or even 7×7 or bigger. For example,

$$\begin{pmatrix} r_1r_2x_0 & r_2y_2 & z_1 & r_0y_0 & r_0r_1x_2 & 0 \\ & r_2x_1 & y_0 & z_2 & r_1y_1 + sx_1 & r_1r_2x_0 \\ & & x_2 & y_1 & z_0 & r_2y_2 \\ & & & x_0 & y_2 & z_1 \\ & & & & r_0x_1 & r_0y_0 + sx_0 \\ & & & & & r_0r_1x_2 + sy_2 \end{pmatrix} \tag{11}$$

and cancel r_0, r_2 from the Pfaffians as necessary. And so on, . . . It is not clear that any of this is useful.

3.2 Matrix of first syzygies

I order the relations L_i and choose their signs as in (8). The matrix M_1 of first syzygies in the approved $(A\,B)$ form of [R2], 2.1 is the transpose of

$$\begin{array}{ccccccccc} . & x_1 & y_0 & z_2 & r_1x_0 & . & . & . & . \\ -x_1 & . & x_2 & y_1 & y_2 & . & . & . & . \\ -y_0 & -x_2 & . & r_2x_0 & z_1 & . & . & . & . \\ -z_2 & -y_1 & -r_2x_0 & . & sx_0 + r_0y_0 & . & . & . & . \\ -r_1x_0 & -y_2 & -z_1 & -sx_0 - r_0y_0 & . & . & . & . & . \\[1ex] . & . & r_2x_1 & . & -sx_1 - r_1y_1 & . & y_0 & -z_2 & . \\ . & . & x_2 & . & y_2 & -y_0 & . & x_0 & . \\ . & . & -y_1 & . & -r_0x_1 & z_2 & -x_0 & . & . \\[1ex] z_0 & . & . & . & . & -sx_2 - r_2y_2 & . & -r_0x_2 & y_1 \\ . & z_0 & r_2y_2 & . & -r_0r_1x_2 - sy_2 & r_1r_2x_0 + sy_0 & . & r_0y_0 & -z_2 \\ . & . & z_0 & . & . & -sx_1 - r_1y_1 & y_2 & -r_0x_1 & . \\ . & . & . & z_0 & . & r_1x_2 & . & -y_2 & x_1 \\ . & . & . & . & z_0 & -r_2x_1 & -x_2 & y_1 & . \\[1ex] y_2 & . & . & -r_0x_2 & . & -z_1 & . & . & x_0 \\ r_1y_1 & . & -r_2y_2 & r_0r_2x_1 + sy_1 & r_0r_1x_2 + sy_2 & . & -z_1 & . & z_2 \\ r_1x_2 & . & . & sx_2 + r_2y_2 & . & . & . & -z_1 & y_0 \end{array} \tag{12}$$

The spinor sets made up by $I = $ (4 out of the first 5 rows, with i omitted) and the complementary $J = I^c$ have spinors of the form $z_1 \operatorname{Pf}_i$.

see website + Fun

Apr 2011

The equations become

$$
\begin{aligned}
L_1 :\ & x_1y_2 = d(x_2^2+\mu^2 a)+\lambda z_2,\\
L_2 :\ & x_2y_1 = c(x_1^2+\lambda^2 b)+\mu z_1,\\
L_3 :\ & y_1y_2 = cdx_1x_2+\lambda cz_1+\mu dz_2-\lambda\mu e\\
& \qquad\equiv c(dx_1x_2+\lambda z_1)+\mu(dz_2-\lambda e)\\
& \qquad\equiv d(cx_1x_2+\mu z_2)+\lambda(cz_1-\mu e),\\
L_4 :\ & x_1z_2 = x_2z_1-\lambda by_2+\mu ay_1,\\
L_5 :\ & y_1z_2 = (cz_1-\mu e)x_1-bd(\lambda cx_2+\mu y_2),\\
L_6 :\ & y_2z_1 = (dz_2-\lambda e)x_2-ac(\lambda y_1+\mu dx_1),\\
L_7 :\ & z_1^2+ay_1^2+ex_1^2+bdx_1y_2-\lambda b(dz_2-\lambda e)=0,\\
L_8 :\ & z_1z_2+acx_1y_1+ex_1x_2+bdx_2y_2-\lambda\mu abcd=0,\\
L_9 :\ & z_2^2+acx_2y_1+ex_2^2+by_2^2-\mu a(cz_1-\mu e)=0.
\end{aligned}
\tag{66}
$$

