MA3E1 Groups and representations

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Abstract

Notes and homework sheets from 3rd year undergraduate lecture course on Groups and representations. I would be very grateful for corrections or suggestions for improvement.

Contents

1 Introduction and the cyclic group $\mathbb{Z}/r$

1.1 General cultural introduction

Group theory is the mathematical description of symmetry:

> "Groups are to symmetry what numbers are to counting"

(slogan due to Trevor Hawkes).

Given some structure Σ in mathematics or physics (etc.), a *symmetry group* of Σ is a set G of transformations of Σ; the axioms of a symmetry group are simply that

(1) G contains the identity transformation Id_Σ.

(2) for every $g \in G$, the inverse g^{-1} is also in G.

(3) for every $g_1, g_2 \in G$, the composite $g_1 \circ g_2$ is also in G.

(These are all closure operations: closure under composition, under inverse, and under doing nothing.)

For example, G might be the Euclidean group (all rotations and reflections of $\mathbb{E}^3$), or the symmetries of a regular polygon or regular polyhedron, or all the permutations S_n of a finite set of n elements. Or it might be all the isomorphisms of a field extension in Galois theory, all the isometries of a metric space, all the Lorentz transformations of Lorentz space $\mathbb{R}^{1,3}$ in special relativity, or the internal symmetries of particles in quantum theory.

When studying symmetry, sooner or later you inevitably face questions about G as an abstract group. As you know, an *abstract group* is the data of

(a) an abstract set G,

(b) a preferred element $e = e_G$ (the "identity" or "neutral element"),

(c) a map $i\colon G \to G$ (the "inverse map"),

(d) a binary operation $m\colon G \times G \to G$ (the "group multiplication" or "group law"; I write $m(g_1, g_2) = g_1 \cdot g_2$ for the present discussion),

satisfying the axioms (that you already have in memory):

(1) identity: $e \cdot g = g \cdot e = g$ for all $g \in G$;

(2) inverse: $i(g) \cdot g = g \cdot i(g) = e$ for all $g \in G$;

(3) associative: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_i \in G$.

Conversely, playing with an abstract group G leads inevitably to questions about whether G arises as a symmetry group, and in how many different ways. For example, we can view the dihedral group D_{2m} as an abstract group: its elements are

$$e, a, a^2, \ldots, a^{m-1}, \quad b, ab, a^2b, \ldots, a^{m-1}b, \tag{1.1}$$

and its multiplication law is determined by the rules

$$a^m = e, \quad b^2 = e \quad \text{and} \quad ba = a^{m-1}b. \tag{1.2}$$

However, D_{2m} also arises as the group of rotations and reflections of a regular m-gon, making its salient features immediately clear. Or we can make the same abstract group D_{2m} act as a permutation group of a set of m elements, with

$$a = m\text{-cycle } (0, 1, 2, \dots, m-1) \quad \text{and} \quad b = \text{inversion } i \leftrightarrow m-i-1 \tag{1.3}$$

[See Ex. 1.5.] This is summarised in the slogan

> "a symmetry group is an abstract group, and an abstract group is a symmetry group."

Symmetries often appear naturally as linear operations; mathematically, this means we can write them as linear transformations of a vector space, or as matrices acting on vectors. A *matrix group* is a set G of matrices that forms a symmetry group in the sense of the above definition, so that (say) the product g_1g_2 of any two matrices $g_1, g_2 \in G$ is still in G. *Representation theory* studies all possible ways of mapping a given abstract group to a matrix group.

A foretaste of the main results of the course: given a finite group G, a finite dimensional linear representation of G over $\mathbb{C}$ breaks up uniquely as a direct sums of irreducible representations, of which there are only finitely many (*up to isomorphism* throughout that sentence). Whereas a linear representation is defined as a set of $m \times m$ matrices corresponding to the elements of G (so a large and cumbersome collection of data), it is determined up to isomorphism by its *character*, which is just a complex valued function on the conjugacy classes of G. The characters of irreducible representations are written as an $r \times r$ array of algebraic numbers that (for a tractable group) forms a kind of entertaining and usually not very difficult crossword puzzle. By the end of the course you will all be experts at solving these.

I hope to persuade you over the course of the term that group representations are just as easy to work with as vector spaces, and a lot easier than much of 2nd year Linear Algebra. I will proceed gently, pausing to make a few major ideological pronouncements. The course is mostly about representations, and I only need basic properties of finite groups.

1.2 The cyclic group $\mathbb{Z}/r$

I start with the cyclic groups $\mathbb{Z}/r$ for several reasons:

(i) They are by a long way the most useful in applications.

(ii) The representation theory of $\mathbb{Z}/r$ appears as an essential component of the character theory of general finite groups.

(iii) We can treat it with almost no prerequisites.

(iv) It motivates and gives direction to the general theory.

(v) At the same time I take the opportunity to remind you of a couple of basic algebraic tricks that you may need.

A representation of $\mathbb{Z}/r$ on a vector space V is a linear map $\varphi\colon V \to V$ such that $\varphi^r = \mathrm{Id}_V$ (that is, φ composed with itself r times is the identity). I work over $\mathbb{C}$, and assume V is finite dimensional, with basis $\{e_1, \dots, e_m\}$. Then φ is given by a matrix M satisfying $M^r = \mathrm{Id}$. It follows from the theory of Jordan Normal Form that M can be diagonalised, with diagonal entries rth roots of 1. However, the result I want is easier, and it is best to go through a proof from first principles.

