Apollonian Circle Packings

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Circle packings are a particularly elegant and simple way to construct quite complicated and elaborate sets in the plane. One systematically constructs a countable family of tangent circles whose radii tend to zero. Although there are many problems in understanding all of the individual values of their radii, there is a particularly simple asymptotic formula for the radii of the circles, originally due to Kontorovich and Oh [4]. In this partly expository note we will discuss the history of this problem, explain the asymptotic result and present an alternative approach.

1 A brief history of Apollonian circles

Apollonius (c. 262 - 190 BC) was born in Perga (now in Turkey) and gave the names to various types of curves still used: ellipse, hyperbola and parabola. However, very little detail is known about his life and, although he wrote extensively on many topics, rather little of his work has survived (perhaps partly because it was considered rather esoteric by his contemporaries). What has survived (partly in the form of translations into arabic) includes seven of his eight books on "conics". These include problems on tangencies of circles.

![Figure 1: The three initial circles $C_1$, $C_2$, $C_3$, and the two mutually tangent circles $C_0$ and $C_4$ guaranteed by Apollonius’ theorem.](image)

The result of Apollonius which is of particular interest to us is the following.

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$^{1}$I am very grateful to Richard Sharp for many discussions on this approach and the details.
Theorem 1.1. Given three mutually tangent circles $C_1, C_2, C_3$ with disjoint interiors there are precisely two circles $C_0, C_4$ which are tangent to each of the original three.

This result is illustrated in Figure 1 (b). The proof is so easy and short that we include it.

Proof. We can apply a Möbius transformation which takes a point of tangency between two of the initial circles to infinity. These two circles are then mapped to two parallel lines, and the third initial circle to a circle between, and just touching, these parallel lines. We can then construct the two new circles by translating the middle circle between the parallel lines and then transforming back. Since a Möbius transformation preserves circles and lines we are done.

In 1643, Descartes wrote to Princess Elizabeth of Bohemia (1618-1680) stating formula he had established on the radii of the tangent circles, and for which she independently provided a proof. The radii are related by the following formula.

Theorem 1.2 (Descartes-Princess Elizabeth). Assume that the radii of the original circles are $a_1, a_2, a_3 > 0$ and the fourth mutually tangent circle has radius $a_4 > 0$ then

$$2 \left( \frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} + \frac{1}{a_4^2} \right) = \left( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \right)^2.$$ 

A simple proof appears in the notes of Sarnak [14].

Princess Elizabeth was a genuine princess by virtue of being the daughter of Queen Elizabeth (1596-1662) and King Frederick V of Bohemia (whose reign lasted a brief 1 year and 4 days). Queen Elizabeth of Bohemia was in turn the daughter of King James I of England. In 1605, King James was the target of an unsuccessful assassination plan (the “gunpowder plot” of Guy Fawkes and co-conspirators, celebrated in England annually on 5 November) and Queen Elizabeth of Bohemia would have become Queen of England (aged 9) had the plot succeeded.
In 1646, Elizabeth's brother, Philip, stabbed to death Monsieur L'Espinay, for flirting with their mother and sister. In the ensuing family rift, Elizabeth wrote to Queen Christina of Sweden for an audience and help reinstating her Father's lands, but Christina invited Descartes to Stockholm instead, which proved unfortunate for him since he promptly died of pneumonia. Finally, Elizabeth entered a convent in Germany for the last few years of her life, where she worked her way up to the top job of abbess.

The formula of Descartes was subsequently rediscovered by Frederick Soddy (1877-1956), which is the reason that the circles are sometimes called “Soddy circles”. Frederick Soddy is more famous (outside of Mathematics) for having won the Nobel Prize for Chemistry in 1921, and having introduced the terms “isotopes” and “chain reaction”. However, most relevant to us, he rediscovered the formula of Descartes and published it in the distinguished scientific journal *Nature* in the form of a poem [15]:

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Figure 3: (a) Princess Elizabeth of Bohemia (artist unknown); (b) Queen Elizabeth of Bohemia (artist Gerrit van Honthorst (1592-1656)); (c) King James I of England (artist John de Critz (1551-1642))

Figure 4: (a) Rene Descartes (artist Frans Hals (1582-1666)); (b) Descartes at the court of Queen Christina (artist Nils Forsberg, after Pierre Louis Dumesnil (1698-1781))
2 Circle Counting