This set of equations comes neatly from $I_0=(L_1,L_2,L_4,L_8)$ (unchanged from (64) except for the unsurprising term ex_1x_2 in L_8) by coloning out $x_1x_2y_1y_2$; its syzygy matrix M is

$$
\begin{array}{ccccccccc}
y_1 & dx_2 & -x_1 & -\mu d & \lambda & . & . & . & . \\
cx_1 & y_2 & -x_2 & \lambda c & . & \mu & . & . & . \\
. & -z_1 & \lambda b & -y_1 & x_1 & . & \mu & . & . \\
\lambda bc & -z_2 & . & -cx_1 & x_2 & . & . & \mu & . \\
-z_1 & \mu ad & . & dx_2 & . & x_1 & . & \lambda & . \\
-z_2 & . & \mu a & y_2 & . & x_2 & . & . & \lambda \\
. & -ay_1 & . & z_1 & . & -\lambda b & -x_2 & x_1 & . \\
by_2 & . & . & z_2 & \mu a & . & . & -x_2 & x_1 \\
 & & & & & & & & \\
. & \mu ae & . & acy_1+ex_2 & . & -by_2 & -\mu ac & z_2 & -z_1 \\
\lambda be & . & . & -bdy_2-ex_1 & -ay_1 & . & -z_2 & z_1 & -\lambda bd \\
acy_1+ex_2 & . & . & -\mu acd & . & z_2 & . & y_2 & -dx_2 \\
-ex_1 & . & -ay_1 & dz_2-\lambda e & . & -z_1 & -y_2 & . & dx_1 \\
. & -ex_2 & -by_2 & -cz_1+\mu e & -z_2 & . & cx_2 & . & -y_1 \\
. & bdy_2+ex_1 & . & \lambda bcd & z_1 & . & -cx_1 & y_1 & . \\
-cz_1+\mu e & . & z_2 & cdx_2 & -y_2 & . & . & . & -\mu d \\
. & dz_2-\lambda e & -z_1 & cdx_1 & . & y_1 & \lambda c & . & .
\end{array}
\tag{67}
$$

(Or ad lib., swap Rows i and $i+8$, or apply orthogonal row operation, adding $x\times$ Row i to Row j and $-x\times$ Row $j+8$ to Row $i+8$, with $i+8$ and $j+8$ taken mod 16.) One checks that it satisfies ${}^tMJM=0$ where $J=\left(\begin{smallmatrix}0&I\\I&0\end{smallmatrix}\right)$

From Fun2, Mar-Apr 2013

If we think of the 9 relations modulo λ, μ in rolling factors format

$$\bigwedge^2 \begin{pmatrix} y_1 & x_1 & dx_2 & z_1 \\ cx_1 & x_2 & y_2 & z_2 \end{pmatrix} \quad \text{and} \quad \begin{aligned} z_1^2 &= \cdots \\ z_1z_2 &= \cdots \\ z_2^2 &= \cdots, \end{aligned} \tag{78}$$

it is natural to arrange them as

$$\begin{array}{ll} L_{23}: & -x_1y_2 + dx_2^2 + \lambda z_2 - \mu^2 ad \\ L_{13}: & -y_1y_2 + cdx_1x_2 + \lambda cz_1 + \mu dz_2 + \lambda\mu e \\ L_{12}: & -x_2y_1 + cx_1^2 + \mu z_1 - \lambda^2 bc \\ L_{14}: & -y_1z_2 + cx_1z_1 + \lambda bcdx_2 + \mu bdy_2 + \mu ex_1 \\ L_{24}: & -x_1z_2 + x_2z_1 + \lambda by_2 - \mu ay_1 \\ L_{34}: & -y_2z_1 + dx_2z_2 + \lambda acy_1 + \mu acdx_1 + \lambda ex_2 \\ F_{11}: & -z_1^2 + ay_1^2 + bd^2x_2^2 + ex_1^2 - \mu^2abd^2 - \lambda^2be \\ F_{12}: & -z_1z_2 + acx_1y_1 + bdx_2y_2 + ex_1x_2 + \lambda\mu abcd \\ F_{22}: & -z_2^2 + ac^2x_1^2 + by_2^2 + ex_2^2 - \lambda^2abc^2 - \mu^2ae \end{array} \tag{79}$$