By all means accept the statement and ignore the proof for the moment if you prefer. The example given after the theorem will be more entertaining and enlightening than the boring algebraic proof.

Theorem *Let V be a finite dimensional vector space over $\mathbb{C}$ and $\varphi\colon V \to V$ a $\mathbb{C}$-linear map such that $\varphi^n = \mathrm{Id}_V$. Write M for the matrix representing φ in a basis $\{e_1, \dots, e_m\}$, so that $M^r = \mathrm{Id}$.*

Then M is diagonalisable. In other words, there is a new basis $\{f_1, \dots, f_m\}$ of V made up of eigenvectors of M, so that $\varphi(f_i) = \lambda_i f_i$, with each λ_i satisfying $\lambda_i^r = 1$.

Step 1 *An eigenvalue of M is an rth root of unity.*

Since $Mv = \lambda v$, we get $M^2 v = \lambda^2 v$ and so on up to $v = M^r v = \lambda^r v$. Q.E.D.

Step 2 If λ is an eigenvalue of M, its *generalised eigenspace* V_λ is given by

$$V_\lambda = \bigl\{ v \in V \bigm| (M - \lambda)^a v = 0 \text{ for some } a > 0 \bigr\}, \tag{1.4}$$

where $M - \lambda$ means the matrix $M - \lambda\,\mathrm{Id}$. *I assert that then $\varphi(V_\lambda) \subset V_\lambda$.*

If $(M - \lambda)^a v = 0$ then $(M - \lambda)^a M v = M(M - \lambda)^a v = 0$; here the matrix $(M - \lambda)^a$ is a linear combination of powers M^i, and of course M commutes with all of these. Therefore $MV_\lambda \subset V_\lambda$. Q.E.D.

Step 3 *Moreover, the restricted map $\varphi\colon V_\lambda \to V_\lambda$ equals λ times the identity.* This is the key simplifying feature arising from $M^r = \mathrm{Id}$.

Write $M'\colon V_\lambda \to V_\lambda$ for the restricted map. By construction $M' - \lambda$ is a nilpotent map on V_λ. A standard argument shows that V_λ has a basis for which $M' - \lambda$ is strictly upper triangular. (Induction on the dimension: $M' - \lambda$ cannot be injective; by induction, its image $(M' - \lambda)(V_\lambda) \subset V_\lambda$ has such a basis. Now choose a complementary basis of V_λ.)

In this new basis, M' is upper triangular with diagonal entries λ. The only way that such a matrix can satisfy $M'^r = 1$ is if $\lambda^r = 0$ and the strictly upper triangular part is zero. Indeed, write $M' = \lambda \operatorname{Id} + X$ with X strictly upper triangular. Then

$$(\lambda \operatorname{Id} + X)^r = \lambda^r \operatorname{Id} + r\lambda^{r-1} X + \tbinom{r}{2} \lambda^{r-2} X^2 + \cdots. \tag{1.5}$$

First, by looking at the diagonal entries, we see that $\lambda^r = 1$. Next, since X is strictly upper triangular, if it is nonzero there is a smallest j with some $X_{i,i+j} \neq 0$. The terms X^2, X^3 etc. in (1.5) cannot contribute to $X_{i,i+j}$ so that $r\lambda^{r-1} X_{i,i+j} = 0$, which contradicts the choice of j. Therefore $X = 0$ and $M' = \lambda \operatorname{Id}$.

Step 4 *Suppose that* $\{\lambda_k\}$ *are* all *the eigenvalues of* M. *Then* V *is the direct sum* $V = \bigoplus V_{\lambda_k}$. This implies the theorem.

For any i, the subspaces V_{λ_i} and $\sum_{k \neq i} V_{\lambda_k}$ have zero intersection (because a λ_i-eigenvector is not a sum of λ_k-eigenvectors for $\lambda_k \neq \lambda_i$). Therefore the sum $\sum V_{\lambda_k}$ is a direct sum $\bigoplus V_{\lambda_k}$.

Now for each i, consider the matrix product

$$E_i = \prod_{k \neq i} \frac{M - \lambda_k}{\lambda_i - \lambda_k}. \tag{1.6}$$

Viewed as a map $E_i \colon V \to V$, it is zero on each V_{λ_k}, because one of the factors is. However, it is the identity on V_{λ_i}: in each factor the numerator $M - \lambda_k$ acts on V_{λ_i} by $\lambda_i - \lambda_k$.

It follows that the matrix $\sum_k E_k$ is the identity on $\bigoplus V_{\lambda_k} \subset V$, with E_k the projection to the components of the direct sum.

Finally the product matrix $Z = \prod (M - \lambda_k)$ maps all the V_{λ_k} to zero. Its image is invariant under M. If this is nonzero, then M must have a new eigenvalue on it, which contradicts the choice of $\{\lambda_k\}$. Therefore $V = \bigoplus V_{\lambda_k}$. Q.E.D.

The discussion of Maschke's theorem in Chapters 2 and 3 include 3 more proofs of Theorem .

