Parallel unprojection equations for $\mathbb{Z}/3$ Godeaux surfaces

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Abstract

I construct a 9-dimensional affine "key variety" $V \subset \mathbb{A}^{13}$ by triple parallel unprojection from a hypersurface. With a basic choice of $\mathbb{G}_m$ action (that is, grading), regular sections of V give rise to a number of varieties, including the universal cover of general $\mathbb{Z}/3$ Godeaux surfaces, together with a small menagerie of related curves, surfaces, 3-folds and 4-folds. The construction includes cases of $\mathbb{Z}/3$ Godeaux surfaces having an involution. As a by-product, the equations and syzygies of V lead to useful exercises illustrating general Gorenstein codimension 4 phenomena.

1 The key variety and the main result

Consider a hypersurface $F = 0$ with F in the intersection of the three codim 2 ideals

$$(x_0, y_0) \cap (x_1, y_1) \cap (x_2, y_2) \tag{1}$$

where $x_0, y_0,\ x_1, y_1,\ x_2, y_2$ are six independent variables (viewed as three pairs), and such that the coefficient of $y_0y_1y_2$ in F equals 1. The general case is

$$y_0y_1y_2 = sx_0x_1x_2 + r_0x_1x_2y_0 + r_1x_0x_2y_1 + r_2x_0x_1y_2. \tag{2}$$

Indeed, any terms with y_1y_2 can be tidied away by doing

$$y_0 \mapsto y_0 + \text{multiples of } x_i. \tag{3}$$

For the moment, consider the coefficients s, r_0, r_1, r_2 also as independent indeterminates. Following Papadakis, treat the subvarieties $(x_i = y_i = 0)$

as unprojection divisors, and introduce the corresponding unprojection variables z_i, that is,

$$\begin{aligned} z_0 &= (y_1y_2 - r_0x_1x_2)/x_0 \\ &= (sx_1x_2 + r_1x_2y_1 + r_2x_1y_2)/y_0 \\ z_1 &= (y_0y_2 - r_1x_0x_2)/x_1 \\ &= (sx_0x_2 + r_0x_2y_0 + r_2x_0y_2)/y_1 \\ z_2 &= (y_0y_1 - r_2x_0x_1)/x_2 \\ &= (sx_0x_1 + r_0x_1y_0 + r_1x_0y_1)/y_2 \end{aligned}$$

The z_i are subject to the linear unprojection equations deduced in the obvious way from these expressions; also, adding two of the z_i gives rise to a 5×5 Pfaffian format, which provides the bilinear relations for z_iz_j: for example

$$\begin{pmatrix} x_1 & y_0 & z_2 & r_1x_0 \\ & x_2 & y_1 & y_2 \\ & & r_2x_0 & z_1 \\ & & & sx_0 + r_0y_0 \end{pmatrix} \tag{4}$$

hence

$$z_1z_2 = sx_0y_0 + r_0y_0^2 + r_1r_2x_0^2, \tag{5}$$

and similarly

$$\begin{aligned} z_0z_2 &= sx_1y_1 + r_1y_1^2 + r_0r_2x_1^2, \\ z_0z_1 &= sx_2y_2 + r_2y_2^2 + r_0r_1x_2^2. \end{aligned}$$

Theorem 1.1 *These* 9 *equations define a codimension* 4 *affine Gorenstein* 9*-fold* $V \subset \mathbb{A}^{13}_{\langle x_i,y_i,z_i,r_i,s\rangle}$. *Its singular locus is* $\mathbb{A}^4_{\langle r_0,r_1,r_2,s\rangle}$ *union the three planes* $\mathbb{A}^2_{\langle r_i,x_i\rangle}$ *for* $i = 0,1,2$. *It has a diagonal action of the torus* $\mathbb{G}_m^6$ *and* S_3 *symmetry permuting the indices. Grading by*

$$\operatorname{wt} x_i = 1, \quad \operatorname{wt} y_i = \operatorname{wt} r_i = 2, \quad \operatorname{wt} z_i = \operatorname{wt} s = 3$$

gives V *canonical weight* -12.