The syzygies are

$$\begin{array}{ccccccccc} y_1 & -x_1 & dx_2 & \lambda & -\mu d & 0 & 0 & 0 & 0 \\ cx_1 & -x_2 & y_2 & 0 & \lambda c & \mu & 0 & 0 & 0 \\ -\mu bd & \lambda b & z_1 & -x_1 & y_1 & 0 & \mu & 0 & 0 \\ \lambda bc & 0 & z_2 & -x_2 & cx_1 & 0 & 0 & \mu & 0 \\ 0 & z_1 & -\lambda e & -dx_2 & 0 & -y_1 & \lambda c & \mu d & 0 \\ -\mu e & z_2 & 0 & -y_2 & 0 & -cx_1 & 0 & \lambda c & \mu d \\ z_1 & 0 & \mu ad & 0 & -dx_2 & -x_1 & 0 & \lambda & 0 \\ z_2 & \mu a & -\lambda ac & 0 & -y_2 & -x_2 & 0 & 0 & \lambda \\ \\ 0 & \lambda abc & \mu ae & -acx_1 & -ex_2 & by_2 & 0 & -z_2 & z_1 \\ \lambda be & \mu abd & 0 & ay_1 & ex_1 & -bdx_2 & z_2 & -z_1 & 0 \\ ex_2 & acx_1 & 0 & -\lambda ac & 0 & -z_2 & 0 & y_2 & -dx_2 \\ -ex_1 & -ay_1 & 0 & -\mu ad & -\lambda e & z_1 & -y_2 & dx_2 & 0 \\ by_2 & 0 & -acx_1 & \mu a & -z_2 & 0 & 0 & -x_2 & x_1 \\ -bdx_2 & 0 & ay_1 & 0 & z_1 & \lambda b & x_2 & -x_1 & 0 \\ 0 & -by_2 & -ex_2 & z_2 & \mu e & -\lambda bc & 0 & cx_1 & -y_1 \\ 0 & bdx_2 & ex_1 & -z_1 & 0 & -\mu bd & -cx_1 & y_1 & 0 \end{array} \tag{80}$$

Modulo λ and μ, the first 8 rows are standard determinantal syzygies, and the last 8 correspond to the rolling factors (e.g., Row 9 certifies $z_1F_{22} - z_2F_{12}$ as an element of the determinantal ideal generated by the first 6 relations). The matrix M satisfies ${}^tMJM = 0$.

Same, massaged for z1

4.5 Approaching final form

Tidy up giving precedence to z_1 and z_2. Modifying the relations as

$$\begin{array}{rl}
-L_3: & y_1y_2 - cdx_1x_2 - \lambda cz_1 - \mu dz_2 + \lambda\mu e, \\
-L_2: & x_2y_1 - c(x_1^2 + \lambda^2 b) - \mu z_1, \\
L_1: & -x_1y_2 + d(x_2^2 + \mu^2 a) + \lambda z_2, \\
L_4: & -x_1z_2 + x_2z_1 + \mu ay_1 - \lambda by_2, \\
L_6: & -y_2z_1 + (dz_2 - \lambda e)x_2 - ac(\mu dx_1 + \lambda y_1), \\
L_5: & -y_1z_2 + (cz_1 - \mu e)x_1 - bd(\lambda cx_2 + \mu y_2), \\
L_9 + acL_2: & z_2^2 + ac^2x_1^2 + by_2^2 + ex_2^2 + \lambda^2 abc^2 + \mu^2 ae, \\
-L_8: & -z_1z_2 - acx_1y_1 - bdx_2y_2 - ex_1x_2 + \lambda\mu abcd, \\
L_7 + bdL_1: & z_1^2 + ay_1^2 + bd^2x_2^2 + ex_1^2 + \mu^2 abd^2 + \lambda^2 be
\end{array} \tag{81}$$

