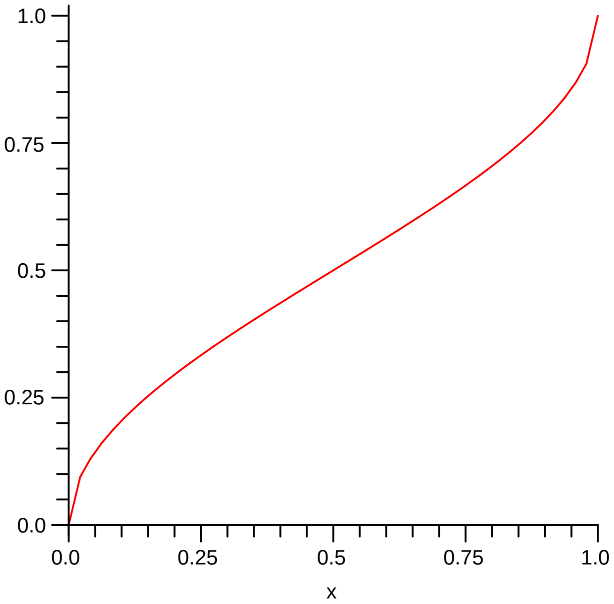

FIGURE 1. A plot of $\frac{2}{\pi}\sin^{-1}\sqrt{x}$

1. PROOF OF THEOREM 1

We begin by recalling some simple results about the winding cycle.

Using duality, we can identify the real first homology group with the linear functionals on cohomology, i.e., $H_1(T_1V, \mathbb{R}) = \mathrm{Hom}(H^1(T_1V, \mathbb{R}), \mathbb{R})$. Given any ϕ-invariant measure m we can associate the asymptotic cycle in $H_1(X, \mathbb{R})$, defined by $\Phi(m) = \int_X \langle X, \omega\rangle dm$

Proposition 1.1. *If m is the Liouville measure then $\Phi(m) = 0$.*

In particular, by the Birkhoff ergodic theorem we have that for almost all $v \in T_1V$ we have that $\frac{1}{T}\int_0^T \langle \omega, X\rangle(\phi_t x) = 0$. Proposition 1.1 is also true for the measure of maximal entropy, for example.

Let $C([0,1], \mathbb{R})$ be the space of continuous functions on the interval $[0,1]$. Recall that the standard Wiener measure W on $C([0,1], \mathbb{R})$ is defined by

$$W\left(\{f(t) : f(t_i) - f(t_{i-1}) \le \alpha_i, i = 1, \cdots, k\}\right)$$
$$= \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \int_{-\infty}^{\alpha_i} e^{-u^2(t_i - t_{i-1})/2}.$$

where $0 = t_0 < t_1 < \cdots < t_k = 1$ and $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$. Let $F : SV \to \mathbb{R}$ be Hölder continuous function with $\int F d\mu = 0$. We are interested in the family $F_T(\cdot, x) \in C([0,1], \mathbb{R})$, where $T > 0$, defined by $F_T(t, x) = \frac{1}{T^{1/2}}\int_0^{tT} F(\phi_t x)dt$.

We say that a Hölder continuous function F is a coboundary if there is a continuous function $G : T_1 V \to \mathbb{R}$ such that $F(x) = \frac{d}{dt} G(\phi_t x)$.

The following result is due to Denker and Philipp [2].

Proposition 1.1 (Weak Invariance Theorem). *Assume that F is not equal to a coboundary and $\int F d\mu = 0$. The sequence of functions $F_T(x.\cdot)$ converges in distribution to the standard Wiener measure $W(t)$ on $C([0,1])$.*

The prefix "weak" refers to convergence in the weak topology cf. [5, p. 148]. In particular, it means that for any set $A \subset C([0,1],\mathbb{R})$ of non-zero Wiener measure, but for which the boundary ∂W has zero measure, we have that

$$\lim_{T\to 0} \mu\{x \,:\, F_T(x.\cdot) \in A\} = W(A).$$

There is a standard way to deduce corollaries to the weak invariance principle, which is the following:

Continuous Mapping Principle. *If ξ_T converges to ξ in distribution and $h : C([0,1],\mathbb{R}) \to \mathbb{R}$ is measurable and continuous (except possibly on a set of Wiener measure zero) then $h(\xi_T)$ converges to $h(\xi)$ in distribution.*

Using the weak invariance principle and the continuous mapping principle leads to the following corollary [5, p.147].