2.1 The asymptotic formulae

We can order the number of circles ordered by (the reciprocal of) their radii. It is easy to see that the sequence \( \left( \frac{1}{a_n} \right) \) tends to infinity or, equivalently, the sequence of radii \( (a_n) \) tends to zero. This is because the total area of the disjoint disks enclosed by the circles \( \sum_{n=1}^{\infty} \pi a_n^2 \) which is in turn is bounded by the area inside the outer circle. A natural question is then to ask: How fast does the sequence \( \left( \frac{1}{a_n} \right) \) grow, or, equivalently, how fast do the radii \( (a_n) \) tend to zero?

We begin with some notation.

Definition 2.1. Given, \( T > 0 \) we denote by \( N(T) \) the finite number of circles with radii greater than \( \frac{1}{T} \).

In particular, we see from our previous comments that \( N(T) \to +\infty \) as \( T \to 0 \). A far stronger result is the following [4], [9].

Theorem 2.2 (Kontorovich-Oh, 2009). There exists \( C > 0 \) and \( \delta > 1 \) such that the number \( N(T) \) is asymptotic to \( CT^\delta \) as \( T \) tends to infinity, i.e.,

\[
\lim_{T \to \infty} \frac{N(T)}{T^\delta} = C.
\]

It is the convention to write \( N(T) \sim CT^\delta \) as \( T \to \infty \).

We can consider two examples we looked at before:

Example 2.3. We can consider the reciprocals of the sequence of radii \( a_0 = -\frac{1}{3}, a_1 = \frac{1}{5}, a_2 = \frac{1}{8} \) and \( a_3 = \frac{1}{8} \). Using Theorem 1.2 we can compute following monotone increasing sequence of reciprocal radii:

\[
\left( \frac{1}{a_n} \right) = 5, 8, 8, 12, 12, 20, 20, 21, 29, 29, 32, 32, \ldots
\]

(see Example 4.13 in the Appendix)
2.2 The exponent $\delta$

Of particular interest is the value of $\delta$ which controls the rate of growth of the radii and has a simple interpretation.

Lemma 2.5. We can write that

$$\delta = \inf \left\{ t > 0 : \sum_{n=1}^{\infty} \frac{1}{a_n^t} < +\infty \right\}.$$ 

We call such numbers the packing exponent. It is actually equal to the Hausdorff Dimension of the limit set (i.e., the closure of the union of the circles).

Remark 2.6 (The numerical value of $\delta$). Unfortunately, there is no explicit expression for $\delta$ and it is rather difficult to estimate. The first rigorous estimates were due to Boyd [2] who, using the definition above, estimated $1 \cdot 300197 < \delta < 1 \cdot 314534$. A well known estimate is due to McMullen [8], which is: $\delta = 1 \cdot 30568 \ldots$

Perhaps surprisingly, there is a unique choice of $\delta$ which is independent of the actual radii of the circles.

Lemma 2.7. For different Apollonian circle packings exactly the same value of $\delta$ arises (independently of the initial choices $a_0, a_1, a_2, a_3, a_4$).

Again the idea of the proof is so simple that we recall the idea so as to dispel any mystery.
Proof. Let \( C_1 \) and \( C_2 \) be any two Apollonian circle packings. We want to deduce the independence of the value \( \delta \) using the following result: If there exists a smooth bijection \( T : C_1 \to C_2 \) then the sets share the same Hausdorff Dimension. Let us identify the plane with \( \mathbb{C} \). Then it is a simple exercise to show that there is a Möbius transformation \( g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) of the form

\[
g(z) = \frac{az + b}{\overline{b}z + \overline{a}}
\]

and \( a, b \in \mathbb{C} \) with \( |a|^2 - |b|^2 = 1 \), such that \( T(C_1) = C_2 \). In particular, this is because Möbius transformations always take circles to circles. \( \square \)

3 Some preliminaries for a proof

We will describe a proof which differs from the original proof of Kontorovich-Oh and other proofs. This approach is more in the spirit of the classical proof of the Prime Number Theorem, except we use approximating Poincaré series in place of zeta functions.