allows me to massage the matrix of syzygies into the form

$$\begin{array}{ccccccccc}
\cdot & acx_1 & by_2 & z_2 & \cdot & \mu a & x_1 & x_2 & \cdot \\
-acx_1 & \cdot & ex_2 & \cdot & z_2 & -\lambda ac & -dx_2 & -y_2 & \cdot \\
-by_2 & -ex_2 & \cdot & -\mu e & \lambda bc & z_2 & y_1 & cx_1 & \cdot \\
-z_2 & \cdot & \mu e & \cdot & -cx_1 & -y_2 & -\mu d & \lambda c & \cdot \\
\cdot & -z_2 & -\lambda bc & cx_1 & \cdot & -x_2 & \cdot & \mu & \cdot \\
-\mu a & \lambda ac & -z_2 & y_2 & x_2 & \cdot & \lambda & \cdot & \cdot \\
-x_1 & dx_2 & -y_1 & \mu d & \cdot & -\lambda & \cdot & \cdot & \cdot \\
-x_2 & y_2 & -cx_1 & -\lambda c & -\mu & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & \\
z_1 & \lambda e & \cdot & \cdot & y_1 & dx_2 & \cdot & -\mu d & \lambda c \\
-\lambda b & z_1 & -\mu bd & -y_1 & \cdot & x_1 & \cdot & \cdot & \mu \\
\cdot & \mu ad & z_1 & -dx_2 & -x_1 & \cdot & \cdot & \lambda & \cdot \\
\cdot & ay_1 & bdx_2 & z_1 & -\lambda b & \cdot & \cdot & -x_1 & -x_2 \\
-ay_1 & \cdot & ex_1 & \lambda e & z_1 & \mu ad & \cdot & dx_2 & y_2 \\
-bdx_2 & -ex_1 & \cdot & \cdot & -\mu bd & z_1 & \cdot & -y_1 & -cx_1 \\
\lambda abc & \mu ae & \cdot & -ex_2 & by_2 & -acx_1 & z_1 & z_2 & \cdot \\
\mu abd & \cdot & -\lambda be & ex_1 & -bdx_2 & ay_1 & \cdot & z_1 & z_2
\end{array} \tag{82}$$

This has lots of symmetry, in effect, many different skew symmetries:

(1) The bottom left 8×8 block has diagonal entries z_1.

Same, massaged for y1

The syzygies become

$$
\begin{array}{ccccccccc}
. & acx_1 & ex_2 & by_2 & \mu ae & z_2 & z_1 & \lambda abc & . \\
-cx_1 & . & \lambda c & -\mu & y_2 & . & . & -x_2 & . \\
-ex_2 & -\lambda ac & . & -z_2 & . & y_2 & dx_2 & acx_1 & . \\
-by_2 & \mu a & z_2 & . & -acx_1 & -x_2 & -x_1 & . & . \\
-\mu e & -y_2 & . & cx_1 & . & -\lambda c & \mu d & z_2 & . \\
-z_2 & . & -y_2 & x_2 & \lambda ac & . & \lambda & -\mu a & . \\
-z_1 & . & -dx_2 & x_1 & -\mu ad & -\lambda & . & . & . \\
-\lambda bc & x_2 & -cx_1 & . & -z_2 & \mu & . & . & . \\
\\
-y_1 & \lambda & -\mu d & . & dx_2 & . & . & -x_1 & . \\
-\lambda be & -ay_1 & -ex_1 & -bdx_2 & . & z_1 & . & \mu abd & z_2 \\
\mu bd & x_1 & -y_1 & . & -z_1 & . & . & \lambda b & -\mu \\
. & dx_2 & . & -y_1 & -\lambda e & \mu d & . & -z_1 & -\lambda c \\
-bdx_2 & . & z_1 & \lambda b & -ay_1 & x_1 & . & . & x_2 \\
. & -z_1 & . & -\mu bd & -ex_1 & -y_1 & . & -bdx_2 & -cx_1 \\
. & z_2 & -\mu e & -\lambda bc & ex_2 & -cx_1 & -y_1 & by_2 & . \\
ex_1 & -\mu ad & -\lambda e & z_1 & . & dx_2 & . & -ay_1 & y_2
\end{array}
\tag{84}
$$

