Parallel unprojection equations
for $\mathbb{Z}/3$ Godeaux surfaces

Miles Reid

Abstract
I construct a 9-dimensional affine “key variety” $V \subset \mathbb{A}^1^3$ by triple parallel unprojection from a hypersurface. With a basic choice of $\mathbb{G}_m$ action (that is, grading), regular sections of $V$ give rise to a number of varieties, including the universal cover of general $\mathbb{Z}/3$ Godeaux surfaces, together with a small menagerie of related curves, surfaces, 3-folds and 4-folds. The construction includes cases of $\mathbb{Z}/3$ Godeaux surfaces having an involution. As a by-product, the equations and syzygies of $V$ lead to useful exercises illustrating general Gorenstein codimension 4 phenomena.

1 The key variety and the main result

Consider a hypersurface $F = 0$ with $F$ in the intersection of the three codim 2 ideals
\[(x_0, y_0) \cap (x_1, y_1) \cap (x_2, y_2)\] (1)
where $x_0, y_0, x_1, y_1, x_2, y_2$ are six independent variables (viewed as three pairs), and such that the coefficient of $y_0y_1y_2$ in $F$ equals 1. The general case is
\[y_0y_1y_2 = sx_0x_1x_2 + r_0x_1x_2y_0 + r_1x_0x_2y_1 + r_2x_0x_1y_2.\] (2)
Indeed, any terms with $y_1y_2$ can be tidied away by doing
\[y_0 \mapsto y_0 + \text{multiples of } x_i.\] (3)
For the moment, consider the coefficients $s, r_0, r_1, r_2$ also as independent indeterminates. Following Papadakis, treat the subvarieties $(x_i = y_i = 0)$
as unprojection divisors, and introduce the corresponding unprojection variables $z_i$, that is,

\[
\begin{align*}
    z_0 &= (y_1 y_2 - r_0 x_1 x_2)/x_0 \\
    &= (s x_1 x_2 + r_1 x_2 y_1 + r_2 x_1 y_2)/y_0 \\
    z_1 &= (y_0 y_2 - r_1 x_0 x_2)/x_1 \\
    &= (s x_0 x_2 + r_0 x_2 y_0 + r_2 x_0 y_2)/y_1 \\
    z_2 &= (y_0 y_1 - r_2 x_0 x_1)/x_2 \\
    &= (s x_0 x_1 + r_0 x_1 y_0 + r_1 x_0 y_1)/y_2
\end{align*}
\]

The $z_i$ are subject to the linear unprojection equations deduced in the obvious way from these expressions; also, adding two of the $z_i$ gives rise to a $5 \times 5$ Pfaffian format, which provides the bilinear relations for $z_i z_j$: for example

\[
\begin{pmatrix}
    x_1 & y_0 & z_2 & r_1 x_0 \\
    x_2 & y_1 & y_2 & r_2 x_0 \\
    r_2 x_0 & z_1 & \end{pmatrix}
\]

hence

\[
z_1 z_2 = s x_0 y_0 + r_0 y_0^2 + r_1 r_2 x_0^2.
\]

and similarly

\[
\begin{align*}
    z_0 z_2 &= s x_1 y_1 + r_1 y_1^2 + r_0 r_2 x_1^2, \\
    z_0 z_1 &= s x_2 y_2 + r_2 y_2^2 + r_0 r_1 x_2^2
\end{align*}
\]

**Theorem 1.1** These 9 equations define a codimension 4 affine Gorenstein 9-fold $V \subset \mathbb{A}^1_{13}^{(x_i, y_i, z_i, r_i, s)}$. Its singular locus is $\mathbb{A}^4_{(r_0, r_1, r_2, s)}$ union the three planes $\mathbb{A}^2_{(r_i, x_i)}$ for $i = 0, 1, 2$. It has a diagonal action of the torus $\mathbb{G}_m^6$ and $S_3$ symmetry permuting the indices. Grading by

\[
\begin{align*}
    \text{wt } x_i &= 1, & \text{wt } y_i &= \text{wt } r_i = 2, & \text{wt } z_i &= \text{wt } s = 3
\end{align*}
\]

gives $V$ canonical weight $-12$.