3.1 An analogy with the prime numbers

Purely for the purposes of motivation, we recall the Prime Number Theorem. Let

\[ 2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots \]

denote the prime numbers. Let \( \pi(x) \) denote the number of primes numbers between 1 and \( x \). Since there are infinitely many primes, we see that \( \pi(x) \to \infty \) as \( x \) tends to infinity. How does \( \pi(x) \) grow as \( x \to +\infty \)? The solution is the classical prime number Theorem [3].

**Theorem 3.1** (Prime Number Theorem: Hadamard, de la Valle Poussin (1896)). \( \pi(x) \sim \frac{x}{\log x} \) as \( x \to +\infty \), i.e.,

\[
\lim_{x \to +\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1.
\]

The essence of the proof of the Prime Number Theorem is to analyse the associated complex function: The *Riemann zeta function*, defined formally by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}.
\]

The Riemann zeta function has the following important basic properties [3].

**Lemma 3.2.** The Riemann zeta function \( \zeta(s) \) converges to a well defined function for \( \text{Re}(s) > 1 \). Moreover:

1. for \( \text{Re}(s) > 1 \) we have that \( \zeta(s) \) is analytic and non-zero;
2. for \( s = \delta + it \) with \( t \neq 0 \) there exists a small neighbourhood in which \( \zeta(s) \) is analytic;
3. \( \zeta(s) \) has a simple pole at \( s = 1 \).

The result then follows by using a Tauberian theorem to convert this information on the domain of \( \zeta(s) \) into information on prime numbers. For completeness, we recall the statement of the Ikehara-Wiener Tauberian Theorem [3].

**Theorem 3.3** (Ikehara-Wiener Tauberian Theorem). Assume that \( \rho : \mathbb{R} \to \mathbb{R} \) is a monotone increasing function for which there exists \( c > 0, \delta > 0 \) such that

\[
\int_0^\infty t^{-s}d\rho(t) - \frac{c}{s-\delta}
\]

is analytic in a neighbourhood of \( \text{Re}(s) \geq \delta \) then \( \lim_{T \to +\infty} \frac{\rho(T)}{T^\delta} = c \).

Remark 3.4. The Prime Number Theorem easily follows from applying the Theorem 3.3 to the auxiliary function \( \rho(T) = \sum_{p \leq T} \log p \) and then relating the Steiltjes integral to \( \zeta'(s)/\zeta(s) \). We refer the reader to [3] for further details of these now standard manipulations.

To adapt the proof of the Prime Number Theorem to the present setting, suggests considering a new complex function

\[
\xi(s) = \sum_{n=1}^\infty a_n^n
\]

where \( a_n \) are the radii of the circles in the Apollonian circle packing. In fact, it is more convenient to study a related function (a Poincaré series) and use an approximation argument to get the final result. However, to analyse such functions, we first introduce a dynamical ingredient.

### 3.2 An iterated function scheme viewpoint

Let us again identify the plane with the complex numbers \( \mathbb{C} \), then we can introduce a transformation which preserves the circle packing \( \mathcal{C} \). We want to define the “reflection” \( R \) in the circle \( \mathcal{C} = C(z_0, r) \) of radius \( r \) centered at \( z_0 \).

![Figure 7: Reflection in a circle](image)

More precisely, let \( z_0 \in \mathbb{C} \) and radius \( r > 0 \) then we associate a transformation

\[
R : \mathbb{C} - \{z_0\} \to \mathbb{C} - \{z_0\}
\]

\[
R(z) = \frac{r^2(z - z_0)}{|z - z_0|^2} + z_0.
\]
Rather than reflecting in the original Apollonian circles, we need to find four “dual circles” which we will reflect in. This point of view has a nice historical context. The original statement of the result was due to Philip Beecroft (1818-1862) who was a school teacher in Hyde, near Manchester, in England, and was the son of a miller and lived with his two elder sisters [1]. In his article he too had recovered Theorem 1.2.

Theorem 3.5 (Philip Beecroft, from “Lady’s and Gentleman’s diary” in 1842). “If any four circles be described to touch each other mutually, another set of four circles of mutual contact may be described whose points of contact shall coincide with those of the first four.”

As in [4], we associate to the four initial Apollonian circles a new family of “dual” tangent circles (the dotted circles in the figure). We can then consider the four associated reflections $R_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ in the four dual circles $K_1, K_2, K_3, K_4$ as shown in the figure ($i = 1, 2, 3, 4$).

