MA4L7 Algebraic curves

Example sheet 4, Deadline Tue 26th Feb

1. Function theory on a hyperelliptic curve Assume that $\frac{1}{2} \in k$, and let C be a hyperelliptic curve of genus $g \geq 2$. It comes with a divisor $|D|$ that gives a g^1_2 and a double cover $\varphi_D \colon C \to \mathbb{P}^1$. Write $f_1, f_2 \in \mathcal{L}(C, D)$ for a basis, where $x = f_1/f_2$ is a parameter on $\mathbb{P}^1$.

The field extension $k(\mathbb{P}^1) \subset k(C)$ is a quadratic extension defined by $z^2 = F_{2g+2}(x)$, and has a hyperelliptic involution that does $i \colon z \mapsto -z$.

The monomials $S^n(f_1, f_2) = \{f_1^n, f_1^{n-1}f_2, \dots, f_2^n\}$ are linearly independent in $\mathcal{L}(nD)$ for each n, because x is transcendental over k. Calculate the dimension of $\mathcal{L}(nD)$ for $n = 1, \dots, g$. [Hint: Start by using the above to show that $(g-1)D$ must be irregular, and deduce that $K_C \sim (g-1)D$. On the other hand, gD must be regular.]

Next, use RR to show that $\mathcal{L}((g+1)D)$ is strictly bigger than $S^{g+1}(f_1, f_2)$. We can choose the complementary basis element g so that $z = g/f_2^{g+1}$ is anti-invariant under the hyperelliptic involution, giving the new generator with $z^2 = F_{2g+2}(x)$.

Show the monomials $S^n(f_1, f_2)$ and $S^{n-g-1}(f_1, f_2) \cdot g$ form a basis of $\mathcal{L}(nD)$ for every n.

2. Curves of genus $g = 4$ Let C be a curve of genus 4, assumed to be nonhyperelliptic. Write $\varphi_K \colon C \hookrightarrow \mathbb{P}^3$ for its canonical embedding and identify C with its image $C \subset \mathbb{P}^3$.

By construction of the canonical embedding, the hyperplanes of $\mathbb{P}^3$ cut out $|K|$ on C. In the same way, quadric surfaces in $\mathbb{P}^3$ cut out divisors of $|2K|$. Calculate the dimension of the space of quadrics in $\mathbb{P}^3$ and $l(2K) = \dim \mathcal{L}(C, 2K)$, and conclude that C is contained in a unique quadric hypersurface $Q \subset \mathbb{P}^3$.

As an irreducible quadric, Q necessarily has rank 3 or 4. If Q has rank 4 (so is $x_1x_2 = x_3x_4$ in appropriate coordinates), prove that C has two different linear systems g^1_3, D_1 and D_2, with $K_C = D_1 + D_2$. Prove that $C \subset Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree $(3, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and so $C \subset Q$ is cut out by a cubic hypersurface, $C = Q \cap F_3$.

If D_1 is a g^1_3 on C, use RR to deduce that $D_2 = K - D_1$ is also a g^1_3. Therefore $K = D_1 + D_2$ is the sum of two linear systems g^1_3. We distinguish two cases: $D_1 \not\sim D_2$, or $D_1 \sim D_2$. Show that the first case corresponds to the canonical image C contained in a quadric of rank 4.

In the second case, write $K = 2D$ with $D = D_1 = D_2$. Write t_1, t_2

for homogeneous coordinates on the target $\mathbb{P}^1$ of $\varphi_D\colon C \to \mathbb{P}^1$. Show that $\mathcal{L}(C,K)$ is based by $x_1, x_2, x_3 = t_1^2, t_1t_2, t_2^2$ and a new variable y. In $\mathcal{L}(2K)$ there is a quadratic relation between the x_1, x_2, x_3, providing the quadric of rank 3 $x_1x_3 = x_2^2$. Calculate the dimension of $\mathcal{L}(3K)$ and show that there must be a cubic relations $y^3 + A_2(x_1,x_2,x_3)y + B_3(x_1,x_2,x_3)$ (here we need $1/3 \in k$ to do the Tschirnhausen transformation).

