Problems on pencils of small genus

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1 Introduction

Let $f\colon X \to B$ be a fibre space of curves over a base curve B; write $F = f^{-1}(P)$ for a fibre of f, and set $g = \operatorname{genus} F \ge 2$ and $b = \operatorname{genus} B$. For genus 2 pencils, the geometric methods of Horikawa and Xiao Gang are very powerful and practical, and my aim is to try to generalise these to genus 3, 4, 5,...

My problems here are concerned with the study of the relative canonical algebra $\mathcal{R}(X/B) = \mathcal{R}(f)$ of f, and how to use it to get geographical information on X. First the definitions: write $K_{X/B} = K_X - f^*K_B$, and set

$$f_*(K_{X/B}^{\otimes n}) = \mathcal{R}_n \qquad \text{for } n \ge 0.$$

Then $\mathcal{R}_n$ is a locally free sheaf of $\mathcal{O}_B$-modules of rank

$$\dim \mathcal{R}_n = h^0(F, nK_F) = \begin{cases} 1 & \text{if } n = 0; \\ g & \text{if } n = 1; \\ 3g-3 & \text{if } n = 2; \\ (2n-1)(g-1) & \text{if } n \ge 2. \end{cases}$$

This is trivial using base change if $n \ge 2$. For $n = 1$, by base change, you need to prove that $h^0(\mathcal{O}_F) = 1$ for every fibre F; this is a kind of vanishing result which needs $\operatorname{char} k = 0$. It follows by an argument relating the coherent cohomology group $H^0(\mathcal{O}_F)$ at a multiple fibre to the topological monodromy (see e.g. [Xiao], pp.1–3]).

The *relative canonical algebra* is of course

$$\mathcal{R}(X/B) = \mathcal{R}(f, K_{X/B}) = \bigoplus_{n \ge 0} \mathcal{R}_n,$$

with multiplication induced by the tensor product $K_{X/B}^{\otimes a} \otimes K_{X/B}^{\otimes b} \to K_{X/B}^{\otimes a+b}$. It's a finitely generated graded $\mathcal{O}_B$-algebra, usually generated in degrees ≤ 3

and related in degrees ≤ 6 (the "1–2–3 conjecture", soon to be a theorem, I hope). By base change, its stalk at $P \in B$ is an $\mathcal{O}_{B,P}$-algebra whose reduction modulo m_P is the k-algebra

$$R(F) = R(F, K_F) = \bigoplus_{n \geq 0} H^0(F, nK_F),$$

where $F = f^{-1}P$ is the scheme-theoretic fibre over P and $K_F = \omega_F = (K_{X/B})_{|F}$. This means that to understand $\mathcal{R}(X/B)$ locally around P, it's enough to write down $R(F)$ and then do a flat deformation; e.g., to prove the 1–2–3 conjecture it's enough to prove the corresponding statement for $R(F)$. For genus 3 fibrations, this (and much more besides) is proved in Mendes Lopes [ML].

The algebra $\mathcal{R}(X/B)$ can be defined in terms of a minimal nonsingular model, but I usually assume that X is a relative canonical model, that is, at worst Du Val singularities and K_X relatively ample. Then, as usual, $X = \operatorname{Proj}_B \mathcal{R}(X/B)$.

When discussing local properties of fibres, it's important to distinguish whether the property depends on the singular fibre alone, or depends on a (complex) neighbourhood of the fibre (the latter is more usual); this is like stalk $\mathcal{F}_P$ versus fibre $\mathcal{F}_P/(m_P \cdot \mathcal{F}_P)$ of a sheaf. I will say "fibre germ" to mean "tubular neighbourhood of fibre" or "germ of fibration" $f \colon X \to B$ around $P \in B$, but I will sometimes be sloppy and just say "fibre".

I write $\mathcal{M}_g$ for the moduli space of stable curves of genus g, the usual compactification of the moduli space of nonsingular curves. $\mathcal{M}_g$ is a coarse moduli space, so that a fibration $X \to B$ with general fibre a nonsingular curve of genus g corresponds to a rational map $B \dashrightarrow \mathcal{M}_g$, which extends to a morphism $B \to \mathcal{M}_g$, since B is a nonsingular curve and $\mathcal{M}_g$ a projective variety. This morphism is called the *modular invariant* of $X \to B$; the fact that it a priori only a rational map has some noteworthy consequences when the germ is deformed (see §3, x.y). Recall that $\mathcal{M}_g$ has a codimension 1 boundary component $\Delta_a = \{E_1 + E_2\}$ for each $a \leq g/2$, corresponding to the degeneration $C \mapsto E_1 + E_2$ of C into two curves of genus a and $g - a$ meeting transversally in one point.

