Dynamical Systems

Let $\sigma : \Sigma_A \to \Sigma_A$ be a subshift of finite type associated to a matrix A. Let $P_m(\sigma)$ denote the number of periodic points $\sigma^m x = x$ of period m.

1. We formally define the zeta function as

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} P_m(\sigma)\right) \quad \text{where } z \in \mathbb{C}.$$

Show that this can be rewritten as an infinite product: $\zeta(z) = \prod_{\gamma} (1 - z^{|\gamma|})^{-1}$, where $\gamma = \{x, \sigma x, \dots, \sigma^{m-1} x\}$ denotes a periodic orbit of prime period $m =: |\gamma|$.

[Hint: Use $\exp\left(\sum_{n=1}^{\infty} z^n/n\right) = \exp(-\log(1-z)) = \frac{1}{1-z}$]

2. Show that $\zeta(z)$ can be rewritten as:
$\zeta(z) = 1/\det(I - zA)$. Deduce that $\zeta(z)$ is rational in z.

3. Calculate the zeta function $\zeta(z)$ when
$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, i.e., the "full shift on 2 symbols".

4. Calculate the zeta function $\zeta(z)$ when
$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

5. Calculate the zeta function $\zeta(z)$ when

$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.