MA4J8 Commutative Algebra II. Worksheet 3

I. Serre's R_1 plus S_2 criterion for normality

Let A be a Noetherian integral domain with fraction field $K = \operatorname{Frac} A$. *Serre's condition* R_1 (regular in codimension 1) says: for every height 1 prime P the localisation A_P is a DVR (height 1 prime means minimal nonzero prime). In what follows, assume A satisfies R_1.

Next, *Serre's condition* S_2 is the statement that the localisation A_P has depth ≥ 2 at every prime P of height ≥ 2. This is vacuous if $\dim A \leq 1$.

Prove **Serre's criterion:** Let A be a Noetherian domain satisfying R_1. Then A *is normal if and only if it satisfies* S_2. Required to prove:

there exist $x \in K$ integral over A but not in A

$\iff$ there exists a prime P of height ≥ 2 for which the local ring $(B, m) = (A_P, PA_P)$ has height $= 1$.

Proof of $\Rightarrow$. No height 1 prime P is an associated prime of K/A, because $x \notin A_P$ implies x is not integral over A_P (it is a DVR), so is not integral over A.

If $x \in K$ is integral over A but not in A then $A[x]$ is finite. The module $A[x]/A$ is finite, so if nonzero it has an associated prime $P \in \operatorname{Spec} A$, and P has height ≥ 2 by the above. Choose $y \in A[x]$ so that $y \notin A$ but $Py \subset A$. Use the "ghost of the departed" argument to prove that $\operatorname{depth} P = 1$.

[For any $s_1 \in P$, consider $s_1 y \in A/(s_1)$. Show it is not zero, but is annihilated by any $s_2 \in P$.]

Proof of the converse $\Leftarrow$. P fails S_2 means that for any nonzero $s_1 \in m$, the maximal ideal m is an associated prime of $B/(s_1)$. If $y \notin s_1 B$ but $my \subset (s_1)$ prove that the fraction $x = y/s_1 \in K$ is integral over B.

Work in 3 steps: first use $my \subset (s_1)$ to deduce that $mx \subset B$.

If $xm \subsetneq B$ then $xm \subset m$, and the determinant trick implies that x is integral over B.

On the other hand $xm = B$ implies that $x^{-1} \in m$ (we are working inside a field), and m is the principal ideal (x^{-1}), which contradicts $\dim B \geq 2$.

II. Past exam question

1. Suppose that A is a Noetherian local ring with maximal ideal m, and let M be a finite A-module. Explain what it means for $s_1, s_2 \in m$ to form a regular sequence of length 2 for M.

2. Give the definition of the Koszul complex $K(s_1, s_2; M)$. Prove that $K(s_1, s_2; M)$ is exact if and only if s_1, s_2 is a regular sequence for M.

3. Consider the ring $A = k[x, y, z, t]/I$, where I is the ideal generated by the four relations

$$xt - yz,\ t^2 - z(1 + z),\ yt - xz(1 + z),\ y^2 - x^2(1 + z).$$

Write $m = (x, y, z, t)$. Prove that $\dim_k m/m^2 = 4$.

You may assume that I is prime. Write $K = \operatorname{Frac} A$ for the field of fractions of the integral domain A. Verify that $u = y/x \in K$ is integral over A.

Prove that $u \notin A$, but that $m \cdot (y/x) \subset A$.

4. Let A, m be a local integral domain of dimension ≥ 2 with field of fraction $K = \operatorname{Frac} A$. Suppose that there exists $f \in K \setminus A$ such that $mf \subset A$. Prove that there does not exist any regular sequence $s_1, s_2 \in m$ of length 2.

III. Assorted questions

Q1. Specialise one section of the proof of the main theorem on dimension to establish that

$$\dim_k m/m^2 \geq \dim A \quad \text{for a local ring } A, m, k.$$

Remark. m/m^2 is a vector space over $k = A/m$. In the geometric case it is the dual of the tangent space to a variety, with $\dim m/m^2 = \dim A$ the condition for nonsingularity.

Q2. Assuming the main theorem on dimension of local rings, prove that $\dim A/(x) = \dim A - 1$ for A a Noetherian ring and $x \in A$ a nonzerodivisor. The issue is to pass from local to A itself.

Q3. Define the *height* of a prime ideal P of A as the Krull dimension $\dim A_P$ of the local ring A_P. Prove that this is the maximum length of all chains

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n = P.$$

Q4. Check that $\operatorname{ht} P = 0$ means that P is a minimal prime. The minimal prime ideals correspond to the irreducible components of $\operatorname{Spec} A$. Recall that

$$\operatorname{rad}(A) = \text{intersection of prime ideals} = \text{intersection of minimal prime ideals}.$$

If a prime ideal P contains a nonzerodivisor x, prove that $\operatorname{ht} P \geq 1$. [Easy: [A&M] Cor. 11.17.]

Q5. For A a Noetherian ring and $x \in A$, let P be a prime that is minimal among prime ideals containing x. Use the main theorem on dimension to prove that $\operatorname{ht} P \leq 1$. (See [A&M] Cor. 11.17, and [Ma], Theorem 13.5. The result is called *Krull's Hauptidealsatz*.)

