Apr 2022 exam, compulsory Question 1

1. Prove that a nilpotent element of a ring A is contained in every prime ideal. If $f \in A$ is not nilpotent prove that there is a prime ideal P not containing f.

What does it mean to say that A has Krull dimension 0? If this holds, deduce that the intersection of all maximal ideals of A equals the nilradical of A.

2. If $\varphi\colon A \to B$ is a ring homomorphism and P a prime ideal of B, prove that $\varphi^{-1}(P)$ is a prime ideal of B.

Let S be a multiplicative set in A. Show that the prime ideals of $S^{-1}A$ are in bijection with the prime ideals of A disjoint from S.

3. Let A be an integral domain and $t \in A$ a nonunit. If $x \in A$ is a nonzero multiple of t, say $x = tx_1$, prove that $(x) \subset (x_1)$ is a strict inclusion of ideals.

If A is Noetherian, deduce that $\bigcap_{n=1}^{\infty}(t^n) = 0$.

4. Let A be a ring and M a Noetherian module on which A acts faithfully (that is, no element $a \in A$ acts on M by 0). Prove that A is a Noetherian ring. [Hint: Consider the A-module homomorphism $\varphi\colon A \to \bigoplus_{i=1}^{n} M$ given by $1_A \mapsto (m_1, \dots, m_n)$ where $m_1, \dots, m_n$ generate M.]

5. Describe the equivalence relations on pairs (m, s) that defines the localisation $S^{-1}M$ of an A-module with respect to a multiplicative set S of A. Describe the homomorphism $M \to S^{-1}M$ and say what is its kernel.

In the case $M = A/I$ for I an ideal of A, determine which primes P have $M_P \neq 0$.

6. Given a ring A and prime ideals P_i of A for $i = 1, \dots, n$, suppose that I is an ideal of A not contained in any of the P_i. Prove that I is not contained in the union $\bigcup_{i=1}^{n} P_i$. [Argue by contradiction, and by induction on n.]

7. Define the Zariski topology on the prime spectrum $X = \operatorname{Spec} A$ of a ring A. Introduce the principal open sets X_f for $f \in A$, and prove that they form a basis for the Zariski topology.

Give a necessary and sufficient condition on a set $\{f_\lambda\}_{\lambda\in\Lambda}$ of elements of A for the principal open sets X_{f_λ} to cover X. If it holds, deduce that X is covered by finitely many of them.

Apr 2022 exam, Question 2

1. Let $A = k[x, y]$ be the polynomial ring over a field k, and M the quotient module $M = A/(x^2y, xy^2)$. For each of the three prime ideals $P_1 = (x)$, $P_2 = (y)$ and $P_3 = (x, y)$ of A, find an element of M whose annihilator equals P_i.

Give the definition of an associated prime of an A-module M.

2. For a nonzero A-module M, consider the set of ideals of the form $\operatorname{ann} m$ for nonzero $m \in M$, ordered by inclusion. Prove that any maximal element of this set is an associated prime of M.

Deduce that a nonzero module M over a Noetherian ring A has an associated prime.

3. Let $\varphi\colon A \to B$ be a homomorphism of Noetherian rings. For a B-module M, let $\varphi^*(M)$ be the module M viewed as an A-module via the homomorphism φ. If $Q \in \operatorname{Spec} B$ is an associated prime of M as B-module, prove that $P = \varphi^{-1}(Q) \in \operatorname{Spec} A$ is an associated prime of $\varphi^*(M)$.

4. In addition to the assumptions of (3), suppose that A is an integral domain, and that M is a B-module for which the A-module $\varphi^*(M)$ introduced in (3) has the prime ideal $P = 0$ as an associated prime. Prove that φ is injective, and that M has a submodule M' for which $\varphi^* M'$ is a torsion-free A-module.

Deduce that M has an associated prime $Q \in \operatorname{Spec} B$ with $P = 0 = \varphi^{-1}(Q)$.

Apr 2022 exam, Question 3

Let A be a Noetherian local ring with maximal ideal m and with the residue field $A/m = k$, and let M be a finite A-module.

1. Give the definition of the m-adic completion $\widehat{A}$ of A and the m-adic completion $\widehat{M}$ of M.

Describe the natural homomorphism $A \to \widehat{A}$. What is the maximal ideal $\widehat{m}$ of $\widehat{A}$? Explain briefly why $\widehat{A}$ is complete in its $\widehat{m}$-adic topology. (No proofs are required.)