1.3 Example: $\mathbb{Z}/5$ and $\mathbb{Z}/r$ for general r

$\mathbb{Z}/r$ is the cyclic group of order r with a chosen generator 1. Its elements are $\{e, 1, 2, \ldots, r-1\}$ where $e = 0$ is the neutral element, and the group law takes $a, b \mapsto a + b \bmod r$.

A representation of $\mathbb{Z}/r$ over a field K is given by a finite dimensional vector space V and a K-linear map $\varphi \colon V \to V$ such that $\varphi^r = \operatorname{Id}_V$. In any basis φ is given by a matrix $M \in \operatorname{GL}(m, K)$ with $M^r = \operatorname{Id}$. The preceding section proved that M can be diagonalised as long as we work over $\mathbb{C}$; that is, there is a basis in which $M = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$. Necessarily each λ_i satisfies $\lambda_i^r = 1$.

Write e_0, e_1, e_2, e_3, e_4 for the standard basis of K^5 (viewed as column vectors). Consider the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \text{so that} \quad Me_0 = e_1, \dots, Me_4 = e_0. \tag{1.7}$$

This M is a *permutation matrix*: it has a single 1 in each row and column, and zeros elsewhere. Applying M to the standard basis performs the 5-cycle $(0, 1, 2, 3, 4)$. Taking the generator $1 \in \mathbb{Z}/5$ to M defines a representation of $\mathbb{Z}/5$ on K^5.

Consider the eigenvalue problem $Mv = \lambda v$ for $v = {}^t(x_0, x_1, \dots, x_4)$ (where ^{t} is transpose, the column vector). One calculates

$$x_0 = \lambda x_1, \quad x_1 = \lambda x_2, \quad x_2 = \lambda x_3, \quad x_3 = \lambda x_4, \quad x_4 = \lambda x_0. \tag{1.8}$$

Thus v is a multiple of $(1, \lambda^4, \lambda^3, \lambda^2, \lambda)$ and $\lambda^5 = 1$.

To proceed, we need the 5th roots of 1, so work over $\mathbb{C}$. Write $\varepsilon = \exp \frac{2\pi i}{5} = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$. The 5 roots of $\lambda^5 = 1$ are the powers $1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4$.

Set $f_j = \sum_i \varepsilon^{-ij} e_i$ for $j = 0, \dots, 4$. The calculation of (1.8) gives $Mf_j = \varepsilon^j f_j$, so that f_j is an eigenvector of M with eigenvalue ε^j. The 5 f_j are the required eigenbasis.

Conversely, starting from an eigenbasis f_j of $\mathbb{C}^5$ with $Mf_j = \varepsilon^j f_j$, if we set $e_k = \frac{1}{5} \sum_j \varepsilon^{jk} f_k$ for $k = 0, \dots, 4$ then M acts as the 5-cycle $(0, 1, 2, 3, 4)$ on the e_k.

Formulas for ε The equation for the 5th roots of unity splits as

$$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1). \tag{1.9}$$

The first factor corresponds to the trivial root $x = 1$, and the second factor to the 4 primitive 5th roots of 1, that is, ε^i for $i = 1, 2, 3, 4$. It is interesting and useful for what I want to do later to derive explicit radical expressions for these. For this, write $y = x + \frac{1}{x}$ and

$$\frac{x^4 + x^3 + x^2 + x + 1}{x^2} = x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = y^2 + y - 1. \tag{1.10}$$

Then the quadratic formula gives $y = \frac{-1+\sqrt{5}}{2}$ or its conjugate $\frac{-1-\sqrt{5}}{2}$; note that $z = y + 1$ satisfies $z^2 = 1 + z$ and is the "Golden Ratio". Then x is a root of $x^2 - yx + 1$, so that $x = \frac{y+\sqrt{y^2-4}}{2}$. A little calculation gives

$$\cos \frac{2\pi}{5} = \frac{y}{2} = \frac{-1 + \sqrt{5}}{4} \approx 0.30902 \text{ and } \sin \frac{2\pi}{5} = \sqrt{\frac{5 + \sqrt{5}}{8}} \approx 0.95105. \tag{1.11}$$

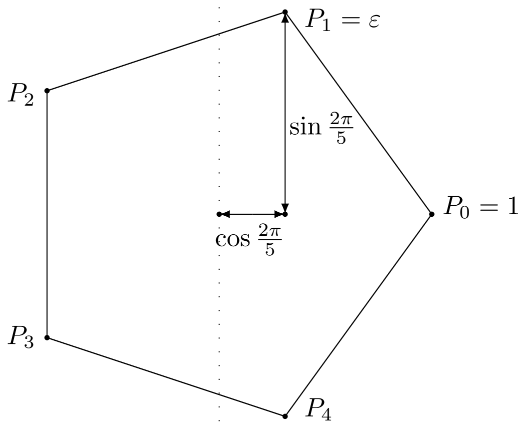


Figure 1: The 5th roots of 1 as the regular pentagon

1.4 The representation theory of $\mathbb{Z}/r$, general conclusions

The basic thing to understand is the rth roots of 1. There are r of these, forming the subgroup $\boldsymbol{\mu}_r \subset \mathbb{C}^\times$ of the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ defined by $x^r = 1$. We can draw them as the vertices of the regular r-gon in $S^1 \subset \mathbb{C}^\times$, as exemplified in Figure 1.