2 Genus 2

Horikawa's method consists of studying the rank 2 vector bundle $\mathcal{R}_1 = f_* K_{X/B}$ and the relative 1-canonical map $\varphi_{K_{X/B}} \colon X \to \mathbb{P}_B(\mathcal{R}_1)$; on the general fibre of f, this is the canonical double cover $F \to \mathbb{P}^1$ of a genus 2 curve. Horikawa [H] shows how to define an invariant $H(P) = H(X/B, P)$ of a neighbourhood of a singular fibre $F = f^{-1}P$ such that

$$K_X^2 = 2\chi(\mathcal{O}_X) - 6\chi(\mathcal{O}_B) + \sum_{P\in B} H(P). \qquad (*)$$

e.g., if $B = \mathbb{P}^1$ and every fibre is 2-connected (and $q = 0$) then $K^2 = 2p_g - 4$.

I call $H(P)$ the *Horikawa number* of a genus 2 fibre germ. I know of several different ways of defining it:

(i) Horikawa's definition is in terms of his analysis of the 1-canonical model:

$$\begin{aligned} \mathrm{I}_k &\mapsto 2k-1; \\ \mathrm{II}_k &\mapsto 2k; \\ \mathrm{III}_k &\mapsto 2k-1; \\ \mathrm{IV}_k &\mapsto 2k; \\ \mathrm{V} &\mapsto 1. \end{aligned}$$

(ii) The degree of the base locus of the relative 1-canonical system. If X is the relative canonical model, the base locus of $f_*K_{X/B}$ is the subscheme Z of X defined by $f^*f_*\mathcal{O}_X(K_{X/B}) = I_Z \cdot \mathcal{O}_X(K_{X/B})$; if you prefer to work with the relative minimal model, there is a similar expression in terms of minus the selfintersection of the fixed part plus the degree of the base locus of the moving part.

(iii) The length of the cokernel of $S^2\mathcal{R}_1 \hookrightarrow \mathcal{R}_2$.

(iv) (Xiao Gang) The "virtual number of fibres of type $E_1 + E_2$" (with E_1, E_2 elliptic curves meeting transversally in 1 point). As it stand, (iv) is only a dynamic definition: it's known (Horikawa, Xiao Gang, Mendes Lopes) that a tubular neighbourhood of a bad fibre has a small deformation $X_u \to$ disc so that for $u \neq 0$ every fibre of X_u is either nonsingular or has a single node; then count the number of reducible fibres in the disc.

Problem 2.1 It would be interesting to make this definition static, as the intersection number of the modular invariant $B \to \mathcal{M}_2$ with the boundary component $\Delta_1 = \{E_1 + E_2\}$ of $\mathcal{M}_2$; although the universal curve of genus 2 over $\mathcal{M}_2$ is not defined, the corresponding relative canonical algebra may well be defined?

It's easy to pass from the concrete definition (i) to the more abstract definitions (ii), (iii) or (iv), or to pass from (ii) or (iii) to the global result $(*)$. For example $|K_X+aF|$ for $a \gg 0$ has base locus exactly Z, and φ_{K_X+aF} is a 2-to-1 map to a normally embedded scroll over B. The proof that my favourite definition (iii) implies $(*)$ is an interesting calculation:

Lemma 2.2

$$K_X^2 = 2\chi(\mathcal{O}_X) - 6\chi(\mathcal{O}_B) + \operatorname{length}\operatorname{coker}\{S^2\mathcal{R}_1 \hookrightarrow \mathcal{R}_2\}$$

Proof The fact that $S^2\mathcal{R}_1 \to \mathcal{R}_2$ is an inclusion comes at once from the fact that on the general fibre φ_K maps F onto $\mathbb{P}^1$; the same argument shows $S^n\mathcal{R}_1 \hookrightarrow \mathcal{R}_n$ for all n. Hence the length of the cokernel equals $\chi(B,\mathcal{R}_2) - \chi(B, S^2\mathcal{R}_1)$.

I use RR on X and the Leray spectral sequence to calculate $\chi(B,\mathcal{R}_1)$ and $\chi(B,\mathcal{R}_2)$: first for $\mathcal{R}_1 = f_*\mathcal{O}_X(K_X - f^*K_B)$, taking account of Grothendieck duality $R^1f_*K_{X/B} = \mathcal{O}_B$, I get

$$\begin{aligned}\chi(\mathcal{O}_X(K_X - f^*K_B)) &= \chi(\mathcal{O}_X) + \frac{1}{2}(K_X - f^*K_B)(-f^*K_B)\\ &= \chi(\mathcal{O}_X) + 2\chi(\mathcal{O}_B)\\ &= \chi(\mathcal{R}_1) - \chi(\mathcal{O}_B).\end{aligned}$$

Similarly

$$\chi(\mathcal{R}_2) = \chi(\mathcal{O}_X(2K_X - 2K_B)) = \chi(\mathcal{O}_X) + K_X^2 + 12\chi(\mathcal{O}_B).$$

Now from $\chi(\mathcal{R}_1) = \chi(\mathcal{O}_X) + 3\chi(\mathcal{O}_B)$ I can calculate $\chi(S^2\mathcal{R}_1)$ by RR on B:

$$\chi(\mathcal{R}_1) = 2\chi(\mathcal{O}_B) + \deg\mathcal{R}_1 \qquad \text{gives} \qquad \deg\mathcal{R}_1 = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_B);$$

and

$$\chi(S^2\mathcal{R}_1) = 3\chi(\mathcal{O}_B) + 3\deg\mathcal{R}_1 = 3\chi(\mathcal{O}_X) + 6\chi(\mathcal{O}_B).$$

The lemma follows by subtraction. QED

For practical purposes, Horikawa's methods are entirely sufficient for working with genus 2 fibration. However, the structure of the relative canonical algebra can also be determined very easily. This following result is proved in [ML], Theorem II.3.1 (or the reader can find his or her own proof as a reasonable exercise).