With this grading, regular sections of V *provide the graded rings over the following varieties (among other possibilities):*

(A) *Set r_i equal to general combinations of x_i, y_i of weight 2, and s equal to a general combinations of x_i, y_i of weight 3; also, set $x_0 + x_1 + x_2 = 0$ and $z_0 + z_1 + z_2 = 0$. Then* Proj *of this ring is a canonical surface $Y \subset \mathbb{P}^6(1,1,2,2,2,3,3)_{\langle x_1,x_2,y_0,y_1,y_2,z_1,z_2\rangle}$ with $p_g = 2$, $K^2 = 3$. It is nonsingular in general.*

Moreover, taking r_i and s symmetric under permuting the indices gives Y a fixed point free action of $\mathbb{Z}/3$, hence a quotient $\mathbb{Z}/3$ Godeaux surface $X = Y/(\mathbb{Z}/3)$ as in [R1]; or an action of S_3, giving X with an involution (see Section 2 for a specific case).

(B) *Omitting the sections $\sum x_i = 0$ and $\sum z_i = 0$ in (1) gives a quasi-smooth Fano 4-fold $F \subset \mathbb{P}^6(1,1,1,2,2,2,3,3,3)_{\langle x_0,x_1,x_2,y_0,y_1,y_2,z_0,z_1,z_2\rangle}$ with $K_F = \mathcal{O}_F(-3)$ and $3 \times \frac{1}{3}(1,1,2,2)$ orbifold points.*

(C) *Omitting the section $x_0+x_1+x_2 = 0$ in (1) gives a nonsingular Calabi–Yau 3-fold containing Y as a hyperplane section.*

(D) *Omitting the section $z_0 + z_1 + z_2 = 0$ in (1) gives a quasismooth Fano 3-fold $W \subset \mathbb{P}^6(1,1,2,2,2,3,3,3)_{\langle x_1,x_2,y_0,y_1,y_2,z_0,z_1,z_2\rangle}$ of index 2 with $-K_W = 2A$, $A^3 = 1$ having $3 \times \frac{1}{3}(1,2,2)$ orbifold points [GRDB], No 40198.*

The 4-fold in (2) and the 3-folds in (3) and (4) can be given fixed point free $\mathbb{Z}/3$ actions, or full S_3 symmetry, while maintaining the stated nonsingularity properties.

Remark 1.1 Specialising r_0, r_1, r_2 to 0 and s to 1 gives

$$\bigwedge^2 \begin{pmatrix} x_0 & y_2 & z_1 \\ z_1 & x_1 & y_0 \\ y_1 & z_0 & x_2 \end{pmatrix} = 0.$$

Thus V is a flat deformation of the cone over Segre($\mathbb{P}^2 \times \mathbb{P}^2$).

The symmetric group S_3 acts on V by permuting the indices $0, 1, 2$. My paper [R1] used the eigenbasis coming from the cyclotomic change of bases to $x_0 + \varepsilon x_1 + \varepsilon^2 x$ with $\varepsilon \in \boldsymbol{\mu}_3$ and similarly for the y_i and z_i.

The general algebraic properties of the key variety V come directly by unprojection from the hypersurface (2). The hypersurface has the obvious $\mathbb{G}_m^6$ action, which is preserved by the unprojection. The singular locus of V

is discussed in the next section, along with the nonsingularity of its sections (A–D). As a preparation, note that the equations include 4 unprojection equations for z_0:

$$x_0 z_0 = \cdots, \quad y_0 z_0 = \cdots, \quad z_1 z_0 = \cdots, \quad z_2 z_0 = \cdots, \tag{6}$$

so that V is nonsingular where $z_0 \neq 0$, and similarly for z_1, z_2. They also include

$$y_0 z_0 = \cdots, \quad y_0 y_2 = \cdots, \quad y_0 y_1 = \cdots, \quad y_0^2 r_0 = \cdots, \tag{7}$$

so that V is also nonsingular where $y_0 \neq 0$, and similarly for y_1, y_2. Thus the singular locus of V is contained in $y_i = z_i = 0$. One sees that these define a reducible subvariety of V with many components, all of dimension ≤ 4. Thus V is at least normal.

2 Nonsingularity

I prove all the nonsingularity results in Theorem 1.1 by brute force computer algebra. I only describe the calculations for (A), since the others are practically identical, and the Magma files doing all of them are online at [], (currently DropBox, `NonSing_Calc_for_God3.txt`), and run in short order on the Magma online calculator `http://magma.maths.usyd.edu.au/calc`.