It is often convenient to give $\boldsymbol{\mu}_r$ the generator $\varepsilon = \varepsilon_r = \exp \frac{2\pi i}{r}$. There are other choices, of course: any of the $\varphi(r)$ primitive roots of 1, given by ε^i for $1 \le i \le r-1$ with i coprime to r. Any such choice provides an isomorphism $\mathbb{Z}/r \cong \boldsymbol{\mu}_r$.

A 1-dimensional representation of $\mathbb{Z}/r$ over $\mathbb{C}$ is a homomorphism $\varphi\colon \mathbb{Z}/r \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$; it is determined by the image g of the generator 1. Since $g^r = 1$, the image $\varphi(g)$ must be in $\boldsymbol{\mu}_r$, so that there are exactly r possibilities: $\varphi(g) = \varepsilon^i$ for $i \in [0, 1, \dots, r-1]$.

The above Theorem implies that every representation of $\mathbb{Z}/r$ is isomorphic to a direct sum of 1-dimensional representations, corresponding to a diagonal matrix with diagonal entries in $\boldsymbol{\mu}_r$.

The rest of the course treats the same questions for a general finite group G. We can state the conclusions so far on $\mathbb{Z}/r$ in terms suggesting the shape of the general case. Every representation (finite dimensional, over $\mathbb{C}$) is isomorphic to a direct sum of irreducibles. There are only finitely many irreducible representations of G up to isomorphism. An irreducible representation of G is uniquely determined by its character (a map $G \to \mathbb{C}$; for a cyclic group the two notions are exactly the same thing). Finally, evaluating the characters of irreducible representations on (conjugacy classes of) elements of G provides an $r \times r$ array of numbers that satisfy the following *orthogonality relations*:

Proposition *Write $\rho_i\colon \mathbb{Z}/r \to \mathbb{C}^\times$ for the representation taking 1 to ε^i. Then* *for* $i, j \in [0, \ldots, r-1]$

$$\sum_{g \in \mathbb{Z}/r} \overline{\rho_i(g)} \rho_j(g) = \begin{cases} r & \text{if } i = j, \\ 0 & \text{else.} \end{cases} \tag{1.12}$$

Appendix: Representation theory is easy

Some introductory remarks emphasising the *easy* aspects of the course.

1.5 The group $\mathbb{Z}/2$

A vector space V is $\mathbb{R}^n$ or $\mathbb{C}^n$, and a linear map $V_1 \to V_2$ is a matrix. Once we choose a basis, an element $v \in V$ is a vector. A group means *symmetry.* A representation of G is a vector space with G symmetry.

Consider first a vector space V with $\mathbb{Z}/2$ symmetry. This means a linear map $\tau\colon V \to V$ with $\tau^2 = \mathrm{Id}_V$. What can we say about it?

First, for any $v \in V$, the vector $u = v + \tau(v)$ is invariant under τ. Next, in the same way, the vector $w = v - \tau(v)$ is anti-invariant under τ, because $\tau(w) = \tau(v) - \tau^2(v) = \tau(v) - v = -w$. In this context, $v \in V$ is τ invariant if and only if v is an eigenvector of τ with eigenvalue $+1$, and anti-invariant if v is an eigenvector of τ with eigenvalue -1,

Finally, we can easily recover v and $\tau(v)$ from u and w: just do

$$u + w = 2v \quad \text{so that} \quad v = \tfrac{1}{2}(u + w), \tag{1.13}$$

and

$$u - w = v + \tau(v) - v + \tau(v) = 2\tau(v), \tag{1.14}$$

so that $\tau(v) = \frac{1}{2}(u - w)$.

Proposition *Write V^+ for the $+1$ eigenspace of τ and V^- for the -1 eigenspace. Then*

(i) $V^+ = \{v + \tau(v) \mid v \in V\}$ *and* $V^- = \{v - \tau(v) \mid v \in V\}$.

(ii) $V = V^+ \oplus V^-$.

(iii) The maps $E^+ = \frac{1}{2}(1 + \tau)$ and $E^- = \frac{1}{2}(1 - \tau)$ are the two projectors of the direct sum. Then $E^+ + E^- = \mathrm{Id}_V$ and

$$(E^+)^2 = E^+, \quad (E^-)^2 = E^-, \quad E^+ \circ E^- = 0 \tag{1.15}$$

(iv) If we choose a basis $e_1, \ldots, e_k$ of V^+ and $f_{k+1}, \ldots, f_n$ of V^- then τ written in this basis has the block diagonal form $\left(\begin{smallmatrix} \mathrm{Id}_k & 0 \\ 0 & -\mathrm{Id}_{n-k} \end{smallmatrix}\right)$.

This result is the entirety of the representation theory of $\mathbb{Z}/2$. (Notice that the quantity $\frac{1}{2} \in \mathbb{R}$ or $\mathbb{C}$ occurs in taking the average $\frac{1}{2}(v + \tau(v))$. All of this would go wrong if I tried to do the same in characteristic 2.)