Proposition 2.2 (ArcSine Law). *The Arcsine Law holds for the geodesic flow on V, i.e., Let $F_T^+(x) = |\{0 \le t \le T-1 \,:\, F_t(x) \ge 0\}$ then for a.e. (μ) $x \in SV$ we have that for any $0 \le \beta < 1$ then*

$$\lim_{T\to +\infty} \mu \left\{ x \in SV \,:\, \frac{F_T^+(x)}{T} \ge \beta \right\} = \frac{2}{\pi} \sin^{-1}(\sqrt{\beta})$$

Proof. We briefly sketch the derivation. We can consider the map $h : C([0,1],\mathbb{R}) \to \mathbb{R}$ be defined by

$$h(f) = \mathrm{Leb}\,\{0 \le t \le 1 \,:\, f(t) \ge 0\} \text{ and for each } f \in C([0,1],\mathbb{R}),$$

i.e., the proportion of the interval for which the function is positive. This function is measurable and continuous except on a set of zero measure [1, Appendix II]. For any $0 \le \alpha < 1$ we can apply the Continuous Mapping Principle to deduce that if $A = \{f \in C([0,1],\mathbb{R}) \,:\, h(f) \ge \alpha\}$. then

$$\lim_{T\to 0} \mu\{x \,:\, h(F_T(\cdot, x)) \ge \alpha\} = W\left(\{f \in C([0,1],\mathbb{R}) \,:\, h(f) \ge 0\}\right).$$

Moreover, the expression on the Right Hand Side of this last identity has been explicitly computed and is equal to the expression in the statement of the proposition. □

Proof of Theorem 1. To complete the proof of the theorem, we want to apply Proposition 2.2 with $F(x) = \langle \omega, X \rangle$. By Proposition 2.1 we can deduce that $\int F d\mu = 0$. Furthermore, we know that F is not a coboundary, otherwise its integral around any closed geodesic would vanish. However, this could only happen if ω is trivial, which leads to a contradiction. □

Remarks.

(i) There are a number of other standard corollaries to the invariance theorem [2, pp.542-543] (e.g., the law of the iterated logarithm) which we could have considered, but the arcsine law seems particularly well suited to homological considerations.

(ii) The proof generalizes easily to the case of weak mixing Anosov flows and Gibbs measures with vanishing asymptotic cycles.

(iii) There is apparently no natural generalization of the Arcsine Law to higher dimensions [3], thus makig it unlikely that there is a natural corresponding result for homology covers.

2. An application

Let V be a compact surface and let $\widehat{V}$ be a $\mathbb{Z}$-cover for V with the lifted metric and the canonical projection $\pi : \widehat{V} \to V$. Assume the $\mathbb{Z}$-cover $\widehat{V}$ for V has the lifted metric. The ergodic behaviour of typical geodesics on $\widehat{V}$ was perhaps first studied in work of Guivarc'h [4] who showed that the geodesic flow on V was exact.

Notation. Choose a closed geodesic τ on $\widehat{V}$ which divides $\widehat{V}$ into two components $V = \widehat{V}^+ \cup \widehat{V}^-$, say. Given any unit tangent vector $\tilde{v} \in T_1\widehat{V}$ we can consider the associated geodesic $\gamma_{\widehat{v}} : \mathbb{R} \to \widehat{V}$ with $\dot{\gamma_{\widehat{v}}}(0) = \widehat{v}$. For any $T > 0$, we can consider the total amount of time

$$\lambda_{\widehat{v}}^+(T) = \mathrm{Leb}\left(\{0 \leq t \leq T : \gamma_{\widehat{v}}(t) \in T_1\widehat{V}^+\}\right)$$

that the segment $\gamma_{\widehat{v}}([0,T])$ spends in the component V^+. Similarly, we can denote by $\lambda_v^-(T)$ the amount of time segment the $\gamma_{\widehat{v}}([0,T])$ spends in V^-. (For almost all $\widehat{v}$ we have that $\lambda_{\widehat{v}}^+(T) + \lambda_{\widehat{v}}^-(T) = T$).

For typical geodesics the distribution of these times satisfies a classical arcsine law. Let $M \subset T_1\widehat{V}$ be a fundamental domain for T_1V. Let $\widehat{\mu}$ be the restriction of the Riemannian volume on $\widehat{V}$ to M, normalized so that $\widehat{\mu}(M) = 1$.

The following is a corollary to Theorem 1.