Claim 2.1 *The S_3 symmetric surface Y that is the univeral cover of the $\mathbb{Z}/3$ Godeaux is nonsingular, and the $\mathbb{Z}/3$ action on it is free.*

Start from the graded polynomial ring $R = k[x_1, x_2, y_0, y_1, y_2, z_1, z_2]$, and define x_0, z_0 by

$$x_0 = -x_1 - x_2 \quad \text{and} \quad z_0 = -z_1 - z_2.$$

I define the sections by

$$\begin{aligned} r_0 &= y_0 + x_0^2 + 7x_1x_2, \\ r_1 &= y_1 + x_1^2 + 7x_0x_2, \qquad \text{and} \quad s = x_0^3 + x_1^3 + x_2^3. \\ r_2 &= y_2 + x_2^2 + 7x_0x_1 \end{aligned}$$

The nine equations of $Y \subset \mathbb{P}^6(1,1,2,2,2,3,3)_{\langle x_1,x_2,y_0,y_1,y_2,z_1,z_2\rangle}$ are then

$$\begin{array}{rr} sx_0x_2 + r_0x_2y_0 + r_2x_0y_2 - y_1z_1, & \\ -r_1r_2x_0^2 - sx_0y_0 - r_0y_0^2 + z_1z_2, & r_0x_1x_2 - y_1y_2 + x_0z_0, \\ sx_0x_1 + r_0x_1y_0 + r_1x_0y_1 - y_2z_2, & -r_0r_2x_1^2 - sx_1y_1 - r_1y_1^2 + z_0z_2, \\ -r_1x_0x_2 + y_0y_2 - x_1z_1, & -sx_1x_2 - r_1x_2y_1 - r_2x_1y_2 + y_0z_0, \\ r_2x_0x_1 - y_0y_1 + x_2z_2, & -r_0r_1x_2^2 - sx_2y_2 - r_2y_2^2 + z_0z_1. \end{array} \tag{8}$$

Write $L = [L_1, \dots, L_9]$ for these equations.

Brute force computer algebra frees us from heavy lifting, so simply define the 9×7 Jacobian matrix $\frac{\partial L_i}{\partial\{x_i,y_i,z_i\}}$ and its set of 4×4 minors J (with $\#J = \binom{9}{4} \times \binom{7}{4} = 4410$). Then Magma takes

1.3 seconds to verify that $z_0^4 \in \langle J \rangle$, the ideal generated by J,

so that the singular locus of Y is contained in $z_0 = 0$, hence also in $z_0 = z_1 = z_2 = 0$. Similarly, it takes

0.8 seconds to verify that $y_0^5 \in \langle J \cup \{z_0, z_1, z_2\}\rangle$, and
0.8 seconds to verify that $x_0^{13} \in \langle J \cup \{z_0, z_1, z_2\} \cup \{y_0, y_1, y_2\}\rangle$.

This proves that Y is nonsingular.

To prove that Y is disjoint from the fixed point locus of $\mathbb{Z}/3$ on V, it is enough to check that, in the same coordinates, L together with the equations $x_0^3 = x_1^3 = x_2^3$, $y_0^3 = y_1^3 = y_2^3$, $z_0 = z_1 = z_2$ defines the empty set in $\operatorname{Proj} R$. Obviously $z_0 + z_1 + z_2 = 0$ and $z_0 = z_1 = z_2$ implies that all the $z_i = 0$. In fact, Magma says at once that the ideal generated by

$$L \cup \{x_1^3 - x_0^3, x_2^3 - x_1^3, y_1^3 - y_0^3, y_2^3 - y_1^3, z_1 - z_0, z_2 - z_1\}$$

defines the empty set of $\operatorname{Proj} R$.

3 Applications to codimension 4 Gorenstein

I have so far applied the variety $V \subset \mathbb{A}^{13}$ to construct various varieties. In the rest of this note, I use it to illustrate the general structure theory of Gorenstein codimension 4 ideals, supporting [R2].

3.1 The 9 equations of V as extended Pfaffians

Adjoining z_0 to the 4×4 Pfaffians of (4) is a Tom_3 unprojection; recall that this means that the 6 entries m_{ij} of the matrix with $i, j \neq 3$ are in unprojection ideal (x_0, y_0, z_1, z_2) (a codimension 4 complete intersection), so that its Pfaffians are also in (x_0, y_0, z_1, z_2). Tom unprojections are usually related to $\mathbb{P}^2 \times \mathbb{P}^2$ (see [TJ], Section 9 for more details), and one can try to accommodate the unprojection equations as the 4×4 Pfaffians of a 6×6 skew matrix with extra symmetry. Since this case is Tom_3, if we put z_0 as the entry m_{36} then in Pfaffians it does not multiply any of the 4 entries in its own Row-and-Column 3, but it does multiply the other 6 entries in the unprojection ideal (x_0, y_0, z_1, z_2).