The aim is to associate to the Apollonian circle packings complex functions, playing the rôle of the zeta function in number theory. These will be defined in terms of a family of contractions (i.e., an associated iterated function scheme) built out of the maps $R_i$ on each of the four curvilinear triangles external to the initial four circles. For definiteness, let us fix the central curvilinear triangle $T$, whose sides are arcs from the circles $C_1, C_2$ and $C_3$ (with the other cases being similar) and let $x_1, x_2, x_3$ denote the vertices. We can consider the three natural linear fractional contractions $f_1, f_2, f_3 : T \to T$ defined by

$$f_i = R_4 \circ R_i, \quad i = 1, 2, 3,$$

each of which fixes the vertex $x_i$ of $T$. A simple calculation gives that:

- $|f'_i(z)| < 1$ for $z \in T - \{x_i\}$ for $i = 1, 2, 3$; and

- $|f'_i(x_i)| = 1$ for $i = 1, 2, 3$ (i.e., $x_i$ is a parabolic point at the point of contact of $K_4$ with $K_1, K_2$ and $K_3$, respectively).
We recall the following explicit example from [6]:

**Example 3.6** (cf. [6]). In the case of the Apollonian circle packing with $a_0 = -1$ and $a_1 = a_2 = a_3$ we can explicitly write:

$$f_1(z) = \frac{az + b}{bz + a} \text{ where } a = -5 \sqrt{\frac{4\sqrt{3} - 3}{78}} \text{ and } b = \sqrt{\frac{100\sqrt{3} - 153}{78}}$$

and $f_2(z) = e^{-2\pi i/3} f_1(e^{2\pi i/3}z)$ and $f_3(z) = e^{-2\pi i/3} f_1(e^{2\pi i/3}z)$.

In particular, one can easily check that:

1. For each $i = 1, 2, 3$ the iterates $f_i^k : T \to T$ ($k \geq 1$) have the effect of mapping the central circle $C_4$ onto a sequence of circles $\{f_i^k(C_4)\}_{k=1}^\infty$ occurring in $\mathcal{C}$ leading into the vertex $x_i$ (cf. Figure 9 (a)); and

2. Any sequence of compositions of these three maps can be naturally written in the form $\overline{f} := f_{i_k}^{n_k} \cdots f_{i_1}^{n_1} : T \to T$, for $n_1, \cdots, n_k \geq 1$ and $i_1, \cdots, i_k \in \{1, 2, 3\}$ with $i_l \neq i_{l+1}$ for $1 \leq l \leq k - 1$.

The relevance of these maps to our present study is that we see that we can rewrite

$$\xi(s) = \sum_{\overline{f}} \text{diam}(\overline{f}(C_0))^s,$$

at least for the contribution of circles in $T$, the other cases being similar, where the summation is over all such compositions $\overline{f} = f_{i_k}^{n_k} \cdots f_{i_1}^{n_1}$ in item 2 above.
3.3 Contracting maps and Poincaré series

The maps described above can be conveniently regrouped as follows:

\[ \mathcal{T} := f_{i_k}^{n_k-1} \circ (f_{i_k} \circ f_{i_{k-1}}^{n_{k-2}}) \circ \cdots \circ (f_{i_2} \circ f_{i_1}^{n_2}) \circ (f_{i_2} \circ f_{i_1}^{n_1}). \]  

(1)

The advantage of this presentation is that at least part of this expression is contracting, in the following sense (cf. [6]).

Lemma 3.7 (after Mauldin-Urbanski). For the Apollonion circle packings we have that the maps \( \phi_j = \phi_j^{(i_j,n_j)} : T \to T \) are uniformly contracting (i.e., sup \( \sup_{z \in T} |\phi_j'(z)| < 1 \)).

This is illustrated in Figure 9 (b) with \( f_3^n f_1, n \geq 1 \).

Unfortunately, considering only compositions of the uniform contractions \( \phi_j \) leads only to some of the circles in the circle packing \( \mathcal{C} \). The rest of the circles require the final application of the maps \( f_{i_k}^{n_k-1} \) in (1), which therefore also needs addressing. Moreover, the counting function we will actually use is a more localized version, which allows us to approximate the counting function for circles by a counting function for derivatives - for which the associated complex functions are easier to analyse. In particular, we want to analyse the following related complex functions.

