

MA4L7 Algebraic curves. First example sheet (not assessed)

The first week's lectures talked around the prerequisites. (Many students who did Christian Boehning's lecture course MA4A5 will find this too easy.)

1. Affine varieties $X \subset \mathbb{A}^n$

Reread UAG, Chap. 2 up to the proof of NSS. Don't worry too much about the Zariski topology, because it is just the cofinite topology if X irreducible and 1-dimensional.

2. Affine coordinate ring

and function field $k(X)$ from UAG, Chap. 3.

The coordinate ring is defined as $k[X] = k[x_{1..n}]/I_X$.

Exercise 0.1 Use the NSS to establish the bijections

$$\begin{aligned} \left\{ \text{maximal ideals of } k[X] \right\} &\longleftrightarrow \left\{ \text{maximal ideal of } k[x_{1..n}] \text{ containing } I_X \right\} \\ &\longleftrightarrow \left\{ m_P = (x_i - a_i \mid i \in [1..n]), \text{ where } P = (a_{1..n}) \in X \right\}. \end{aligned}$$

and

$$\begin{aligned} \left\{ \text{prime ideals of } k[X] \right\} &\longleftrightarrow \left\{ \text{prime ideal of } k[x_{1..n}] \text{ containing } I_X \right\} \\ &\longleftrightarrow \left\{ I_Y \text{ with } Y \subset X \text{ irreducible subvariety.} \right\} \end{aligned}$$

These have the flavour "the ring $k[X]$ knows everything about X ", and will justify writing $X = \text{Spec } X$ (with a small abuse of terminology concerning the single prime ideal $0 \subset k[X]$).

Exercise 0.2 X affine irreducible with affine coordinate ring $k[X]$ and function field $k(X)$. Prove that if $f \in k(X)$ is regular at every $P \in X$, then $f \in k[X]$. That is rational plus everywhere regular implies polynomial.

Moreover for $0 \neq g \in k[X]$, if $f \in k(X)$ is regular at every $P \in X$ with $g(P) \neq 0$ then $f \in k[X]_{[g]}$.

In either case, you need to use NSS. This will be used later as a step in going from birational (geometry up to birational equivalence) to biregular (geometry up to isomorphism).

3. DVR

Recall the definition of DVR from lectures or one of the textbooks.

Exercise 0.3 *Prove that $P \in X$ is a nonsingular point of a curve if and only if the local ring $\mathcal{O}_{X,P}$ is a DVR.*

This is more or less the definition, but you have to get all the words right.

4. Integral closure.

Exercise 0.4 *Show that \mathbb{Z} is integrally closed in \mathbb{Q} . More generally, if A is a UFD, prove that A is integrally closed. (That is, any element of $K = \text{Frac } A$ that satisfies a monic polynomial equation over A is actually in A .)*

Deduce that a DVR is integrally closed.

I will prove later that a 1-dimensional Noetherian local ring A that is integrally closed in its field of fractions $K = \text{Frac } A$ is a DVR.

Exercise 0.5 *Prove the following lemma: consider $A \subset A[x]/f$ where A is a ring and $f \in A[x]$ a monic polynomial. Then*

$$A \text{ is a field} \iff A[x]/f \text{ is a field.}$$

5. Rational functions on \mathbb{P}^1 and the “baby case” of RR.

Let u, v be homogeneous coordinates on \mathbb{P}^1 , and write $x = v/u$ for the affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$, and write $P = (1 : 0)$ and $Q = (0 : 1) \in \mathbb{P}^1$. The vector space $k[x]_{\leq d}$ has dimension $1 + d$ (with a basis you can easily guess). If we view it as the space of rational functions with pole $\leq dQ$, it is the ideal first case of RR space $\mathcal{L}(\mathbb{P}^1, dQ)$. The equality $l(\mathbb{P}^1, dQ) = 1 - g + d$ (with $g = 0$) holds for all $d \geq -1$, and fails by 1 for $d = -2$.

By considering $(x - a)/(x - b)$, show that $k(\mathbb{P}^1)$ contains a function with $\text{div } f = P_1 - P_2$ for any $P_1, P_2 \in \mathbb{P}^1$. More generally, if $\sum m_i P_i$ and $\sum n_j Q_j$ have $\sum m_i = \sum n_j$, then there exists $f \in k(\mathbb{P}^1)$ with $\text{div } f = \sum m_i P_i - \sum n_j Q_j$.

Prove that $l(\mathbb{P}^1, D) = 1 - g + \text{deg } D$ for any $D = \sum m_i P_i$ of degree $d = \sum m_i$.