With this grading, regular sections of $V$ provide the graded rings over the following varieties (among other possibilities):
(A) Set \( r_i \) equal to general combinations of \( x_i, y_i \) of weight 2, and \( s \) equal to a general combinations of \( x_i, y_i \) of weight 3; also, set \( x_0 + x_1 + x_2 = 0 \) and \( z_0 + z_1 + z_2 = 0 \). Then \( \text{Proj} \) of this ring is a canonical surface \( Y \subset \mathbb{P}^6(1,1,2,2,2,3,3)_{(x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2)} \) with \( p_g = 2, K^2 = 3 \). It is nonsingular in general.

Moreover, taking \( r_i \) and \( s \) symmetric under permuting the indices gives \( Y \) a fixed point free action of \( \mathbb{Z}/3 \), hence a quotient \( \mathbb{Z}/3 \) Godeaux surface \( X = Y/(\mathbb{Z}/3) \) as in [R1]; or an action of \( S_3 \), giving \( X \) with an involution (see Section 2 for a specific case).

(B) Omitting the sections \( \sum x_i = 0 \) and \( \sum z_i = 0 \) in (1) gives a quasismooth Fano 4-fold \( F \subset \mathbb{P}^6(1,1,1,2,2,2,3,3)_{(x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2)} \) with \( K_F = \mathcal{O}_F(-3) \) and \( 3 \times \frac{1}{3}(1,1,2,2) \) orbifold points.

(C) Omitting the section \( x_0 + x_1 + x_2 = 0 \) in (1) gives a nonsingular Calabi–Yau 3-fold containing \( Y \) as a hyperplane section.

(D) Omitting the section \( z_0 + z_1 + z_2 = 0 \) in (1) gives a quasismooth Fano 3-fold \( W \subset \mathbb{P}^6(1,1,2,2,2,3,3)_{(x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2)} \) of index 2 with \( -K_W = 2A, A^3 = 1 \) having \( 3 \times \frac{1}{3}(1,1,2,2) \) orbifold points [GRDB], No 40198.

The 4-fold in (2) and the 3-folds in (3) and (4) can be given fixed point free \( \mathbb{Z}/3 \) actions, or full \( S_3 \) symmetry, while maintaining the stated nonsingularity properties.

Remark 1.1 Specialising \( r_0, r_1, r_2 \) to 0 and \( s \) to 1 gives

\[
\bigwedge^2 \begin{pmatrix} x_0 & y_2 & z_1 \\ z_1 & x_1 & y_0 \\ y_1 & z_0 & x_2 \end{pmatrix} = 0.
\]

Thus \( V \) is a flat deformation of the cone over \( \text{Segre}(\mathbb{P}^2 \times \mathbb{P}^2) \).

The symmetric group \( S_3 \) acts on \( V \) by permuting the indices 0, 1, 2. My paper [R1] used the eigenbasis coming from the cyclotomic change of bases to \( x_0 + \varepsilon x_1 + \varepsilon^2 x \) with \( \varepsilon \in \mu_3 \) and similarly for the \( y_i \) and \( z_i \).

The general algebraic properties of the key variety \( V \) come directly by unprojection from the hypersurface (2). The hypersurface has the obvious \( \mathbb{G}_m^6 \) action, which is preserved by the unprojection. The singular locus of \( V \)
is discussed in the next section, along with the nonsingularity of its sections (A–D). As a preparation, note that the equations include 4 unprojection equations for $z_0$:

$$x_0z_0 = \cdots, \quad y_0z_0 = \cdots, \quad z_1z_0 = \cdots, \quad z_2z_0 = \cdots,$$

(6)

so that $V$ is nonsingular where $z_0 \neq 0$, and similarly for $z_1, z_2$. They also include

$$y_0z_0 = \cdots, \quad y_0y_2 = \cdots, \quad y_0y_1 = \cdots, \quad y_0^2r_0 = \cdots,$$

(7)

so that $V$ is also nonsingular where $y_0 \neq 0$, and similarly for $y_1, y_2$. Thus the singular locus of $V$ is contained in $y_i = z_i = 0$. One sees that these define a reducible subvariety of $V$ with many components, all of dimension $\leq 4$. Thus $V$ is at least normal.

2 Nonsingularity

I prove all the nonsingularity results in Theorem 1.1 by brute force computer algebra. I only describe the calculations for (A), since the others are practically identical, and the Magma files doing all of them are online at [], (currently Dropbox, NonSing_Calc_for_Godid.txt), and run in short order on the Magma online calculator http://magma.maths.usyd.edu.au/calc.