Theorem 2.3 *Near a 2-connected fibre,*

$$\mathcal{R}(X/B) = \mathcal{O}_B[x_1, x_2, z]/(z^2 = h_6(x_1, x_2));$$

near a 2-disconnected fibre,

$$\mathcal{R}(X/B) = \mathcal{O}_B[x_1, x_2, y, z]/(q, z^2 = h_6(x_1, x_2, y)),$$

where q is a relation in degree 2, which can be normalised to $q = (ay - x_1x_2)$ or $q = (ay - x_1^2)$ for some $a \in m_P \subset \mathcal{O}_B$.

In these expressions, x_1, x_2 and y generate the invariant subalgebra $\mathcal{R}^+$ of the hyperelliptic involution, and the -1-eigensheaf is given by $\mathcal{R}^- = \mathcal{R}^+ \cdot z$.

This result can be described as follows: the multiplication map

$$\mu\colon S^2\mathcal{R}_1 \to \mathcal{R}_2$$

is generically an isomorphism, but fails to be surjective at the finitely many 2-disconnected fibres; this failure is measured by the Horikawa number $H(P)$, a local invariant of the fibre germ over P. The invariant subalgebra $\mathcal{R}^+$ is determined by $\mathcal{R}_1$, $\mathcal{R}_2$ and μ. The corresponding projective scheme $Q = \operatorname{Proj}_B \mathcal{R}^+ \to B$, the quotient of $X \to B$ by the hyperelliptic involution, is a conic bundle over B, in the sense that every fibre is isomorphic to a plane conic: where μ is an isomorphism it is $\cong \mathbb{P}^1 = \mathbb{P}(\mathcal{R}_1)$, or more precisely the second Veronese embedding of $\mathbb{P}(\mathcal{R}_1)$; where μ is not surjective $\operatorname{Proj}_B \mathcal{R}^+$ is locally the relative conic in $\mathbb{P}(1,1,2)$ given by $q = 0$. The global quadratic equation q can be deduced from $\mu\colon S^2\mathcal{R}_1 \to \mathcal{R}_2$; in local terms $\mathcal{R}_2$ is the $\mathcal{O}_B$-module generated by $u_0 = x_1^2$, $u_1 = x_1x_2$, $u_2 = x_2^2$ and y with the linear relation $q = (ay - x_1x_2)$ or $(ay - x_1^2)$, and the quadratic relation on $\mathcal{R}_2$ is $u_0u_2 = u_1^2$.

The fact that the canonical algebra is a relative complete intersection is very convenient. Thus the set of all fibre germs is just the set of pairs (q, h) of relations over $\mathcal{O}_B$ in degrees 2 and 6 satisfying the open condition that the complete intersection $V(q, h) \subset \mathbb{P}_B(1,1,2)$ has at worst Du Val singularities. Horikawa's analysis of the 1-canonical model can be obtained from this, by eliminating y from the relations defining $\mathcal{R}(X/B)$ using q. Xiao Gang's Morsification conjecture follow immediately, since the deformation theory of complete intersections is unobstructed: just modify q, h so that $q = (b(t)y - x_1x_2)$, where t is a local parameter on B, the function $b(t) \in \mathcal{O}_B$ has simple roots, and the curve $h_6(x_1, x_2, y) = 0$ on the conic bundle $q = 0$ is nonsingular and doesn't pass through $(0,0,1)$ over the zeros of $b(t)$.

Note that coker $S^2\mathcal{R}_1 \to \mathcal{R}_2$ is an $\mathcal{O}_{B,P}$-module, not necessarily killed by m_P. This means that $H(X/B,P)$ is not determined by the fibre F of the relative canonical model over P. For example, all of Horikawa's models I_k, II_k have fibres F of the same nature – namely, two irreducible curves of arithmetic genus 1 meeting transversally at a point Q. In this case coker $S^2\mathcal{R}_1 \to \mathcal{R}_2$ is a module of the form $\mathcal{O}_{B,P}/t^H$, where t is a local parameter and $H = H(X/B,P)$. In other words, $H = H(X/B,P)$ is determined not just by the base locus of $|K_F|$, but by how many infinitesimal steps this base locus sticks out of the fibre over P. In fact in this case $H = H(X/B,P)$ is determined by the number of -2-curves in the fibre of the relative minimal model (Horikawa's definition), or by the type of the singularity $Q \in X$ of the relative canonical model; despite the fact that the scheme-theoretic base locus of $|K_{X/B}|$ is not contained in the fibre in general.

3 Genus 3 pencils

3.1 Invariants of fibrations

Let $f\colon X \to B$ be a genus g fibration. At least while the canonical algebra of the general fibre has a concrete description (e.g., complete intersection), it makes sense to try to define invariants of the fibration in terms of this description and the way it degenerates.