You have already seen this argument many times when $V = \mathrm{Mat}(n \times n, \mathbb{R})$ is the vector space of $n \times n$ matrices and τ is the transpose. Any matrix M is uniquely expressed $M = A + B$ with A symmetric and B skew: as above, just take

$$A = \tfrac{1}{2}(M + \tau(M)) \quad \text{and} \quad B = \tfrac{1}{2}(M - \tau(M)). \tag{1.16}$$

1.6 Representations of $\mathbb{Z}/3$

A representation of $\mathbb{Z}/3$ is a vector space V and a linear map $\tau\colon V \to V$ such that $\tau^3 = \mathrm{Id}_V$. Exactly as above, for any $v \in V$, we check that $v + \tau(v) + \tau^2(v)$ is invariant under τ.

So what? Can we find complementary elements playing the role of $v - \tau(v) \in V^-$ in the above? When discussing $\mathbb{Z}/2$ we used ± 1, which are the complex square roots of 1. For $\mathbb{Z}/3$ we need some basic stuff about the complex cube roots of 1. (More generally, for $\mathbb{Z}/n$ we need the nth roots of 1.) I am sure that you know all this, but roots of 1 are central to the course, and a little reminder doesn't do any harm:

The polynomial $x^3 - 1$ splits as $(x-1)(x^2+x+1)$. The quadratic factor has two roots $\frac{-1 \pm \sqrt{-3}}{2}$. Writing i for one of the complex square roots of -1 expresses these as

$$\omega = \frac{-1 + \sqrt{-3}}{2} = \exp \frac{2\pi i}{3} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \qquad \text{and} \quad \omega^2 = \overline{\omega}, \tag{1.17}$$

so that $x^3 - 1$ splits as $(x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$. I write $\boldsymbol{\mu}_3 = \{1, \omega, \omega^2\}$ (LaTeX `{\boldsymbol\mu}`). These 3 elements form a subgroup of the multiplicative group $\mathbb{C}^\times$ isomorphic to $\mathbb{Z}/3$. Notice that the coefficients of $x^3 - 1$ are the elementary symmetric functions in the 3 elements, for example $1 + \omega + \omega^2 = 0$. In the Argand plane $1, \omega, \omega^2$ are the vertices of a regular triangle centred at zero.

After these preparations, the representation theory of $\mathbb{Z}/3$ is a mechanical extension of the above treatment for $\mathbb{Z}/2$. Namely, for any $v \in V$, define

$$u_1 = \tfrac{1}{3}(v + \tau(v) + \tau^2(v)), \quad \text{and} \quad \begin{aligned} u_\omega &= \tfrac{1}{3}(v + \omega^2\tau(v) + \omega\tau^2(v)), \\ u_{\omega^2} &= \tfrac{1}{3}(v + \omega\tau(v) + \omega^2\tau^2(v)). \end{aligned} \tag{1.18}$$

Then $v = u_1 + u_\omega + u_{\omega^2}$, and one calculates easily that

$$\tau(u_1) = u_1 \quad \text{and} \quad \tau(u_\omega) = \omega u_\omega \quad \text{and} \quad \tau(u_{\omega^2}) = \omega^2 u_{\omega^2}. \tag{1.19}$$

Proposition *Let V be a vector space over $\mathbb{C}$ and $\tau\colon V \to V$ a linear map such that $\tau^3 = \mathrm{Id}_V$. Then the eigenvalues of τ are elements of $\boldsymbol{\mu}_3$. Write $V_1, V_\omega, V_{\omega^2}$ for the eigenspaces. Then* *(i)* $V_1 = \{v + \tau(v) + \tau^2(v) \bigm| v \in V\}$, *and*

$$\begin{aligned} V_\omega &= \{v + \omega^2\tau(v) + \omega\tau^2(v) \bigm| v \in V\}, \\ V_{\omega^2} &= \{v + \omega\tau(v) + \omega^2\tau^2(v) \bigm| v \in V\}. \end{aligned} \tag{1.20}$$

(ii) $V = V_1 \oplus V_\omega \oplus V_{\omega^2}$.

(iii) *The linear maps* $E_1 = \frac{1}{3}(1+\tau+\tau^2)$, $E_\omega = \frac{1}{3}(1+\omega^2\tau+\omega\tau^2)$ *and* $E_{\omega^2} =$ *sim. are the projectors of the direct sum. They satisfy* $E_1 + E_\omega + E_{\omega^2} = \mathrm{Id}_V$,

$$\begin{gathered} (E_1)^2 = E_1, \quad (E_\omega)^2 = E_\omega, \quad (E_{\omega^2})^2 = E_{\omega^2}, \\ E_1 \circ E_\omega = E_1 \circ E_{\omega^2} = E_\omega \circ E_{\omega^2} = 0. \end{gathered} \tag{1.21}$$

(iv) *Written in a basis of* V *obtained by concatenating bases of* V_1, V_ω *and* V_{ω^2}, *the map* τ *takes the block diagonal form* $\begin{pmatrix} \mathrm{Id}_{n_1} & 0 & 0 \\ 0 & \omega\,\mathrm{Id}_{n_\omega} & 0 \\ 0 & 0 & \omega^2\,\mathrm{Id}_{n_{\omega^2}} \end{pmatrix}$.

1.7 Homework to Chapter 1

1.1. Rotations of $\mathbb{R}^2$ For an angle $\theta \in [0, 2\pi)$, the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.22}$$

gives the anticlockwise rotation of $\mathbb{R}^2$ through θ.