Definition 3.8. Given \( z_0 \in T \) and an allowed word \( j = (j_1, \ldots, j_N) \), with \( j_r \neq j_{r+1} \) for \( r = 1, \ldots, N-1 \), we can associate a localised Poincaré function

\[ \eta^j(s) = \sum_{k=0}^{\infty} \sum_{\phi} |(f_i^k \circ \phi \circ \phi_j)'(z_0)|^s \]

(2)

where:

1. we first apply a fixed contraction \( \phi_j = \phi_j^{(i_j,n_j)} \);
2. we next sum over all subsequent allowed hyperbolic compositions \( \phi := \phi_i \circ \cdots \circ \phi_{i_{N+1}} : T \to T \); and, finally,
3. we sum over the “parabolic tails” \( f_i^k \) (where \( i \) is associated to \( \phi_i = f_i \circ f_i^n \), say).

The need to consider the contribution from different \( \phi_j \) is an artefact of our method of approximation in the proof.

Remark 3.9. Poincaré series are more familiar in the context of Kleinian groups \( \Gamma \) acting on three dimensional hyperbolic space and its boundary, the extended complex plane \( \hat{\mathbb{C}} \). Our analysis applies to the Poincaré series of many such groups. In the particular case of classical Schottky groups the analysis is easier, since one can dispense with the parabolic tail (i.e., item 3 above).

As we will soon see, each such Poincaré series satisfies the hypotheses of Theorem 3.3, which allows us to estimate the corresponding counting function defined as follows.

Definition 3.10. We define an associated counting function

\[ M^j(T) = \# \{ f_i^k \circ \phi \circ \phi_j : |(f_i^k \circ \phi \circ \phi_j)'(z_0)| \leq T \} \text{ for } T > 0. \]
Let $\Sigma = \{(i_n)_{n=1}^{\infty} : i_n \neq i_{n+1} \text{ for } n \geq 0\}$ and consider the cylinder

$$[j] = \{(i_n)_{n=1}^{\infty} \in \Sigma : i_r = j_r, \text{ for } 1 \leq r \leq N\}.$$ 

In particular, in the next section we will use the Poincaré series to deduce the following.

**Proposition 3.11.** There exists $C > 0$ and a measure $\mu$ on $\Sigma$ such that $M(j)(T) \sim C\mu([j])T^\delta$ as $T \to +\infty$, i.e.,

$$\lim_{T \to +\infty} \frac{M(j)(T)}{T^\delta} = 1.$$ 

There may be some circles whose radii we don’t seem to capture with this coding, but their contribution doesn’t affect the basic asymptotic.

### 4 The proof of Theorem 2.2

To complete the proof of Theorem 2.2 we need to complete the proof of Proposition 3.11 (in subsection 4.1 below) and then perform the approximation of the counting functions for circles by those for derivatives (in subsection 4.2 below).

#### 4.1 Extending the Poincaré series

By the chain rule we can write

$$(f_i^k \circ \phi \circ \phi_j)'(z_0) = (f_i^k)'(\phi \circ \phi_jz_0)\phi'(\phi_jz_0)\phi_j'(z_0)$$

and, in particular, we can now rewrite the expression for $\eta(j)$ in (2) as:

$$\eta(j)(s) = |\phi_j'(z_0)|^s \sum_{n=0}^{\infty} \sum_{|\phi|=n} \sum_{l=0}^{\infty} |(f_i^k)'(\phi \circ \phi_jz_0)\phi'(\phi_jz_0)|$$

$$= |\phi_j'(z_0)|^s \sum_{n=0}^{\infty} \sum_{|\phi|=n} |\phi'(z_0)|^n h_s(\phi(z_0))$$

(3)

where the function $h_s : T \to \mathbb{C}$ is defined by the summation

$$h_s(z) := \sum_{l=0}^{\infty} |(f_i^l)'(z)|^s \in C^1(T)$$

is analytic in $s$. In particular, we see from the following lemma that $h_s(z)$ converges to a well defined function for $Re(s) > \frac{1}{2}$.