Note that even a smooth fibre may be degenerate in terms of the way the canonical algebra is generated: e.g., a smooth hyperelliptic or trigonal fibre will mess up the generators and relations. **History:** The importance of hyperelliptic fibres for genus 3 pencils became clear in work of K. Konno and T. Ashikaga, e.g. [Ko1], and modifies a conjectural picture due to Xiao Gang and myself. I believe that Horikawa has a substantial body of knowledge on genus 3 pencils (dating from around 1976–80), covering probably most of what I know or guess, and I hope that his results will eventually be published.

Things get very complicated as the relative genus g gets bigger, and it seems a tall order to analyse all the degenerate fibre germs at one go, so Xiao Gang proposes to break the study into two steps:

Step 1 Work first under the assumption that the singular fibre germs are *atomic*, that is, "suffer only one accident" in terms of their singularities and special linear systems. This means the fibre F has either a single node, or is a multiple of a nonsingular fibre, or has some other combination of singularities forced by the monodromy, or has a linear system special-in-the-sense-of-moduli (but not a combination of these accidents). The idea is that atomic fibre germs should be stable in the sense of singularity theory: a small deformation of the germ is of the same nature. See below for examples.

Step 2 Try to understand more general degenerations as a sum of atoms. e.g., in many cases a bad fibre germ $f\colon X \to \text{disc}$ will have a *Morsification*, a small deformation $X_u \to \text{disc}$ so that for $u \neq 0$ every fibre of X_u is atomic (the *Morsification conjecture*). It may make sense to handle a complicated singularity as a sum of atoms even in cases where there may not exist a deformation between them, as in some other phenomenons of singularity theory, so that proving the Morsification conjecture is not necessarily a prerequisite for studying this problem, although it would certainly help a lot.

3.2 Genus 3, nonhyperelliptic

In this case the relative canonical algebra is generically a quartic hypersurface, that it, $\mathcal{R} \cong \mathcal{O}_B[x_1, x_2, x_3]/(h_4)$. The 1-canonical model is a divisor $\overline{X}$ in the scroll $\mathbb{P}_B(\mathcal{R}_1)$, the $\mathbb{P}^2$-bundle corresponding to the rank 3 locally free sheaf $\mathcal{R}_1$, with each fibre a plane curve of degree 4.

Since there are no quadratic relations at the generic fibre, $S^2\mathcal{R}_1 \hookrightarrow \mathcal{R}_2$ is an injection, and generically an isomorphism. I can therefore define the *Horikawa number* $H(X/B, P)$ of a fibre germ to be

$$H(X/B, P) = \operatorname{length}\operatorname{coker}\{S^2\mathcal{R}_1 \hookrightarrow \mathcal{R}_2\}_P.$$

The same calculation as in Lemma 2.2 proves that for $f\colon X \to B$ a nonhyperelliptic genus 3 fibration,

$$K_X^2 = 3\chi(\mathcal{O}_X) - 10\chi(\mathcal{O}_B) + \sum_{P \in B} H(X/B, P). \qquad (*)$$

e.g., if $B = \mathbb{P}^1$ and every fibre is 2-connected and nonhyperelliptic (and $q = 0$) then $K^2 = 3p_g - 7$. The definition is of course rather cheap, since it may not be possible to calculate $H(X/B, P)$ except in very simple cases. However, it fits into a general pattern: important invariants in singularity theory can often be defined as the length of some finite module, e.g., the Milnor number. Note also that since K^2 is a deformation invariant of a surface, $\sum H(X/B, P')$ (summed over a small disc in the base) must be a deformation invariant of a fibre germ, so that $H(X/B, P)$ can be calculated by dynamic methods if a Morsification is available.

3.3 Genus 3 atoms

If my calculations are correct, there are exactly 4 atomic fibre germs in genus 3:

(0) F is an irreducible curve with a single node, and is nonhyperelliptic;

(1) F is a nonsingular hyperelliptic curve;

(2) $F = E + C$ where E and C are irreducible curves of respective genus 1 and 2 meeting transversally in a point Q;

(3) $F = 2C$ where C is a nonsingular genus 2 curve, and $\mathcal{O}_C(C)$ is a nonzero 2-torsion class.

In case (0), $S^2\mathcal{R}_1 \to \mathcal{R}_2$ is surjective so $H(X/B, P) = 0$. Although this case does not contribute to K_X^2 in $(*)$, it contributes -1 to the Euler number of the minimal model, or to $c_2(X)$.