Find the eigenvalues and eigenvectors of $R(\theta)$. For what values of θ can these be real?

1.2. Past exam question (Jun 2016, Q1(i–iv)

(1) Describe the set of complex cube roots of 1. Give the equation satisfied by the primitive cube roots, and show how to solve it using the methods of algebra and the methods of analysis. Prove that this set is a subgroup of $\mathbb{C}^\times$. **[3]**

(2) Determine the irreducible representations of $\mathbb{Z}/3$. **[3]**

(3) Consider the action of $\mathbb{Z}/3$ on $\mathbb{R}^3$ given by the cyclic 3-fold rotation $x \mapsto y \mapsto z \mapsto x$. Find all $\mathbb{R}$-vector subspaces invariant under this action. **[3]**

(4) Consider the action of $\mathbb{Z}/3$ on $\mathbb{C}^3$ given by the same formula $x \mapsto y \mapsto z \mapsto x$. Determine its decomposition into irreducible representations. **[3]**

No solutions will be given.

1.3. Describe the subgroup $\boldsymbol{\mu}_6 \subset \mathbb{C}^\times$ in similar terms. Write out the coordinates of the 6 elements in trig functions and in terms of surds. Determine which of these are primitive 6th roots of 1, and write out the equation they satisfy.

1.4. Let $\varepsilon = \exp \frac{2\pi}{5}$. Write $r_5 = 1 + 2\varepsilon + 2\varepsilon^4 = 1 + 4\cos\frac{2\pi}{5}$. Calculate r_5^2.

1.5. With ε as in Ex. 1.4, consider the matrices

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon & \varepsilon^3 \\ 1 & \varepsilon^3 & \varepsilon & \varepsilon^4 & \varepsilon^2 \\ 1 & \varepsilon^4 & \varepsilon^3 & \varepsilon^2 & \varepsilon \end{pmatrix} \tag{1.23}$$

and $N = \overline{{}^tM}$ (transpose complex conjugate – note that $\overline{\varepsilon} = \varepsilon^{-1} = \varepsilon^4$). Show that $MN = 5\,\mathrm{Id}_5$. Show that $\frac{1}{\sqrt{5}}M$ is a *unitary* matrix, i.e., $M \cdot \overline{{}^tM} = \mathrm{Id}_5$.

1.6. Let $\varepsilon \in \mathbb{C}$ be a primitive rth root of 1. Prove that

$$\sum_{i=0}^{r-1} \varepsilon^{ij} = \begin{cases} r & \text{if } j \equiv 0 \bmod r, \\ 0 & \text{else.} \end{cases} \tag{1.24}$$

1.7. Dihedral group D_{2m} Here are three different descriptions of the *dihedral group D_{2m}*:

1. Generators and relations:

$$\left\langle a, b \;\middle|\; a^m = e, b^2 = e \text{ and } aba = b \right\rangle. \tag{1.25}$$

2. Rotations and reflections of a regular m-gon in the Euclidean plane.

3. Subgroup of the symmetric group S_m generated by the cyclic permutation $(1\ 2\ \dots\ m)$ and the reflection fixing 1, namely $i \leftrightarrow m + 2 - i$, that is

$$\begin{cases} (1)(2\ m)(3\ m-1)\cdots(\frac{m+1}{2}\ \frac{m+3}{2}) & \text{if } m \text{ is odd, or} \\ (1)(2\ m)(3\ m-1)\cdots(\frac{m}{2}\ \frac{m}{2}+2)(\frac{m}{2}+1) & \text{if } m \text{ is even.} \end{cases} \tag{1.26}$$

Verify that these describe the same group.

1.8 The distinction in (1.26) between m even or odd appears in several contexts.

Prove in particular that $D_{4m} \cong D_{2m} \times \mathbb{Z}/2$ if and only if m is odd.

1.9 In Figure 1, the vertices of the regular pentagon are 1 and the primitive 5th roots of 1, with $P_1 = \varepsilon$ given in (1.11) as a radical expression via two uses of the quadratic formula. Calculate $P_2, \dots, P_4$ in similar terms. The $\pm$ in the quadratic formula means that these 4 points are "conjugate" over $\mathbb{Q}$.

Exercises to Appendix

1.10 Let $V = \mathrm{Mat}(n \times n, \mathbb{C})$ and $\tau \colon V \to V$ the transpose. Determine the τ invariant and τ anti-invariant subspaces V^+, V^- and calculate their dimensions.

1.11 Check the statements $(E^+)^2 = E^+$ and so on in Proposition, iii.

1.12 More generally, define a *projector* or an *idempotent* of V to be a linear map $e \colon V \to V$ such that $e^2 = e$. If e is an idempotent, show that there is another idempotent f such that $e + f = \mathrm{Id}_V$, and that $ef = 0$. Prove that $V = e(V) \oplus f(V)$, with e, f acting as the projections to the two direct summands.

1.13 Let V_1, V_2 be representations of $\mathbb{Z}/2$, and $V_i = V_i^+ \oplus V_i^-$ as in the text. Calculate $\mathrm{Hom}_G(V_1, V_2)$.