**Lemma 4.1.** We can estimate $||(f_i^l)'||_{\infty} = O(l^{-2})$.

We recall the simple proof (cf. [7]).
4.1 Extending the Poincaré series

Proof. By a linear fractional change of coordinates (mapping the vertex of \( T \) to infinity) the map \( f_i \) becomes a translation. Transforming this back to convenient coordinates we can write, say,

\[ f_i^l(z) = \frac{(\sqrt{3} - l)z + l}{(-lz + l + \sqrt{3})}. \]

From this we see that

\[ |(f_i^l)'(z)| = \frac{1}{|-lz + l + \sqrt{3}|^2} \]

and the required estimate follows.

The Poincaré series have the useful feature that they can be expressed simply in terms of linear operators on appropriate Banach spaces of functions.

Definition 4.2. Let \( C^1(T) \) be the Banach space of \( C^1 \) functions on \( T \). We can consider the transfer operators \( \mathcal{L}_s : C^1(T) \to C^1(T) \) \((s \in \mathbb{C})\) given by

\[ \mathcal{L}_sw(x) = \sum_i |\phi_i'(x)|^s w(\phi_i x) \]

where \( w \in C^1(T) \). This converges provided \( \text{Re}(s) > \frac{1}{2} \).

We are actually spoilt for choice of Banach spaces. Although the continuous functions \( C^0(T) \) is too large a space for our purposes, we could also work with Hölder continuous functions or suitable analytic functions (on some neighbourhood of the complexification of \( T \) thought of as a subset of \( \mathbb{R}^2 \)). The choice of \( C^1(T) \) is perhaps the more familiar.

The approach in the rest of this subsection is now relatively well known (cf. [11], [7], [5], for example) and is a variant on the symbolic approach to Poincaré series and the hyperbolic circle problem [12], [13].

Lemma 4.3. The operators are well defined provided \( \text{Re}(s) > \frac{1}{2} \). Moreover, for \( \text{Re}(s) > \delta \) we have that the spectral radius satisfies

\[ \rho(\mathcal{L}_s) := \limsup_{n \to +\infty} \|\mathcal{L}_s^n\|^{\frac{1}{n}} < 1. \]

In particular, we see from the definition of \( \mathcal{L}_s \) that we can write

\[ \mathcal{L}_s^n w(z) = \sum_{\phi} |\phi'(z)|^s w(\phi z), \text{ for } n \geq 2, \]

where the summation is over allowed compositions of contractions \( \phi = \phi_{i_n} \circ \cdots \circ \phi_{i_1} \). We can now rewrite the expression for the Poincaré series in (3) more concisely as

\[ \eta^\dagger(s) = |\phi_j'(z_0)|^s \sum_{n=0}^\infty \mathcal{L}_s^n h_{\phi_j z_0}(\phi_j z_0). \]

In order to construct the required extension of \( \eta^\dagger(s) \), we recall the following simple lemma improving on the result in Lemma 4.3.
Lemma 4.4. Let $\text{Re}(s) = \delta$. Then

1. for $s = \delta + i t$ with $t \neq 0$ we have that the spectral radius satisfies $\rho(L_s) < 1$; and
2. for $s = \delta$ we can write $L_\delta = Q + U$ where
   
   (a) $Q$ is a (one dimensional) eigenprojection with $QU = UQ = 0$, $Q^2 = Q$, and
   (b) and $\limsup_{n \to +\infty} ||U^n||^{1/n} < 1$.

Remark 4.5. The spectral properties of $L_s$ can be seen when the operator acts on $C^1$ functions. Alternatively, we could have looked at bounded analytic functions on a small enough neighbourhood $T \subset U \subset \mathbb{C}^2$ in the complexification (cf. [5]).

We can now deduce almost immediately from Lemma 4.3 and Lemma 4.4 the following corollary for this Poincaré series.

Corollary 4.6. The Poincaré series converges to an well defined analytic function provided $\text{Re}(s) > \delta$. Moreover,

1. for $s = \delta + i t$ with $t \neq 0$ there exists a small neighbourhood in which $\eta_L(s)$ is analytic; and
2. $s = \delta$ is a simple pole for $\eta_L(s)$.