3.4

In case (1), the canonical ring of the fibre $R(F)$ is a double cover of a conic $q(x_1, x_2, x_3) = 0$, that is $R(F) = k[x_1, x_2, x_3, y]/(q, y^2 = h_4(x_i, y))$. In order for the fibre to be atomic, I require that the relation in degree 2 in $\mathcal{R}_1$ is $q + a(t)y$, where $a(t) \in m_P$ has a simple zero at $P \in B$, or in other words, that the 1-canonical image $\overline{X} \subset \mathbb{P}(\mathcal{R}_1)$ has q as an ordinary double curve (not a curve of tacnodes or worse). This correspond to the modular invariant having contact of order 2 with the hyperelliptic divisor $\mathcal{H}_3 \subset \mathcal{M}_3$; the moduli space $\mathcal{M}_3$ should be considered as a $\mathbb{Z}/2$-orbifold around $\mathcal{H}_3$, and the modular invariant of every fibration $X \to B$ with a smooth hyperelliptic curve as fibre has even order of contact with $\mathcal{H}_3$. The atomic assumption is that $ty \in S^2\mathcal{R}_1$, so that $\mathcal{R}(X/B)$ requires only 1 generator y in degree 2, so that this atomic fibre has $H(X/B, P) = 1$.

3.5

In case (2), the canonical ring for the fibre F is covered by the case of "1 elliptic tail" in Mendes Lopes' analysis [ML], Theorem III.4.1. Incidentally, the ring needs 6 generators, so has codimension 4, and has 9 relations yoked by 16 syzygies; it can be written in a semideterminantal "format", part of a pattern with Gorenstein rings in codimension 4 that is familiar from many other examples. In the present case, the ring can be obtained fairly simply by an obvious elimination from the Cartesian product graded algebra $R(E, \mathcal{O}_E(P)) \times R(C, \mathcal{O}_C(K_C + P))$. From Mendes Lopes' results (or by direct calculation) one sees that it needs 2 generators in degree 2, so that $H(X/B, P) \geq 2$. By analogy with Horikawa's family I_k, II_k, I believe that the fibre is atomic if and only if $Q = E \cap C \in X$ is a nonsingular point of the relative canonical model, or if and only if the modular invariant $B \to \mathcal{M}_3$ intersects the boundary component $\Delta_1 = \{E_1 + E_2\}$ transversally, and in these cases $H(X/B, P) = 2$.

Although from the point of view of the modular invariant this fibre appears very naturally, its 1-canonical model $\overline{X} \subset \mathbb{P}(\mathcal{R}_1)$ seems at first sight to be too horribly degenerate to be "atomic". The relative canonical map blows up the base point Q to a -1-curve ℓ, then contracts E to an elliptic Gorenstein singularity of degree 2 (i.e., like $x^2 + y^4 + z^4$), and maps C as a double cover of $\mathbb{P}^1$. Thus the fibre of the 1-canonical image $\overline{X}$ is a double line pair, of which the first line is an ordinary double locus of $\overline{X}$, the second is a locus of ordinary tangency of $\overline{X}$ to the fibre $\mathbb{P}^2$, and $\overline{X}$ has an elliptic Gorenstein singularity of degree 2 at one point of the second line. It's not really the fibre that's pathological in this case, nor its relative canonical algebra, but rather the process of eliminating the generators of $\mathcal{R}(X/B)$ in degree ≥ 2, corresponding to projecting to the 1-canonical model. This explains in part my preference for the abstract projective model given by the relative canonical algebra, rather than more concrete models. Another reason will appear shortly.

3.6

In case (3), the relative canonical algebra is covered by the case of double fibre written down in [ML], Theorem III.5.1. In the present case, her result can be derived very simply as follows. Suppose that $X \to B$ is a genus 3 fibrations having a double fibre $F = 2C$, with C be a nonsingular curve of genus 2. Let $\widetilde{B} \to B$ be the double cover branched at P, corresponding to $\tau = \sqrt{t}$ where t is a local parameter on B, and $\widetilde{X} \to \widetilde{B}$ the normalised pullback. The fibre $\widetilde{C}$ is a hyperelliptic curve of genus 3 having a free involution ι, so it's a double cover of a plane conic, with 8 branch points invariant under an involution. From this it's not hard to write down the canonical ring $R(\widetilde{C}, K_{\widetilde{C}})$ together with the action of ι:

$$R(\widetilde{C}, K_{\widetilde{C}}) = k[\xi_1, \xi_2, \xi_3, \eta]/(\xi_2\xi_3 = \xi_1^2, \eta^2 = h_4),$$

with ξ_1 invariant, and ξ_2, ξ_3, η in the $-$ eigenspace of ι; here $h_4(\xi_1, \xi_2, \xi_3)$ is the invariant polynomial of degree 4 which intersects the conic $\xi_2\xi_3 = \xi_1^2$ in the 8 branch points. Now the relative canonical algebra of $\widetilde{X} \to \widetilde{B}$ is clearly

$$\mathcal{R}(\widetilde{X}/\widetilde{B}) = \mathcal{O}_{\widetilde{B}}[\xi_1, \xi_2, \xi_3, \eta]/(\xi_2\xi_3 = \xi_1^2 + \tau q_2', \eta^2 = h_4 + \tau h_4').$$

where q_2' and h_4' are polynomials of the indicated degrees in the -1 eigenspace of ι. Now $\mathcal{R}(X/B)$ is the ring of invariants of the $\mathbb{Z}/2$-action; it's generated by the invariant coordinate ξ_1 and the quadratic monomials in the $-$ eigencoordinates τ, ξ_2, ξ_3 and η, that is:

$$\begin{aligned}
&\text{in degree 1:} \quad x_1 = \xi_1, x_2 = \tau\xi_2, x_3 = \tau\xi_3;\\
&\text{in degree 2:} \quad y_1 = \tau\eta, y_2 = \xi_2^2, y_3 = \xi_3^2;\\
&\text{in degree 3:} \quad z_1 = \xi_2\eta, z_2 = \xi_3\eta;
\end{aligned}$$

here $\tau^2 = t \in \mathcal{O}_B$, and the two relations

$$\xi_2\xi_3 = \xi_1^2 + \tau q_2' = Q \qquad \text{and} \qquad \eta^2 = h_4(\xi_1, \xi_2, \xi_3) + \tau h_4' = H$$