Here the bottom block has y_1 down the diagonal, and the top block is "skew with multipliers": it would be strictly skew if we multiplied rows 2, 5 and 8 by a (and divide rows 10, 13 and 16 by a to preserve isotropy).

All 7 of the 6×6 Pfaffians of the submatrix $[1, 2, 3, 4, 5, 6, 8]$ are divisible by the 7th relation for z_2^2; all 7 of the 6×6 Pfaffians of the submatrix $[2, 3, 4, 5, 6, 7, 8]$ are divisible by L_1.

4.7 Alternative giving precedence to λ or μ

One can attempt a similar thing for λ or μ. The matrix can be massaged by column operations and admissible row operations into the form

$$\begin{array}{ccccccccc}
0 & -y_1 & -z_1 & cx_1 & -bdx_2 & -\mu bd & 0 & bcdx_1 & ex_1 \\
y_1 & 0 & -z_2 & cx_2 & -by_2 & 0 & -cz_1+\mu e & 0 & ex_2 \\
z_1 & z_2 & 0 & -\mu ac & 0 & by_2 & acy_1+ex_2 & 0 & -\mu ae \\
-x_1 & -x_2 & \mu a & 0 & 0 & 0 & z_2 & by_2 & 0 \\
dx_2 & y_2 & 0 & 0 & 0 & -z_2 & -\mu acd & acy_1+ex_2 & 0 \\
\mu d & 0 & -y_2 & 0 & z_2 & 0 & cdx_2 & -cz_1+\mu e & 0 \\
0 & z_1 & -ay_1 & -z_2 & \mu abd & -bdx_2 & -ex_1 & -bdz_2+\lambda be & 0 \\
-dx_1 & 0 & 0 & -y_2 & -ay_1 & z_1 & dz_2-\lambda e & -ex_1 & 0 \\
\lambda & 0 & 0 & 0 & -\mu a & x_2 & -y_2 & z_2 & 0 \\
0 & \lambda & 0 & 0 & 0 & -x_1 & dx_2 & -z_1 & -\mu ad \\
0 & 0 & \lambda & 0 & -x_1 & 0 & -\mu d & y_1 & -dx_2 \\
0 & 0 & 0 & \lambda c & -z_1 & -y_1 & cdx_1 & 0 & -dz_2+\lambda e \\
0 & 0 & x_1 & \mu & \lambda b & 0 & -y_1 & 0 & z_1 \\
0 & x_1 & 0 & -x_2 & 0 & \lambda b & z_1 & 0 & ay_1 \\
0 & 0 & 0 & 0 & -x_2 & -\mu & \lambda c & cx_1 & -y_2 \\
0 & \mu & x_2 & 0 & 0 & 0 & -cx_1 & \lambda bc & z_2
\end{array} \tag{85}$$

Here the top 6×6 block is skew with multipliers, and the top 8×8 block would be after the O(16) row operations $7 \mapsto 7 - \frac{e}{c} \times 16$ and $8 \mapsto 8 + \frac{e}{c} \times 15$. It corresponds to ordering the relations

$$\begin{array}{l}
z_2^2 + y_2^2 b + x_2 y_1 ac - \mu z_1 ac + x_2^2 e + \mu^2 ae \\
-z_1 z_2 - x_1 y_1 ac - x_2 y_2 bd + \lambda\mu abcd - x_1 x_2 e \\
y_1 z_2 - x_1 z_1 c + \mu y_2 bd + \lambda x_2 bcd + \mu x_1 e \\
-z_1^2 - y_1^2 a - x_1 y_2 bd + \lambda z_2 bd - x_1^2 e - \lambda^2 be \\
y_1 y_2 - \lambda z_1 c - \mu z_2 d - x_1 x_2 cd + \lambda\mu e \\
-y_2 z_1 - \lambda y_1 ac + x_2 z_2 d - \mu x_1 acd - \lambda x_2 e \\
-x_2 z_1 + x_1 z_2 - \mu y_1 a + \lambda y_2 b \\
x_1 y_2 - \lambda z_2 - x_2^2 d - \mu^2 ad \\
-x_2 y_1 + \mu z_1 + x_1^2 c + \lambda^2 bc
\end{array} \tag{86}$$