Q6. In the same way, prove that if $I = (a_1, \ldots, a_r)$ and P is a minimal prime divisor of I then $\operatorname{ht} P \le r$.

Here "prime divisors of I" means $P \in \operatorname{Ass} A/I$, so there is some $y \in A/I$ such that $P = \operatorname{ann} y$, or $A \cdot y \cong A/P \subset A/I$. This needs primary decomposition and $\operatorname{Ass} M$, for example [UCA], Chap 7.

Q7. (One of Nagata's famous examples, [A&M Ex 11.4, p. 126]). Start from the polynomial ring $A = k[x_1, \ldots, x_n, \ldots]$ in countably many variables, and choose a sequence of integers a_i with difference $a_{i+1} - a_i$ growing to infinity (for example $i = j^2$ for $j \in \mathbb{N}$). Each ideal

$$P_i = (x_j \mid j \in [a_i + 1, a_{i+1}])$$

is prime. The localisation A_{P_i} at P_i is the polynomial ring in the variables $\{x_j \mid j \in [a_i + 1, a_{i+1}]\}$ over the field of rational functions in all the x_i not in that range.

Check that the complement $S = A \setminus \bigcup P_i$ is a multiplicative set of A and set $B = S^{-1}A$. Each localisation A_{P_i} at P_i is a localisation of B, so that B has Krull dimension $\dim B = \infty$.

The more inscrutable point is that, although its construction involves countable infinities, $B = S^{-1}A$ is still Noetherian: every nontrivial ideal I of B is the localisation of an ideal of $T^{-1}A$ where T is the complement of all but finitely many of the P_i. That is, for any choice of ideal $0 \ne I \subsetneq B$, the localisation divides into two steps $A \mapsto T^{-1}A \mapsto B$, the first of which puts all but finitely of the x_i many into a function field K, with $T^{-1}A$ a polynomial ring $K[x_i]$ in just finitely many variables.

In fact, a nonzero element of B is a/s with $s \notin P_i$, and it is a nonunit if and only if $a \in P_i$ for some i.

By construction of the P_i as generated by disjoint set of variables in a polynomial ring, it follows that $P_i \cap P_j = P_i \cdot P_j$, so an element a/s is only in finitely many of the P_i.

Now for $I \subset B$ a nontrivial ideal there is a nonempty finite set J of ideals P_j such that $I \subset S^{-1}P_j$. (The j can only include the finitely many P_j for a fixed $a/s \in I$, and if $J = \emptyset$ then $I = B$.) Now $S^{-1}A$ is a localisation of $T^{-1}A$ where $T = \mathcal{C}J$ is the complement of J.

Finally $I \subset S^{-1}A$ is the localisation of an ideal of $T^{-1}A$.

Q8. Write Σ for the 3 coordinate axes in $\mathbb{A}^3$. The ideal I_Σ is generated by (xy, xz, yz), so that the coordinate ring $k[\Sigma] = k[x, y, z]/(xy, xz, yz)$.

Find sets of linear forms

$$(a(x, y, z), b(x, y, z), c(x, y, z))$$

such that $axy + bxz + cyz \equiv 0$. [Hint: this is too easy. Look first at the Koszul syzygies between two generators, then cancel.] Write out a minimal free resolution

$$\begin{array}{l} P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow 0 \\ \downarrow \\ k[\Sigma] \end{array}$$

where $P_0 = A$, $P_1 = 3A$, $P_2 = 2A$, the first map $P_0 \leftarrow P_1$ is (xy, xz, yz). P_2 is the module of syzygies holding between the 3 generators of I_Σ, and has basis 2 sets of linear forms (a, b, c) as above.

Q9. [UAG] (3.11) gives the example of the ideal $I = (f, g, h)$ in $k[x, y, z]$ generated by

$$f = xz - y^2, \quad g = x^3 - yz, \quad h = z^2 - x^2y.$$

Is h in the ideal (f, g), and why not? It would work if you were allowed to cancel a bit. Use this idea to find two syzygies holding between the three relations, and determine the minimal free resolution of $k[\Gamma] = k[x, y, z]/I$ in the shape

$$\begin{array}{l} P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow 0 \\ \downarrow \\ k[\Gamma] \end{array}$$

You know you have won if you can write the homomorphism $P_1 \leftarrow P_2$ as a 2×3 matrix that has f, g, h as its 2×2 minors.

Q10. Same question for $f, g, h = y^2 - xz, x^4 - yz, z^2 - x^3y$.
Hint: Plug the code below into the online Magma calculator
`http://magma.maths.usyd.edu.au/calc`

```
R<x,y,z> := PolynomialRing(Rationals(),3);
L := [y^2-x*z, x^4-y*z, z^2-x^3*y];
SyzygyModule(L); // or better still
MinimalBasis(SyzygyModule(L));
```

Figure out what is going on, and how you would do it by hand calculation.

Q11. Prove that the prime ideals in a Noetherian ring A satisfy the d.c.c. That is, a descending chain of prime ideals eventually stabilises