(where Q and H are expressible as polynomials in $x_1, \dots, y_3$) save me the effort of writing down the two monomials $\xi_2\xi_3$ and η^2 as new generators. By construction of the ring of invariants, the relations in $\mathcal{R}(X/B)$ can be expressed as $\operatorname{rank} M \le 1$, where M is the following 4×4 symmetric matrix:

$$\begin{pmatrix} t & x_2 & x_3 & y_1 \\ x_2 & y_2 & Q & z_1 \\ x_3 & Q & y_3 & z_2 \\ y_1 & z_1 & z_2 & H \end{pmatrix} \qquad \text{with entries of degrees} \qquad \begin{matrix} 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 4 \end{matrix}$$

In particular, 3 generators y_1, y_2, y_3 are required in degree 2, so that $H(X/B, P) \ge 3$. It's fairly clear that this fibre is atomic if and only if the monomial y_1 appears in Q with nonzero coefficient; then by writing out the 20 relations $\operatorname{rank} M \le 1$ explicitly, it's easy to see that y_1, y_2 and y_3 generate $\operatorname{coker} S^2\mathcal{R}_1 \to \mathcal{R}_2$, and hence $H(X/B, P) = 3$.

The above form of the equations means that $\mathcal{R}(X/B)$ is a codimension 2 complete intersection in a generic symmetric determinantal (given by setting Q and H to be function of $x_1, \dots, t$). I believe that the other double fibres of [ML], Theorem III.5.1 are also symmetric determinantals, although there is some unpleasantness if you want to put all the cases in a single parametrised family, as she does. The point is that the quadratic relation, which I assumed to be $\xi_1^2 = \xi_2\xi_3$, can be a line pair or double line. Incidentally, $R(F, K_F)$ is exactly the same ring as $R(C, K_{X|C})$ where X is a Godeaux surface with

Tors $= \mathbb{Z}/2$ and $C \in |K_X + 1|$ its unique paracanonical curve, so that the rather complicated proof of [ML], Theorem III.5.1 kills two birds with one stone.

Excercise 3.1 *(i) Show how to eliminate y_i, z_i to get the equation of the 1-canonical model;*

(ii) make some sarcastic remarks about how horrid it is compared to the nice determinantal format of the entire canonical ring.

3.7

Consider the following two cases:

(4) $F = 2C$ with $C \cong \mathbb{P}^1$, and X has 8 nodes on C;

(5) $F = 2E$ with E and elliptic curve, and X has 4 nodes on E.

Cases (1), (3), (4) and (5) all correspond to the modular invariant of an atomic germ $X \to B$ tending to the hyperelliptic divisor $\mathcal{H}_3$, with various possibilities for the monodromy, differing by a $\mathbb{Z}/2$ twist. For an atomic germ in case (3), $X \to B$ intersects the $\mathcal{H}_3$ transversally. An etale double cover of a nonsingular curve of genus 2 as in case (3) fits in the Galois tower corresponding to $\sqrt{(P_1 + P_2)}$, $\sqrt{(P_3 + \cdots + P_6)}$; the other steps of the tower correspond to cases (1), (4) and (5). In other words, as local analytic fibre germs, the 4 cases (3), (5), and (1), (4) for special values of the moduli can all be obtained from one another by base change of order 2 and taking a quotients by a group of order 2, analogous to the duality of Kodaira elliptic fibres.

3.8

It's very instructive to understand exactly why the two cases (1) and (3) are atomic, whereas (4) and (5) are not:

In case (4), the relative canonical algebra is just a quartic hypersurface: the fibre F is a double conic in $\mathbb{P}^2$. Typically, $\overline{X} \subset \mathbb{P}(\mathcal{A}_1)$ is $q^2 + th_4$ where $q = q(x_1, x_2, x_3)$ is a nonsingular conic, $h = h(x_1, x_2, x_3)$ a quartic, and t is a local parameter along B. A deformation of the germ with parameter u is given by $q^2 + th_4 + uh'_4$, and obviously for fixed value of u this will have only Morse singularities. However, doesn't the modular invariant give rise to a contradiction? Case (4) corresponds to the modular invariant crossing the hyperelliptic divisor $\mathcal{H}_3 \subset \mathcal{M}_3$ transversally. Making a normalised base-change by a branched cover of order 2 of a fibre germ in case (4) gives case (1). Surely, since hyperelliptic curves form a divisor $\mathcal{H}_3 \subset \mathcal{M}_3$, and the germ with $u = 0$ has modular invariant tending to a point of $\mathcal{H}_3$ at $t = 0$, the modular invariant for other values of t should also cross $\mathcal{H}_3$? This is just false: the modular invariant is not a morphism, and from the equation $q^2 + th_4 + uh'_4$ you can see at once that the value of the modular invariant at $(t, u) = (0, 0)$ depends on the way you approach $(0, 0)$: if you approach along the tangent direction $t = \lambda u$ then you get the hyperelliptic curve branched in the 8 points $\lambda h + h' = 0$ of $q = 0$.