Proposition 2.1. *The probability that a geodesic spends a proportion of its time greater than* $0 < x < 1$ *in* V^+ *is given by*

$$\lim_{T\to+\infty} \mu\left\{v \in M : \frac{\lambda_v^+(T)}{T} \geq x\right\} = \frac{2}{\pi}\sin^{-1}(\sqrt{x}).$$

Proof. Let ω be a differential 1-form corresponding to the cover $\widehat{V}$. By adding a boundary element df we can can assume, without loss of generality that ω vanishes on the geodesic γ. Consider a geodesic $\widehat{\gamma}_v$ for which $\widehat{\gamma}_v(t_1), \widehat{\gamma}_v(t_2) \in \widehat{\gamma}$, where $t_1 < t_2$. The concatenation of the curve $\widehat{\gamma}_v([t_1,t_2])$ with a piece of the geodesic $\widehat{\gamma}$ joining the end points is a curve on $\widehat{V}$ which projects to a curve on V which is null in homology. In particular, $\int_{t_1}^{t_2} \langle \omega, X\rangle(\dot{\gamma}_v(t))dt = 0$, showing that the integral of $\langle \omega, X\rangle$ changes sign as the geodesic crosses the curve γ.

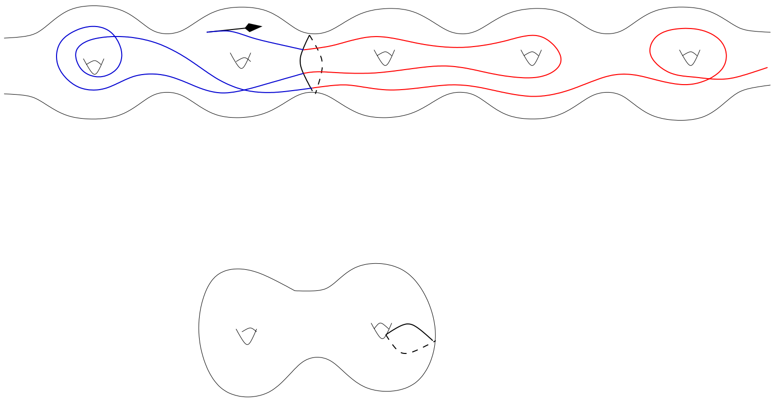

FIGURE 1. A typical geodesic on a $\mathbb{Z}$-cover of a negatively curved surface

Remark. Similarly, the probability that a geodesic spends a proportion of its time greater than $0 \leq x < 1$ in V^- is given by

$$\lim_{T\to+\infty} \mu\left\{v \in M : \frac{\lambda_v^-(T)}{T} \geq x\right\} = \frac{2}{\pi}\sin^{-1}(\sqrt{x}).$$

3. A second ArcSine Law

There is a second Arcsine law that relates to the last time $t \leq T$ that the associated element was zero.

Notation. For any $T > 0$, we can consider the largest $0 \leq t \leq T$ such that $c(v,T) = 0$. We denote this by $l(v,T)$

Theorem 2. *For any $0 < \alpha < 1$ we have that*

$$\lim_{T\to+\infty} \mu\left\{v \in T_1V : \frac{l(v,T)}{T} \geq \alpha\right\} = \frac{2}{\pi}\sin^{-1}(\sqrt{\alpha}).$$

In particular, the probability that a typical geodesic has $l(v,T) \geq 3T/4$ is $\frac{1}{3}$.

Proof of Theorem 2. The proof of this theorem is very similar to the proof of Theorem 1. We can consider the map $g : C([0,1],\mathbb{R}) \to \mathbb{R}$ be defined by

$$g(f) = \sup\{0 \leq t \leq 1 : f(t) = 0\} \text{ and for each } f \in C([0,1],\mathbb{R}),$$

This function is again measurable and continuous except on a set of zero measure [1, Appendix II]. For any $0 \leq \alpha < 1$ we can apply the Continuous Mapping Principle to deduce that if $A = \{f \in C([0,1],\mathbb{R}) : g(f) \geq \alpha\}$. then

$$\lim_{T\to 0} \mu\{x : g(F_T(\cdot,x)) \geq \alpha\} = W\left(\{f \in C([0,1],\mathbb{R}) : g(f) \geq 0\}\right).$$

Moreover, the expression on the Right Hand Side of this last identity has been explicitly computed and is again equal to the expression in the statement of the Proposition. □

References

1. P. Billingsley, *Convergence of probability measures* (1968), John Wiley, New York.
2. M. Denker and W. Philipp, *Approximation by Brownian motion, for Gibbs measures and flows under a function*, Ergod. Th. and Dynam. Sys. **4** (1984), 541-552.
3. N. Bingham and R. Doney, *On higher-dimensional analogues of the arc-sine law*, J. Appl. Probab. **25** (1988), 120–131.
4. Y. Guivarc'h, *Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés*, Ergodic Theory Dynam. Systems **9** (1989), 433–453..
5. C. Heyde, *Invariance principles in statistics*, Internat. Statist. Rev. **49** (1981), 143-152.
6. M. Thaler and R. Zweimuller, *Distributional limit theorems in infinite ergodic theory*, Probab. Theory Related Fields **135** (2006).
15–52.

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