This gives

$$\begin{pmatrix} r_1x_0 & y_2 & z_1 & r_0y_0 + sx_0 & r_0r_1x_2 + sy_2 \\ & x_1 & y_0 & z_2 & r_1y_1 + sx_1 \\ & & x_2 & y_1 & z_0 \\ & & & r_2x_0 & r_2y_2 \\ & & & & r_0r_2x_1 \end{pmatrix}, \tag{9}$$

which contains all the equations except that $x_0z_0 - y_1y_2 + r_0x_1x_2$ only appears after cancelling r_1 or r_2. This is a common phenomenon. The general philosophical point is that the unprojection structure is basic, whereas the matrix format is secondary – the equation $z_0x_0 = \cdots$ is one of the unprojection equations, but it is not completely captured by the matrix.

In this case, the factor r_2 in the bottom 456 triangle floats over to the top 123 triangle to give

$$\begin{pmatrix} r_1r_2x_0 & r_2y_2 & z_1 & r_0y_0 + sx_0 & r_0r_1x_2 + sy_2 \\ & r_2x_1 & y_0 & z_2 & r_1y_1 + sx_1 \\ & & x_2 & y_1 & z_0 \\ & & & x_0 & y_2 \\ & & & & r_0x_1 \end{pmatrix}. \tag{10}$$

The r_2 should not really be included in the matrix, but should be thought of as a crazy-Pfaffian multiplier coming between 123 and 456.

The equations admit other partial expressions as extended Pfaffians, 6×6

or even 7×7 or bigger. For example,

$$\begin{pmatrix} r_1r_2x_0 & r_2y_2 & z_1 & r_0y_0 & r_0r_1x_2 & 0 \\ & r_2x_1 & y_0 & z_2 & r_1y_1 + sx_1 & r_1r_2x_0 \\ & & x_2 & y_1 & z_0 & r_2y_2 \\ & & & x_0 & y_2 & z_1 \\ & & & & r_0x_1 & r_0y_0 + sx_0 \\ & & & & & r_0r_1x_2 + sy_2 \end{pmatrix} \tag{11}$$

and cancel r_0, r_2 from the Pfaffians as necessary. And so on,... It is not clear that any of this is useful.

3.2 Matrix of first syzygies

I order the relations L_i and choose their signs as in (8). The matrix M_1 of first syzygies in the approved $(A\,B)$ form of [R2], 2.1 is the transpose of

$$\begin{array}{ccccccccc}
\cdot & x_1 & y_0 & z_2 & r_1x_0 & \cdot & \cdot & \cdot & \cdot \\
-x_1 & \cdot & x_2 & y_1 & y_2 & \cdot & \cdot & \cdot & \cdot \\
-y_0 & -x_2 & \cdot & r_2x_0 & z_1 & \cdot & \cdot & \cdot & \cdot \\
-z_2 & -y_1 & -r_2x_0 & \cdot & sx_0 + r_0y_0 & \cdot & \cdot & \cdot & \cdot \\
-r_1x_0 & -y_2 & -z_1 & -sx_0 - r_0y_0 & \cdot & \cdot & \cdot & \cdot & \cdot \\[6pt]
\cdot & \cdot & r_2x_1 & \cdot & -sx_1 - r_1y_1 & \cdot & y_0 & -z_2 & \cdot \\
\cdot & \cdot & x_2 & \cdot & y_2 & -y_0 & \cdot & x_0 & \cdot \\
\cdot & \cdot & -y_1 & \cdot & -r_0x_1 & z_2 & -x_0 & \cdot & \cdot \\[6pt]
z_0 & \cdot & \cdot & \cdot & \cdot & -sx_2 - r_2y_2 & \cdot & -r_0x_2 & y_1 \\
\cdot & z_0 & r_2y_2 & \cdot & -r_0r_1x_2 - sy_2 & r_1r_2x_0 + sy_0 & \cdot & r_0y_0 & -z_2 \\
\cdot & \cdot & z_0 & \cdot & \cdot & -sx_1 - r_1y_1 & y_2 & -r_0x_1 & \cdot \\
\cdot & \cdot & \cdot & z_0 & \cdot & r_1x_2 & \cdot & -y_2 & x_1 \\
\cdot & \cdot & \cdot & \cdot & z_0 & -r_2x_1 & -x_2 & y_1 & \cdot \\[6pt]
y_2 & \cdot & \cdot & -r_0x_2 & \cdot & -z_1 & \cdot & \cdot & x_0 \\
r_1y_1 & \cdot & -r_2y_2 & r_0r_2x_1 + sy_1 & r_0r_1x_2 + sy_2 & \cdot & -z_1 & \cdot & z_2 \\
r_1x_2 & \cdot & \cdot & sx_2 + r_2y_2 & \cdot & \cdot & \cdot & -z_1 & y_0
\end{array} \tag{12}$$