Claim 2.1 The $S_3$ symmetric surface $Y$ that is the univeral cover of the $\mathbb{Z}/3$ Godeaux is nonsingular, and the $\mathbb{Z}/3$ action on it is free.

Start from the graded polynomial ring $R = k[x_1, x_2, y_0, y_1, y_2, z_1, z_2]$, and define $x_0, z_0$ by

$$x_0 = -x_1 - x_2 \quad \text{and} \quad z_0 = -z_1 - z_2.$$

I define the sections by

$$r_0 = y_0 + x_0^2 + 7x_1x_2, \quad r_1 = y_1 + x_1^2 + 7x_0x_2, \quad \text{and} \quad s = x_0^3 + x_1^3 + x_2^3.$$
The nine equations of $Y \subset \mathbb{P}^6(1,1,2,2,2,3,3)_{(x_1,x_2,y_0,y_1,y_2,z_1,z_2)}$ are then

\[
\begin{align*}
sx_0x_2 + r_0x_2y_0 + r_2x_0y_2 - y_1z_1, \\
-r_1r_2x_0^2 - sx_0y_0 - r_0y_0^2 + z_1z_2, \\
sx_0x_1 + r_0x_1y_0 + r_1x_0y_1 - y_2z_2, \\
-r_1x_0x_2 + y_0y_2 - x_1z_1, \\
r_2x_0x_1 - y_0y_1 + x_2z_2,
\end{align*}
\]

\[
\begin{align*}
r_0x_1x_2 - y_1y_2 + x_0z_0, \\
r_0r_2x_1^2 - sx_1y_1 - r_1y_1^2 + z_0z_2, \\
-ssx_1x_2 - r_1x_2y_1 - r_2x_1y_2 + y_0z_0, \\
-r_0r_1x_2^2 - sx_2y_2 - r_2y_2^2 + z_0z_1.
\end{align*}
\] (8)

Write $L = [L_1, \ldots, L_9]$ for these equations.

Brute force computer algebra frees us from heavy lifting, so simply define the $9 \times 7$ Jacobian matrix $\frac{\partial L_0}{\partial (x_i,y_i,z_i)}$ and its set of $4 \times 4$ minors $J$ (with $\#J = \binom{9}{4} \times \binom{4}{1} = 4410$). Then Magma takes

1.3 seconds to verify that $z_0^4 \in \langle J \rangle$, the ideal generated by $J$, so that the singular locus of $Y$ is contained in $z_0 = 0$, hence also in $z_0 = z_1 = z_2 = 0$. Similarly, it takes

0.8 seconds to verify that $y_0^5 \in \langle J \cup \{z_0, z_1, z_2\} \rangle$, and

0.8 seconds to verify that $x_0^{13} \in \langle J \cup \{z_0, z_1, z_2\} \cup \{y_0, y_1, y_2\} \rangle$.

This proves that $Y$ is nonsingular.

To prove that $Y$ is disjoint from the fixed point locus of $\mathbb{Z}/3$ on $V$, it is enough to check that, in the same coordinates, $L$ together with the equations $x_0^3 = x_1^3 = x_2^3, y_0^3 = y_1^3 = y_2^3, z_0 = z_1 = z_2$ defines the empty set in $\text{Proj} \, R$. Obviously $z_0 + z_1 + z_2 = 0$ and $z_0 = z_1 = z_2$ implies that all the $z_i = 0$. In fact, Magma says at once that the ideal generated by

\[ L \cup \{x_0^3 - x_1^3, x_2^3 - x_1^3, y_0^3 - y_1^3, y_2^3 - y_1^3, z_1 - z_0, z_2 - z_1 \} \]

defines the empty set in $\text{Proj} \, R$.

## 3 Applications to codimension 4 Gorenstein

I have so far applied the variety $V \subset \mathbb{A}^{13}$ to construct various varieties. In the rest of this note, I use it to illustrate the general structure theory of Gorenstein codimension 4 ideals, supporting [R2].
3.1 The 9 equations of $V$ as extended Pfaffians

Adjoining $z_0$ to the $4 \times 4$ Pfaffians of (4) is a Tom$_3$ unprojection; recall that this means that the 6 entries $m_{ij}$ of the matrix with $i, j \neq 3$ are in unprojection ideal $(x_0, y_0, z_1, z_2)$ (a codimension 4 complete intersection), so that its Pfaffians are also in $(x_0, y_0, z_1, z_2)$. Tom unprojections are usually related to $\mathbb{P}^2 \times \mathbb{P}^2$ (see [TJ], Section 9 for more details), and one can try to accommodate the unprojection equations as the $4 \times 4$ Pfaffians of a $6 \times 6$ skew matrix with extra symmetry. Since this case is Tom$_3$, if we put $z_0$ as the entry $m_{36}$ then in Pfaffians it does not multiply any of the 4 entries in its own Row-and-Column 3, but it does multiply the other 6 entries in the unprojection ideal $(x_0, y_0, z_1, z_2)$.