2. If $\varphi\colon M \to N$ is a homomorphism between two finite A-modules, explain how φ induces a homomorphism $\widehat{\varphi}\colon \widehat{M} \to \widehat{N}$. If φ is injective, prove from first principles that $\widehat{\varphi}$ is also injective. (You may assume that $\bigcap_{i=1}^{\infty} m^i = 0$, and similarly for $\bigcap m^i M$ and $\bigcap m^i N$.)

Deduce that an element $a \in A$ that is a nonzerodivisor has image $\widehat{a} \in \widehat{A}$ that is also a nonzerodivisor.

3. Let (A, m) be a local ring that is m-adically complete. Give the correct assumptions and conclusion of Hensel's lemma concerning a polynomial $f \in A[x]$ whose image $\overline{f}$ modulo $mA[x]$ has a factorisation $\overline{f} = \overline{g}\overline{h}$. (The proof is not required.)

Hence or otherwise show that for k a field of characteristic $\neq 2$, there exists a formal power series $y \in k[[z]]$ with $y^2 = 1 + z$.

4. You may assume that the polynomial $f = y^2 - x^2 - x^3$ is irreducible in the polynomial ring $k[x, y]$. Explain why its image in the completion of $k[x, y]$ at the maximal ideal (x, y) is no longer irreducible.

Give an example of a local integral domain (A, m) whose m-adic completion has zerodivisors.

Assorted questions

1. For an A-module M and ideal I, consider the quotient $M \twoheadrightarrow \overline{M} = M/IM$ and elements $e_i \in M$ with $e_i \mapsto \overline{e}_i \in \overline{M}$.

Find an example in which $\overline{e}_i$ generate $\overline{M}$, but e_i do not generate M.

Prove that $\overline{e}_i$ generate $\overline{M}$ implies e_i generate M under the additional conditions that A is I-adically complete and M is I-adically separated. [Hint: work by successive approximation, as in the proof of Hensel's lemma (but easier). Compare [Ma] Theorem 8.4.]

2. The first two items are easy prerequisites.

1. If A is a Noetherian ring, and S a multiplicative set in A, prove that $S^{-1}A$ is again Noetherian.
2. Let A be a ring intermediate ring between $\mathbb{Z}$ and $\mathbb{Q}$. Is A Noetherian? Write down a proof or a counterexample.
3. Prove A Noetherian implies the formal power series ring $A[\![x]\!]$ is again Noetherian.

3. Let $u\colon M \to M$ be a homomorphism of A-modules. Consider the iteration u^n (that is, u composed with itself n times). Prove that $\{\ker u^n\}$ is an increasing chain of A-submodules and $\{M_n = \operatorname{im} u^n(M) subset M\}$ a decreasing chain.

Now suppose M is Noetherian. Prove that both chains terminate. Determine a submodule $M_0 \subset M$ such that the restriction $u_{|M_0}\colon M_0 \to M_0$ is an isomorphism.

Do the same arguments work if we assume instead that M is Artinian?

4. Let N_1, N_2 be submodules of an A-module M. Prove that M/N_1 and M/N_2 both Noetherian implies that so is $M/(N_1 \cap N_2)$.

Does $M/(N_1 \cap N_2)$ Noetherian imply anything about M/N_1 and M/N_2?

The same question for Artinian.

5. Exercise on the Zariski topology of $\operatorname{Spec} A$. If A is a Noetherian ring then the topology of $\operatorname{Spec} A$ is Noetherian (has the d.c.c. for closed sets, as for affine algebraic sets in [UAG]). Use the d.c.c to prove that $\operatorname{Spec} A$ is the union of finitely many irreducible closed sets (its irreducible components). Deduce that a Noetherian ring has only finitely many minimal prime ideals.

6. State and prove the result that the localisation $f\colon A \to S^{-1}A$ has the Universal Mapping Property (UMP) for ring homorphisms that map elements of S to units. Compare [Ma, Thm 4.3].