Remark 4.7. In fact, we can deduce a little more which, if a little technical looking, is needed in the approximation argument below. In particular, we can also show that the simple pole for $\eta_L(s)$ at $s = \delta$ has a residue of the form $C_j := \frac{|(\phi_j)'(x_0)| \mu(h_\delta)}{\lambda(s)}$ where:

(i) $\lambda(t)$ is an isolated eigenvalue equal to the spectral radius of $L_t$ ($t \in \mathbb{R}$); and
(ii) $Q(h) = \mu(h) k$ where $k$ is an associated eigenfunction, i.e., $L_1 k = k$.

If we now write

$$\eta_L(s) = \int_1^\infty t^{-s} dN_L(t)$$

then comparing Corollary 4.6 with Theorem 3.3 gives the asymptotic formula for $N_L(T)$ in Proposition 3.11.

Let us now move onto the final step in the proof of Theorem 2.2.

4.2 The approximation argument

We can now approximate the values $\text{rad}(g(C_4))$ by suitably scaled values of $1/|g'(x_0)|$, where $g = f_i \circ \widetilde{\varphi} \circ \phi_{2i}$. Without loss of generality we can choose coordinates so that $C_4$ is the unit circle.

As a prelude to this we consider some simple geometric estimates on the sizes of the images of circles.
The approximation argument

Figure 10: The radius of \( g(C_4) \) is related to the derivative \( |g'(0)| \) by the value of \( g^{-1}(\infty) \).

Lemma 4.8. If \( g(z) = \frac{az+b}{cz+d} \) with \( ad-bc = 1 \) and \( a, b, c, d \in \mathbb{C} \), then the radius of the image circle \( C = g(C_4) \) is equal to

\[
\frac{1}{||c|^2 - |d|^2|} = \frac{|g'(0)|}{||\frac{c}{d}|^2 - 1|}
\]

The proof is a reassuringly elementary exercise:

Proof. For the first part, we see that the image circle \( g(C_0) \) has centre \( z_c = \frac{a\overline{c} - b\overline{d}}{||c|^2 - |d|^2|} \) and radius \( \frac{1}{||c|^2 - |d|^2|} \) since we can check that for \( e^{i\theta} \in C_4 = \{ z \in \mathbb{C} : |z| = 1 \} \):

\[
|g(e^{i\theta}) - z_c| = \left| \frac{ae^{i\theta} + b}{ce^{i\theta} + d} - \frac{a\overline{c} - b\overline{d}}{||c|^2 - |d|^2|} \right| = \frac{1}{||c|^2 - |d|^2|}.
\]

We then observe that \( |g'(z)| = |cz + d|^{-2} \) and thus \( |g'(0)| = |d|^{-2} \). Thus by the above we see that the radius of the image circle \( C \) is:

\[
\text{rad}(C) = \frac{1}{||c|^2 - |d|^2|} = \frac{|g'(0)|}{||\frac{c}{d}|^2 - 1|}.
\]

as claimed. \( \square \)

We can write \( g^{-1}(z) = \frac{d \overline{c} - b}{c \overline{z} + a} \) and thus \( g^{-1}(\infty) = \frac{d}{c} \).

Finally, we come to the crux of the approximation argument. The essential idea is to approximate the (technically more convenient) weighting of elements \( g \) by \( |g'(z_0)| \), with a weighting by the more geometric weighting by reciprocals of the radii \( \text{rad}(g(C_4)) \). One simple approach is as follows. We are taking \( z_0 = 0 \), for definiteness, and then we want to use Proposition 3.11 to localise the counting to regions where

\[
\frac{|g'(0)|}{\text{rad}(g(C_4))} = \left| \frac{c}{d} \right|^2 - 1\]
4.3 Generalizations

The approach to counting circles is more analytical than geometrical, and thus is somewhat oblivious to the specific setting of circle packings. In particular, the same method of proof works in a number of related settings where we ask for the radii of circles which are images under a suitable Kleinian group. For example:
4.3 Generalizations

1. Other circle packings for which the circles can be generated by the image of circles under reflections;

2. The radii of the images $g(C)$ of a circle $C$, where $\Gamma \subset SL(2, \mathbb{C})$ is a Schottky group (i.e., a convex cocompact Kleinian group generated by reflections in a finite number of circles with disjoint interiors);

3. The radii of the images $g(C)$ of a circle $C$, where $\Gamma \subset SL(2, \mathbb{C})$ is a quasi-Fuchsian group.

The same basic method can also be used to prove other statistical properties of the radii of the circles.