This gives

$$
\begin{pmatrix}
  r_1x_0 & y_2 & z_1 & r_0y_0 + sx_0 & r_0r_1x_2 + sy_2 \\
  x_1 & y_0 & z_2 & r_1y_1 + sx_1 \\
  x_2 & y_1 & r_2x_0 & z_0 \\
  r_2x_0 & r_2y_2 & r_1y_1 + sx_1 & r_0r_2x_1
\end{pmatrix},
$$

which contains all the equations except that $x_0z_0 - y_1y_2 + r_0x_1x_2$ only appears after cancelling $r_1$ or $r_2$. This is a common phenomenon. The general philosophical point is that the unprojection structure is basic, whereas the matrix format is secondary – the equation $z_0x_0 = \cdots$ is one of the unprojection equations, but it is not completely captured by the matrix.

In this case, the factor $r_2$ in the bottom 456 triangle floats over to the top 123 triangle to give

$$
\begin{pmatrix}
  r_1r_2x_0 & r_2y_2 & z_1 & r_0y_0 + sx_0 & r_0r_1x_2 + sy_2 \\
  r_2x_0 & y_0 & z_2 & r_1y_1 + sx_1 \\
  x_2 & y_1 & z_0 & r_2y_2 \\
  x_0 & y_2 & x_1 & r_0x_1
\end{pmatrix}.
$$

The $r_2$ should not really be included in the matrix, but should be thought of as a crazy-Pfaffian multiplier coming between 123 and 456.

The equations admit other partial expressions as extended Pfaffians, $6 \times 6$
or even $7 \times 7$ or bigger. For example,

\[
\begin{pmatrix}
  r_1 r_2 x_0 & r_2 y_2 & z_1 & r_0 y_0 & r_0 r_1 x_2 & 0 \\
  r_2 x_1 & y_0 & z_2 & r_1 y_1 + sx_1 & r_1 r_2 x_0 \\
  x_2 & y_1 & z_0 & r_2 y_2 \\
  x_0 & y_2 & z_1 & r_0 y_0 + sx_0 \\
  r_0 r_2 x_2 + sy_2
\end{pmatrix}
\]  

(11)

and cancel $r_0$, $r_2$ from the Pfaffians as necessary. And so on, . . . It is not clear that any of this is useful.

### 3.2 Matrix of first syzygies

I order the relations $L_i$ and choose their signs as in (8). The matrix $M_1$ of first syzygies in the approved $(A B)$ form of [R2], 2.1 is the transpose of

\[
\begin{pmatrix}
  . & x_1 & y_0 & z_2 & r_1 x_0 & . & . & . & . \\
  -x_1 & . & x_2 & y_1 & y_2 & . & . & . & . \\
  -y_0 & -x_2 & . & r_2 x_0 & z_1 & . & . & . & . \\
  -z_2 & -y_1 & -r_2 x_0 & . & sx_0 + r_0 y_0 & . & . & . & . \\
  -r_1 x_0 & -y_2 & -z_1 & -sx_0 - r_0 y_0 & . & . & . & . & . \\
  . & . & r_2 x_1 & . & -sx_1 - r_1 y_1 & . & y_0 & -z_2 & . \\
  . & . & x_2 & . & y_2 & . & -y_0 & . & x_0 & . \\
  . & . & -y_1 & . & -r_0 x_1 & . & z_2 & . & -x_0 & . \\
  . & . & . & . & . & . & . & . & . & . \\
  . & . & z_0 & . & -sx_2 - r_2 y_2 & . & -r_0 x_2 & y_1 & . \\
  . & z_0 & r_2 y_2 & . & -r_0 r_1 x_2 - sy_2 & r_1 r_2 x_0 + sy_0 & . & r_0 y_0 & -z_2 & . \\
  . & . & z_0 & . & . & -sx_1 - r_1 y_1 & y_2 & -r_0 x_1 & . \\
  . & . & . & z_0 & . & r_1 x_2 & . & -y_2 & x_1 & . \\
  . & . & . & . & z_0 & -r_2 x_1 & -x_2 & y_1 & . \\
  y_2 & . & . & -r_0 x_2 & . & . & . & . & x_0 \\
  r_1 y_1 & . & -r_2 y_2 & r_0 r_2 x_1 + sy_1 & r_0 r_1 x_2 + sy_2 & . & -z_1 & . & z_2 & . \\
  r_1 x_2 & . & sx_2 + r_2 y_2 & . & . & . & . & -z_1 & y_0 & .
\end{pmatrix}
\]