Let B be a ring and suppose that the localisation map f factors as $g\colon A \to B$ followed by $h\colon B \to S^{-1}A$. Assume that every $b \in B$ can be written $b = g(s) \cdot a$ with $s \in S$ and $a \in A$. Prove that

$$S^{-1}A = T^{-1}B \quad \text{where } T = \bigl\{b \in B \mid h(b) \text{ is a unit of } S^{-1}A\bigr\}.$$

In other words, we can also view $S^{-1}A$ as the localisation $T^{-1}B$ of B.

7. Let A be a local ring of Krull dimension r. Prove that A has localisations $S_i^{-1}A$ at different multiplicative sets S_i with $\dim S_i^{-1}A = i$ for every i with $0 \le i \le r$.

8. Let $\widehat{A}$ be the I-adic the completion of A for an ideal I. When does $\dim A = \dim \widehat{A}$? Give a counterexample, then additional conditions under which it holds.

If A, m is a local ring and $\operatorname{Gr}_m A = \bigoplus I^i/I^{i+1}$ its associated graded ring. Prove that $\dim A = \dim \operatorname{Gr}_m A$. Compare [Ma Thm 13.9].

9. Prove that $\dim A \le \dim m/m^2$ for a local ring A, m, k (use the Main Theorem on dimension). Look up the definition of regular local ring for the case of equality. The Zariski tangent space of A, m is the k-dual vector space of m/m^2, and $\dim m/m^2$ is called the *embedding dimension* of A, m, especially in singularity theory.

9. Characterisation of graded in terms of $\mathbb{G}_m$ action. Write $\mathbb{G}_m(k)$ for the multiplicative group $k^\times$ of an infinite field k. For a k-vector space V, a $\mathbb{Z}$-grading on V is a direct sum decomposition $V = \bigoplus_{m\in\mathbb{Z}} V_m$. This defines an action of $\mathbb{G}_m(k)$ on V with $\lambda \in \mathbb{G}_m$ acting by $\lambda \cdot v = \lambda^m v$. Under reasonable extra conditions, the converse holds: a $\mathbb{G}_m$ action on V defines a grading (this holds for example if the action is compatible with a filtration having finite dimensional quotients). This fits under the slogan that $\mathbb{G} - m$ is reductive.

As exercises, do [Ma Ex 13.1–3].

10. Let $R = \bigoplus_{n\in\mathbb{N}} R_n$ be an $\mathbb{N}$-graded ring. Prove the if and only if condition for R to be Noetherian.

For $I \subset R$ an ideal, prove the equivalent conditions for I to be a graded ideal or homogeneous ideal: (i) generated by homogeneous elements of R; (ii) $I = \bigoplus I_n$ with the usual condition on multiplication $R_{n_1} I n_2$; (iii) Every $f \in I$ is a sum of homogeneous elements that are still in I.

10. Let $R = [x_0, \dots, x_n]/I$ where I is a graded ideal. (The usual "straight" case is that all the generators x_i have degree 1.) An ideal of R is *irrelevant* if it contains $\bigoplus_{n>0} R_n$. Show that the only irrelevant prime ideal is $(x_0, \dots, x_n)$.

Compared to $\operatorname{Spec} R$, the homogeneous or graded spectrum $X = \operatorname{Proj} R$ of R is defined to be the set of homogeneous prime ideals *excluding irrelevant ideals*. In other words, for $P \in \operatorname{Proj} R$ the multiplicative set $S = R \setminus P$ is required to contain homogeneous elements of degree $n > 0$. For $g \in R$ homogeneous of degree $d > 0$, define the *principal open set* $X_g \subset X$ to be the set of $P \in \operatorname{Proj} R$ such that $g \notin P$. Check that these form a basis for the Zariski topology on X.

A point $P \in X$ has a local ring

$$\mathcal{O}_{X,P} = \left\{ \frac{f}{g} \;\middle|\; \begin{array}{l} f, g \text{ homogeneous of the} \\ \text{same degree, and } g \notin P \end{array} \right\},$$

and a principal open set X_g has an affine coordinate ring

$$\Gamma(X_g, \mathcal{O}_X) = \left\{ \frac{f}{g^n} \;\middle|\; f \in R_{nd}. \right\}$$

Show that the homogeneity conditions on f/g or f/g^n in these definitions amount simply to invariance under the $\mathbb{G}_m$ action of (Q9).

Show how the above high-flown description of $\operatorname{Proj} R$ boils down to ordinary projective varieties $V \subset \mathbb{P}^n$ and their standard open pieces V_{x_i} as in [UAG].