1.14 If $\tau^3 = \mathrm{Id}_V$ show that $\frac{1}{3}(1 + \tau + \tau^3)$ is idempotent. [Hint: Calculate $\tau \circ (1 + \tau + \tau^3)$ and $\tau^2 \circ (1 + \tau + \tau^3)$.]

1.15 Cyclic permutation $(x, y, z) \mapsto (y, z, x)$ defines a representation of $\mathbb{Z}/3$ on $\mathbb{R}^3$ or $\mathbb{C}^3$. Find its decomposition into eigenspaces.

1.16 Restricting the permutation $(x, y, z) \mapsto (y, z, x)$ to the invariant plane $x + y + z = 0$ gives a representation of $\mathbb{Z}/3$ on $\mathbb{R}^2$ or $\mathbb{C}^2$ that we might describe as $x \mapsto y$ and $y \mapsto -x - y$, or the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$.

1.17 Check that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)^3 = \mathrm{Id}_3$. Calculate its eigenvalues and the corresponding eigenvectors. Show that they base the invariant plane $\mathbb{C}^2$ defined by $x + y + z = 0$. The form of the matrix breaks the (x, y, z) cyclic symmetry. Notice that for many purposes, including the calculation of eigenvectors you just did, it is simpler to keep the 3 variables.

2 Groups and basic theory of KG-modules

2.1 Definitions from group theory

To be complete, I repeat some of what you already know from group theory. I repeated the definition of (abstract) group G in Chapter 1. A *subgroup* $H \subset G$ is a subset of G that is itself a group (with the same unit element e_G as G, the same inverse map i_G and the same multiplication m_G). We know that for a subset H to be a subgroup, it is enough to check that $e_G \in H$ and that $g_1, g_2 \in H$ implies $g_1 g_2^{-1} \in H$.

A subgroup $H \subset G$ partitions G into left cosets $gH = \{gh \mid h \in H\}$; multiplying gH on the right by elements of H takes gH to itself. The left cosets of H are disjoint, and each is in bijection with H. The set of left cosets of H is written G/H. We have $G = \bigsqcup_{g \in G/H} gH$ (LaTeX `\bigsqcup`). The definition of right cosets Hg is similar; the set of right cosets is written $H \backslash G$ (LaTeX `\backslash`).

If $\varphi\colon G_1 \to G_2$ is a group homomorphism, the image of a subgroup $H \subset G_1$ is a subgroup $\varphi(H) \subset G_2$. The *kernel* of φ is the set of $g \in G$ that map to the neutral element of G_2, that is, $\ker \varphi = \{g \in G_1 \mid \varphi(g) = e_{G_2}\}$. A subgroup $H \subset G$ is *normal* if $gHg^{-1} = H$ for all $g \in G$; it is equivalent to say that $gH = Hg$ for all $g \in G$, that is, each left coset of H is equal to a right coset. We write $H \lhd G$ for H a normal subgroup of G (LaTeX `\lhd`).

Lemma *The kernel of φ is a normal subgroup.*

Conversely, if $H \lhd G$ then setting $(g_1H) \cdot (g_2H) = g_1g_2H$ defines a group structure on the set $\overline{G} = G/H$ of left cosets such that the projection map $\pi\colon G \to \overline{G}$ is a surjective group homomorphism with kernel equal to H.

The only little point in the proof is that in the product g_1Hg_2H, the condition that H is normal gives $Hg_2 = g_2H$, so that $g_1Hg_2H = g_1g_2H$. Q.E.D.

Remark Suppose we ask for a group Γ and a homomorphism $q\colon G \to \Gamma$ such that $H \subset \ker q$. Then $\pi\colon G \to G/H = \overline{G}$ is the universal solution to this problem: it has the stated property, and for any other solution $q\colon G \to \Gamma$, the map q factors in a unique way as $G \xrightarrow{\pi} \overline{G} \xrightarrow{\overline{q}} \Gamma$. That is, π does what is necessary to kill H, but no more. "Sets of cosets" is a device in set theory that is useful to *construct* $\overline{G} = G/H$. However, it is usually simpler to ignore the cosets and think of $\overline{G}$ as a new group with elements $\overline{g} = g$ *up to equivalence* $g \sim gh$.

2.2 Some nice examples of groups

The immediate aim is to give some easy examples of groups. I choose these because their representations can be discussed in terms of explicit matrices.

Binary dihedral group BD_{4m} Let m be an integer and $\varepsilon = \exp\frac{2\pi i}{2m}$ a primitive $2m$th root of 1. (Note the $2m$, not m; the point is that $\varepsilon^m = -1$.) Consider the two matrices

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.1}$$

Check that $A^{2m} = e$, $A^m = B^2 = -1$ and $ABA = B$. Rearrange the second of these to get $BAB^{-1} = A^{-1}$. The group generated by A and B is called the *binary dihedral group*. Its elements are listed as

$$A^i \quad \text{and} \quad A^iB \qquad \text{for } i = 0, \dots, 2m-1, \tag{2.2}$$

and its multiplication law comes from the relations just stated.

The element $A^m = B^2 = -1$ commutes with A and B (either because as a matrix it is scalar diagonal, or because it is a power both of A and of B). The subgroup $\{\pm 1\}$ is the *centre* of BD_{4m} (for $m \geq 2$), and the quotient $\mathrm{BD}_{4m}/\{\pm 1\} = D_{2m}$ is the dihedral group discussed in Chapter 1.