(12)

The spinor sets made up by $I = (4$ out of the first 5 rows, with $i$ omitted) and the complementary $J = I^c$ have spinors of the form $z_1 P f_i$.  

7
// Magma: Matrix of first syzygies
RR<r0, r1, r2, s, x0, x1, x2, y0, y1, y2, z0, z1, z2>
   := PolynomialRing(Rationals(), [2, 2, 3, 1, 1, 1, 2, 2, 2, 3, 3, 3]);
L := [
s*x0*x2 + r0*x2*y0 + r2*x0*y2 - y1*z1,
   -r1*r2*x0^2 - s*x0*y0 - r0*y0^2 + z1*z2,
   s*x0*x1 + r0*x1*y0 + r1*x0*y1 - y2*z2,
   -r1*x0*x2 + y0*y2 - x1*z1,
   r2*x0*x1 - y0*y1 + x2*z2,
   r0*x1*x2 - y1*y2 + x0*z0,
   -r0*r2*x1^2 - s*x1*y1 - r1*y1^2 + z0*z2,
   -s*x1*x2 - r1*x2*y1 - r2*x1*y2 + y0*z0,
   -r0*r1*x2^2 - s*x2*y2 - r2*y2^2 + z0*z1
];
Mat := Matrix(9, [0, x1, y0, z2, r1*x0, 0, 0, 0, 0,
   -x1, 0, x2, y1, y2, 0, 0, 0, 0,
   -y0, -x2, 0, r2*x0, z1, 0, 0, 0, 0,
   -z2, -y1, -r2*x0, 0, s*x0+r0*y0, 0, 0, 0, 0,
   -r1*x0, -y2, -z1, -s*x0-r0*y0, 0, 0, 0, 0, 0,
   0, 0, r2*x1, 0, -s*x1-r1*y1, 0, y0, -z2, 0,
   0, 0, x2, 0, y2, -y0, 0, x0, 0,
   0, 0, -y1, 0, -r0*x1, z2, -x0, 0, 0,
   z0, 0, 0, 0, 0, -s*x2-r2*y2, 0, -r0*x2, y1,
   0, z0, r2*y2, 0, -r0*r1*x2-s*y2, r1*r2*x0+s*y0, 0, r0*y0, -z2,
   0, 0, z0, 0, 0, -s*x1-r1*y1, y2, -r0*x1, 0,
   0, 0, z0, 0, r1*x2, 0, -y2, x1,
   0, 0, 0, 0, z0, -r2*x1, -x2, y1, 0,
   y2, 0, 0, -r0*x2, 0, -z1, 0, 0, x0,
   r1*y1, 0, -r2*y2, r0*r2*x1+s*y1, r0*r1*x2+s*y2, 0, -z1, 0, z2,
   r1*x2, 0, 0, s*x2+r2*y2, 0, 0, 0, -z1, y0]);

Matrix(9, L)*Transpose(Mat); // check Mat is made of syzygies
printf("--------\n");

J0 := ZeroMatrix(RR, 16, 16);
for i in [1..8] do J0[i, i+8] := 1; end for;
J := J0 + Transpose(J0);
Transpose(Mat)*J*Mat; // check M satisfies ^tM*J*M=0
printf("--------\n");
L;
printf("--------\n");
Mat;

for i in [1..5] do
  I := Remove([1..5],i); J := [j+8 : j in [1..8] | j notin I];
  SquareRoot((-1)^i*Determinant(Submatrix(Mat,I cat J,[1..8])))
    div L[9]);
end for;

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Miles Reid,
Math Inst., Univ. of Warwick,
Coventry CV4 7AL, England
e-mail: miles@maths.warwick.ac.uk
web: www.maths.warwick.ac.uk/~miles