The contrast with cases (1) and (3) is very pointed. In case (4), because the algebra is generated in degree 1, the moduli space $\mathcal{M}_3$ of curves of genus 3 has really nothing to do with the situation—the set of germs only depends on the moduli space of plane quartics, and all hyperelliptic curves correspond to one point. On the other hand, in case (1), the modular invariant is a morphism to $\mathcal{M}_3$, taking a well-defined value (special fibre) $\in \mathcal{H}_3$ at $P \in B$; then if the germ is deformed, the modular invariant changes continuously, and must continue to touch $\mathcal{H}_3$ to order 2. In case (3) also, the value of the modular invariant at P is determined purely by the special fibre $F = 2C$, as the double cover of C corresponding to the 2-torsion class $\mathcal{O}_C(C)$, so that the modular invariant is a morphism.

I view this as another argument for working with the relative canonical algebra $\mathcal{R}(X/B)$, rather than a concrete model such as the 1-canonical model; in each case, the thing which is really deformed is the algebra $R(F, K_F)$.

3.9

The above is a local analysis of atomic fibre germs. It's clear by considerations of the modular invariant $B \to \mathcal{M}_3$ and monodromy that the 4 types of atomic fibre germs (0–3) are not small deformations of one another. I believe the Morsification conjecture for genus 3 fibrations; it implies the following (due essentially to Xiao Gang):

Conjecture 3.2 *There is a calculus which associates to each bad fibre germ of a genus* 3 *fibration* $X \to B$ *an invariant* (a_0, a_1, a_2, a_3), *in which* a_i *is the virtual number of atomic fibres in case* (i).

And $X \to B$ *should have the global numerical properties of a genus* 3 *fibration with these atomic fibres as its only singular fibres. More precisely,* *the Chern numbers of $X \to B$ should be given by*

$$\begin{aligned} c_2(X) &= \textit{Euler number of nonsingular minimal model} \\ &= 8(b-1) + a_0 + a_2 + 2a_3; \end{aligned} \tag{1}$$

(here the Euler number of the base is $2-2b$ and that of the general fibre is -4; each fibre in case (0) or (2) decreases the Euler number by 1, and in case (3) by 2); and using $(*)$ together with the values of $H(X/B, P)$ obtained in (3.3–5) gives

$$K_X^2 = 3\chi(\mathcal{O}_X) - 10\chi(\mathcal{O}_B) + a_1 + 2a_2 + 3a_3. \tag{2}$$

Noether's formula and elementary arithmetic gives

$$\chi(\mathcal{O}_X) = 2(b-1) + \frac{1}{9}(a_0 + a_1 + 3a_2 + 5a_3) \tag{3}$$

and

$$K_X^2 = 16(b-1) + \frac{1}{3}(a_0 + 4a_1 + 9a_2 + 14a_4). \tag{4}$$

3.10

The conjectural picture of the geography of genus 3 fibrations which emerges is as follows: fix K^2, $\chi(\mathcal{O}_X)$ and $b = \text{genus}(B)$. Then there are a finite number of solutions of

$$a_1 + 2a_2 + 3a_3 = K_X^2 - 3\chi(\mathcal{O}_X) + 10\chi(\mathcal{O}_B). \tag{5}$$

Fixing (a_1, a_2, a_3), consider genus 3 fibrations $f\colon X \to B$ as in the conclusion of the conjecture. These will usually form one or more irreducible components of the moduli space. By analogy with Horikawa's work, I expect that for fairly small values of K^2, say $K^2 \le 4\chi$, there is normally just one component, and sometimes an extra one corresponding to some extreme possibility for the vector bundle $\mathcal{R}_1 = f_*\omega_{X/B}$; in any case, the number of irreducible components of the moduli space should be approximately the number of solutions of (5). If this is true, there is an area in surface geography in which the number of components of the moduli space grows as 1/6 times the square of $K_X^2 - 3\chi(\mathcal{O}_X)$.

3.11

Some nontrivial results in this area of geography have been obtained by Konno and Ashikaga. For example, [A] proves that there exist surfaces with a genus 3 pencil and assigned values of K^2 and $\chi(\mathcal{O}_X)$ in the range $3\chi - 10 \le K^2 \le 8\chi - 78$; the big values of K^2 come from the base curve B having large genus, so over $\mathbb{P}^1$ he only gets $3\chi - 10 \le K^2 \le 4\chi - 16$. The examples are constructed from divisors in a 3-fold scroll having lots of elliptic Gorenstein singularities of degree 2 (i.e., like $x^2 + y^4 + z^4$), rather similar to Ulf Persson's first explorations. Ashikaga's singular fibres consist of two elliptic curves $E_1 + E_2$ meeting transversally in 2 points; in terms of the invariants a_i of the above conjecture, each of these corresponds to $a_0 = 2$, $a_1 = 1$, that is, it has a Morsification with two irreducible nodal fibres and one nonsingular hyperelliptic fibre.