The spinor sets made up by $I = (4$ out of the first 5 rows, with i omitted) and the complementary $J = I^c$ have spinors of the form $z_1\,\mathrm{Pf}_i$.

```
// Magma: Matrix of first syzygies
RR<r0,r1,r2,s,x0,x1,x2, y0,y1,y2, z0,z1,z2>
 := PolynomialRing(Rationals(),[2,2,2,3,1,1,1,2,2,2,3,3,3]);
L := [
s*x0*x2 + r0*x2*y0 + r2*x0*y2 - y1*z1,
-r1*r2*x0^2 - s*x0*y0 - r0*y0^2 + z1*z2,
s*x0*x1 + r0*x1*y0 + r1*x0*y1 - y2*z2,
-r1*x0*x2 + y0*y2 - x1*z1,
r2*x0*x1 - y0*y1 + x2*z2,
r0*x1*x2 - y1*y2 + x0*z0,
-r0*r2*x1^2 - s*x1*y1 - r1*y1^2 + z0*z2,
-s*x1*x2 - r1*x2*y1 - r2*x1*y2 + y0*z0,
-r0*r1*x2^2 - s*x2*y2 - r2*y2^2 + z0*z1
];
Mat := Matrix(9,[0, x1, y0, z2, r1*x0, 0, 0, 0, 0,
-x1, 0, x2, y1, y2, 0, 0, 0, 0,
-y0, -x2, 0, r2*x0, z1, 0, 0, 0, 0,
-z2, -y1, -r2*x0, 0, s*x0+r0*y0, 0, 0, 0, 0,
-r1*x0, -y2, -z1, -s*x0-r0*y0, 0, 0, 0, 0, 0,
0, 0, r2*x1, 0, -s*x1-r1*y1, 0, y0, -z2, 0,
0, 0, x2, 0, y2, -y0, 0, x0, 0,
0, 0, -y1, 0, -r0*x1, z2, -x0, 0, 0,
z0, 0, 0, 0, 0, -s*x2-r2*y2, 0, -r0*x2, y1,
0, z0, r2*y2, 0, -r0*r1*x2-s*y2, r1*r2*x0+s*y0, 0, r0*y0, -z2,
0, 0, z0, 0, 0, -s*x1-r1*y1, y2, -r0*x1, 0,
0, 0, 0, z0, 0, r1*x2, 0, -y2, x1,
0, 0, 0, 0, z0, -r2*x1, -x2, y1, 0,
y2, 0, 0, -r0*x2, 0, -z1, 0, 0, x0,
r1*y1, 0, -r2*y2, r0*r2*x1+s*y1, r0*r1*x2+s*y2, 0, -z1, 0, z2,
r1*x2, 0, 0, s*x2+r2*y2, 0, 0, 0, -z1, y0]);

Matrix(9,L)*Transpose(Mat); // check Mat is made of syzygies
printf("---------\n");

J0 := ZeroMatrix(RR,16,16);
for i in [1..8] do J0[i,i+8] := 1; end for;
J := J0 + Transpose(J0);
```

```
Transpose(Mat)*J*Mat; // check M satisfies ^tM*J*M=0
printf("---------\n");
L;
printf("---------\n");
Mat;

for i in [1..5] do
  I := Remove([1..5],i); J := [j+8 : j in [1..8] | j notin I];
  SquareRoot((-1)^i*Determinant(Submatrix(Mat,I cat J,[1..8]))
    div L[9]);
end for;
```

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