3. Clifford's theorem Prove that $d \geq 2r$ for any irregular divisor D defining a g_d^r (here irregular means that the irregularity $l(K-D) \neq 0$). In other words, the fastest growth of $l(D)$ among all curves C and divisors D is given by the hyperelliptic curves discussed in Q1.

[Hints: (1) use the following *linear-bilinear lemma*: let $\varphi\colon V_1 \times V_2 \to W$ be a bilinear map from vector spaces V_1, V_2 of dimension l_1, l_2. Suppose $\varphi(v_1, v_2) \in W$ is nonzero for every nonzero $v_1 \in V_1$ and $v_2 \in V_2$. Then the image of φ spans a subspace of dimension $\geq l_1 + l_2 - 1$ in W. Proof: Tensors of rank 1 $\{v_1 \otimes v_2\}$ form a subvariety of dimension $l_1 + l_2 - 1$ in $V_1 \otimes V_2$. The kernel of $\varphi\colon V_1 \otimes V_2 \to W$ intersects it in 0 only.

(2) Consider the multiplication map $\mathcal{L}(D) \times \mathcal{L}(K-D) \to \mathcal{L}(K)$, and put together the inequality of the lemma with the RR formula.]

4. Degree 4 divisor on curve of genus 2 Let $\Gamma_4 \subset \mathbb{P}^2_{\langle x,y,z\rangle}$ be a plane quartic curve with a node or cusp at $(1,0,0)$ and no other singularities. We can assume that its equation is $x^2a_2 + xb_3 + c_4$, with a, b, c forms in y, z of the stated degree. Show that projection from P defines a 2-to-1 cover from the resolution $C \to \mathbb{P}^1_{\langle y,z\rangle}$ ramified in the discriminant sextic $b^2 - 4ac$, so that C is a hyperelliptic curves of genus 2.

Recall that K_C is the final irregular divisor. Prove that for any curve C of genus ≥ 2 and any $P, Q \in C$, we have $l(K+P+Q) - l(K) = 1$, so the morphism φ_D corresponding to $D = K + P + Q$ cannot distinguish the two points P, Q, that is, $\varphi_D(P) = \varphi_D(Q)$.

Now suppose that $g = 2$, and let D be any divisor of degree 4. Show that $l(D - K_C) > 0$, so that D is linearly equivalent to $K + P + Q$. Prove that $\varphi_D\colon C \to \mathbb{P}^2$ either maps C to a quartic curve $\Gamma_4 \subset \mathbb{P}^2$ with a node at $\varphi(P) = \varphi(Q)$ (resp., cusp if $P = Q$), or is a double cover of a plane conic (in the case $D - K_C = g_2^1$, that is, $D = 2g_2^1$).

5. Genus 6 Let C be a curve of $g = 6$, and assume it has no g_2^1, g_3^1 or g_5^2. If D is a g_4^1, show that $|K - D|$ is a g_6^2 (that is, $K - D$ has degree 6 and

$l(K-D) = 3$, and $K-D|$ is a free linear system), so defines a morphism $\varphi_{K-D}\colon C \to \mathbb{P}^2$.

Let $\Gamma_6 \subset \mathbb{P}^2$ be a sextic having double points (nodes or cusps) at the 4 points $(1,0,0),(0,1,0),(0,0,1),(1,1,1)$ of the standard projective frame of reference. By considering the linear system of cubics of $\mathbb{P}^2$ passing through the 4 points, show that the resolution C has a linear system of dimension ≥ 6 and degree ≤ 10.

Given that its resolution $C \to \Gamma_6$ is a curve of genus 6. Show that C has 5 g^1_4s and complementary g^2_6s. [Hint: Four of them are fairly obvious. The fifth comes from the pencil of conics through the 4 points.]

It is a fact that any curve of genus 6 is given either by this construction, or a different construction adapted to the case that C has a g^1_2, g^1_3 or g^2_5, or is a double cover of curve of $g = 1$. (The g^2_5 case correspond to a plane quintic $C_5 \subset \mathbb{P}^2$.) Unfortunately, it would be something of a detour from the main course to discuss this rigorously or comprehensibly.