4.8 Alternative giving precedence to x_1, cx_1

$$
\begin{array}{ccccccccc}
. & z_1 & y_1 & dx_2 & \mu d & \lambda c & \lambda bcd & . & \lambda e \\
-z_1 & . & -by_2 & . & z_2 & -\mu ac & . & acy_1 + x_2e & -\mu ae \\
-y_1 & by_2 & . & -z_2 & . & cx_2 & . & -cz_1 + \mu e & x_2e \\
-dx_2 & . & z_2 & . & y_2 & . & acy_1 + x_2e & -\mu acd & . \\
-\mu d & -z_2 & . & -y_2 & . & . & -cz_1 + \mu e & cdx_2 & . \\
-\lambda & \mu a & -x_2 & . & . & . & z_2 & -y_2 & . \\
-\lambda bd & . & . & -ay_1 & z_1 & -z_2 & \lambda be & -bdy_2 - x_1e & . \\
. & -ay_1 & z_1 & \mu ad & -dx_2 & y_2 & bdy_2 + x_1e & \lambda e & . \\
x_1 & . & . & \mu a & -x_2 & . & by_2 & z_2 & . \\
. & x_1 & . & \lambda & . & . & y_1 & -\mu d & -dx_2 \\
. & . & x_1 & . & \lambda & . & -z_1 & dx_2 & -\mu ad \\
. & -\lambda b & . & x_1 & . & \mu & . & -y_1 & z_1 \\
. & . & -\lambda b & . & x_1 & -x_2 & . & z_1 & ay_1 \\
. & . & . & -z_1 & -y_1 & cx_1 & . & -\lambda bcd & bdy_2 + x_1e \\
. & x_2 & \mu & . & . & . & cx_1 & \lambda c & -y_2 \\
. & . & . & -x_2 & -\mu & . & -\lambda bc & cx_1 & -z_2]);
\end{array}
\tag{87}
$$

with cokernel the vector of relations, ordered and signed as

$$
\begin{array}{l}
acx_2y_1 + by_2^2 - \mu acz_1 + z_2^2 + \mu^2ae + x_2^2e \\
-cdx_1x_2 + y_1y_2 - \lambda cz_1 - \mu dz_2 + \lambda\mu e \\
-\mu acdx_1 - \lambda acy_1 - y_2z_1 + dx_2z_2 - \lambda x_2e \\
-\lambda bcdx_2 - \mu bdy_2 + cx_1z_1 - y_1z_2 - \mu x_1e \\
-\lambda\mu abcd + acx_1y_1 + bdx_2y_2 + z_1z_2 + x_1x_2e \\
ay_1^2 + bdx_1y_2 + z_1^2 - \lambda bdz_2 + \lambda^2be + x_1^2e \\
\mu^2ad + dx_2^2 - x_1y_2 + \lambda z_2 \\
\mu ay_1 - \lambda by_2 + x_2z_1 - x_1z_2 \\
-\lambda^2bc - cx_1^2 + x_2y_1 - \mu z_1
\end{array}
\tag{88}
$$

The final spinor here is the determinant of the bottom left 8×8 block, with diagonal entries x_1 or cx_1, and is equal to $x_1^2 \times L_9^3$.

The top 6×6 block is skew with multiplier c, and the top 8×8 block would be after the O(16) row operations $7 \mapsto 7 - \frac{e}{c} \times 16$ and $8 \mapsto 8 + \frac{e}{c} \times 15$. To spell this out, `AddRow(AddRow(MultiplyRow(MultiplyRow`