![Figure 11: Sets generated by reflection in: (1) tangent circles (2) disjoint circles, and (3) overlapping circles](image)

**Appendix: The case of reciprocal integer circles**

The following is an interesting corollary to Descartes' Theorem.

**Corollary 4.11.** If $\frac{1}{a_0}, \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \in \mathbb{Z}$ then $\frac{1}{a_4} \in \mathbb{Z}$.

**Proof.** In particular, this is a quadratic polynomial in $\frac{1}{a_4} > 0$, so given the radii of the initial circles $a_1, a_2, a_3$ we have two possible solutions

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \pm 2\sqrt{\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4}}.$$

and we denote these $\frac{1}{a_4} > 0$ (and $\frac{1}{a_0} < 0$). We use the convention that the smaller inner circle has radius $a_4 > 0$ and the larger outer circle has a negative “radius” $a_4$ (meaning its actually radius is $|a_4| > 0$ and the negative sign just tells us it is the outer circle). Adding these two solutions gives:

$$\frac{1}{a_0} + \frac{1}{a_4} = 2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right).$$

Since we have seen that

$$\frac{1}{a_0} + \frac{1}{a_4} = 2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right)$$

we easily deduce the result. \qed
Proceeding inductively, for any subsequent configuration of four circles with radii $a_n, a_{n+1}, a_{n+2}, a_{n+3}$, for $n \geq 0$, we can similarly write

$$\frac{1}{a_{n+4}} = 2 \left( \frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} + \frac{1}{a_{n+3}} \right) - \frac{1}{a_n}.$$ 

Proceeding inductively, then one gets infinitely many circles. Moreover, if the reciprocals of the initial four circles are integers then we easily see that this is true for all subsequent circles.

**Corollary 4.12.** If four initial Apollonian circles have that their radii are reciprocals of integers then so do all of the rest.

**Example 4.13.** Let us start with an example with $a_0 = -\frac{1}{3}, a_1 = \frac{1}{5}, a_2 = \frac{1}{8}, a_3 = \frac{1}{8}$. We then iteratively inscribe a circle into each curved triangle formed by three tangent circles. For the purposes of illustration we write $a_n$ inside the circle of radius $\frac{1}{a_n}$ for each iteration.

![Figure 12](image)

Figure 12: We iteratively inscribe additional circles starting with circles of radii $a_0 = -\frac{1}{3}, a_1 = \frac{1}{5}, a_2 = \frac{1}{8}, a_3 = \frac{1}{8}$

**Example 4.14.** Let us also consider the example with $a_0 = -\frac{1}{2}, a_1 = \frac{1}{3}, a_2 = \frac{1}{6}, a_3 = \frac{1}{7}$. We then again iteratively inscribe a circle into each curved triangle formed by three tangent circles.

![Figure 13](image)

Figure 13: We iteratively inscribe additional circles starting with circles with radii $a_0 = -\frac{1}{2}, a_1 = \frac{1}{3}, a_2 = \frac{1}{6}, a_3 = \frac{1}{7}$

**Remark 4.15.** An easy consequence of the fact $\delta > 1$ is that then $\frac{1}{a_n} \in \mathbb{N}$ some value must necessarily have high multiplicity (since we need to fit approximately $C e^{-\delta}$ inverse diameters into the first $[\epsilon^{-1}]$ natural numbers and the “pigeonhole principle” applies). In subsequent work in 2010, Oh-Shah showed that similar results are true for other sorts of circle packing. Oh-Shah also tidied up the original proof of Kontorovich-Oh using ideas of Roblin.
Remark 4.16. Another question we might ask is: It we remove the repetitions in the sequence \((a_n)\) then how many distinct diameters are greater than \(\epsilon\)? The following result was proved by Bourgain and Fuchs: There exists \(C > 0\) such that

\[
\#\{\text{distinct diameters} \ a_n : a_n \geq \epsilon\} \geq \frac{C}{\epsilon}
\]

for all sufficiently large \(\epsilon\). Previously, Sarnak had proved the slightly weaker result that there exists \(C > 0\) such that

\[
\#\{\text{distinct diameters} \ a_n : a_n \geq \epsilon\} \geq \frac{C}{\epsilon \sqrt{\log \epsilon}}
\]

References