4 Conclusion, more problems

A nonhyperelliptic curve of genus 4 is canonically an intersection of a quadric and cubic, so that for a genus 4 pencil $X \to B$, the relative 1-canonical image $\overline{X} \subset \mathbb{P}(\mathcal{R}_1)$ lives in a $\mathbb{P}^3$-bundle over B, and is contained in a uniquely defined relative quadric $\overline{X} \subset Q \subset \mathbb{P}(\mathcal{R}_1)$. Because of this relative quadratic equation, $\mathcal{L} = \ker\{S^2\mathcal{R}_1 \to \mathcal{R}_2\}$ is a line bundle. The value of K^2 is of course determined as in §2 and (3.1) by coker $S^2\mathcal{R}_1 \to \mathcal{R}_2$ and $\deg \mathcal{L}$:

$$K^2 = 4\chi(\mathcal{O}_X) - 12\chi(\mathcal{O}_B) - \deg \mathcal{L} + \text{length coker}\,.$$

However, because of the term $\deg \mathcal{L}$, this is not purely an invariant of the fibre germ. Also, if $X \to B$ varies in a family, it seems quite likely that length coker and $\deg \mathcal{L}$ will vary upper-semicontinuously; e.g., $X \to B$ may acquire a nonsingular hyperelliptic fibre, giving coker $S^2\mathcal{R}_1 \to \mathcal{R}_2$, and the degree of the hypersurface Q will drop.

Genus 4 pencils are very interesting for two reasons: Consider surfaces for which φ_K is birational, and just above the continental rift given by the Castelnuovo–Horikawa inequality $K^2 \ge 3p_g - 7$. I conjecture that up to some bound $K^2 \le ap_g - b$ (with some value I can't remember, something like $a = 10/3$, $b = 10$), (?) any such surface X has a genus 3 pencil $X \to B$; since these are canonically plane quartics, the canonical image of X is certainly *not an intersection of cubics*. After that, and up the next rift $K^2 \le ap_g - b$ (ditto, maybe $a = 7/2$), (?) X has a pencil of curves of genus 3 or 4. The reason for believing this is that in the range $K^2 < 4p_g - 12$, canonical surfaces (?) are contained in a 3-fold $\overline{X} \subset W \subset \mathbb{P}^{p_g-1}$, and when $\deg W$ is small, it is (?) ruled by planes or quadric surfaces. Thus the genus 4 pencils give (?) the surfaces of smallest degree for which the canonical image can be birationally cut out by cubics. (See [R] for some of these conjecture.) Surfaces with a genus 5 pencil $X \to \mathbb{P}^1$, and no essential singular fibres have $K^2 = 4p_g - 12$.

Secondly, if X_0 is a Godeaux surface and $|2K_{X_0}|$ has no fixed part then blowing up the codimension 2 base locus gives a genus 4 pencil $X \to \mathbb{P}^1$. In this case $\mathcal{R}_1 = 4\mathcal{O}_{\mathbb{P}^1}(-1)$ (because its degree is known, and $p_g = 0$). The 2-disconnected fibres come from the torsion group of X_0. For the known classes of Godeaux surfaces, when the torsion is big, $\operatorname{coker} S^2\mathcal{R}_1 \to \mathcal{R}_2$ is also big, and $\overline{X} \subset Q \subset \mathbb{P}^1 \times \mathbb{P}^3$, where Q has bidegree $(a, 2)$ with a small, so that X can be constructed by writing down Q, then X as a divisor on Q; in other words, X is more or less like a complete intersection inside $\mathbb{P}^1 \times \mathbb{P}^3$. But when the torsion is small, Q has bidegree $(a, 2)$ with a large, and $X \subset Q$ is in a divisor class $|\mathcal{O}_Q(b, 2)|$ with b negative; in other words, to construct X, I have to make a residual intersection with a cubic containing many fibres of Q. In this case, the question of how many hyperelliptic curves of genus 4 there are among the fibres, and whether this number changes in the moduli space of Godeaux surfaces, seems very interesting.

References

[A] T. Ashikaga, A remark on the geography of surfaces with birational canonical morphism

[H] E. Horikawa, On algebraic surfaces with pencils of curves of genus 2, in Complex analysis and algebraic geometry, volume dedicated to K. Kodaira, Iwanami and C.U.P., 1977, pp. 79–90

[Ko1] K. Konno, Algebraic surfaces of general type with $c_1^2 = 3p_g - 6$, Tōhoku preprint (1989)

[ML] Margarida Mendes Lopes, The relative canonical algebra for genus 3 fibrations, Warwick thesis

[R] M. Reid, Quadrics through a canonical surface, in LNM **1417**

[Xiao] XIAO Gang, Surfaces fibrées en courbes de genre 2, LNM **1137**