The word *binary* refers to this *central extension* by $\{\pm 1\}$. We view the quotient map $\mathrm{BD}_{4m} \to D_{2m}$ as a "double cover". Lots of groups have interesting central extensions or double covers.

Binary tetrahedral group BT_{24} I introduce the following groups V_4, A_4, H_8 and BT_{24}.

$$\begin{array}{ccc} H_8 & \subset & \mathrm{BT}_{24} \\ \downarrow & & \downarrow \\ V_4 & \subset & A_4 \end{array} \tag{2.3}$$

Here A_4 is the alternating group on 4 elements, or the group of rotations of the regular tetrahedron (not reflections). The quaternion group

$$H_8 = \{\pm 1, \pm i, \pm j, \pm k\} \tag{2.4}$$

consists of the standard unit quaternions, with multiplication determined by

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2.5}$$

Check that $ij = k$ and $ji = -k$ etc. As above, the subgroup $\{\pm 1\}$ is the centre of H_8, and the quotient $H_8/\{\pm 1\} = V_4 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is the *Klein* 4*-group* (or *Vierergruppe*). The presentation $V_4 = \langle a, b, c \mid a^2 = b^2 = c^2 = abc = e \rangle$ is symmetric in the 3 nonzero elements $a = (1, 0)$, $b = (0, 1)$, $c = (1, 1)$, and at the same time makes clear that $V_4 = H_8/\{\pm 1\}$.

The alternating group A_4 has a surjective homomorphism $A_4 \to \mathbb{Z}/3$. To see it, consider the action of A_4 on the 3 unordered pairs of unordered pairs $\{[12, 34], [13, 24], [14, 23]\}$: if you are number 1, you are paired with number 2 or 3 or 4; the 3-cycle $(1)(2, 3, 4) \in A_4$ thus acts as a 3-cycle on the three pairings. One checks that the kernel of this action is the normal subgroup $\{e, (12)(34), (13)(24), (14)(23)\}$ of index 3, isomorphic to V_4.

The quaternion group H_8 has the same kind of 3-fold symmetry given by the 3-cycle (i, j, k). I construct $\mathrm{BT}_{24} = \langle H_8, a\rangle$ as the abstract group obtained by adjoining a new generator a to H_8 such that

$$a^3 = -1 \quad \text{and} \quad aia^{-1} = j, \quad aja^{-1} = k, \quad aka^{-1} = i \tag{2.6}$$

(so conjugacy by a performs the 3-cycle (i, j, k)). A little calculation using these relations show that $H_8 \lhd \mathrm{BT}_{24}$ is a normal subgroup of index 3, with the left coset partition $\mathrm{BT}_{24} = H_8 \sqcup aH_8 \sqcup a^2H_8$. The group multiplication in BT_{24} is determined by that in H_8 and the relations (2.6). One checks as an extended exercise that there is a surjective group homomorphism $\mathrm{BT}_{24} \to A_4$ with kernel -1 defined by $i \mapsto (12)(34)$ and $a \mapsto (234)$, so that in (2.3), the double cover $H_8 \to V_4$ extends to a double cover $\mathrm{BT}_{24} \to A_4$. (For example, $a^3 \to e$ so that $-1 \mapsto e$. Then $aia^{-1} \mapsto (234)(12)(34)(243) = (13)(24)$, etc.)

2.3 H_8 and BT_{24} as matrix groups

So far these are abstract groups or permutation groups. I can also make H_8 into a matrix group $H_8 \subset \mathrm{GL}(2, \mathbb{C})$, by taking i, j, k to the matrices

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad K = IJ = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.7}$$

that satisfy the same relations (2.5). (They are called "Pauli matrices" in quantum mechanics.) As a matrix group, BT_{24} consists of H_8 together with the 16 matrices

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{with} \quad a, b = \frac{\pm 1 \pm i}{2}. \tag{2.8}$$

To see what is happening, consider

$$A = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}. \tag{2.9}$$

One checks that $A^2 = \frac{1}{2}\left(\begin{smallmatrix} -1+i & 1+i \\ -1+i & -1-i \end{smallmatrix}\right)$, $A^3 = -1$, and

$$\begin{gathered} AI = \frac{1}{2}\begin{pmatrix} -1+i & 1-i \\ -1-i & -1-i \end{pmatrix} = JA, \\ \text{and similarly } AJ = KA,\ AK = IA. \end{gathered} \tag{2.10}$$

Therefore A, I, J, K satisfy the same relations as a, i, j, k in (2.6). The matrix group in $\mathrm{GL}(2, \mathbb{C})$ generated by A, I, J, K contains H_8 as a normal subgroup (by the conjugacy (2.10)). Since the matrix A has order 6, its order must be divisible by 3, so ≥ 24. However, it is contained in $H_8 \cup AH_8 \cup A^2H_8$, which implies that it is of order ≤ 24. Therefore A, I, J, K generate a matrix group in $\mathrm{GL}(2, \mathbb{C})$ of order 24, containing H_8 and with AH_8, A^2H_8 as its cosets.

More discussion of these groups and their matrix models to follow…

2.4 KG-modules

Our object of study has many names, more or less synonymous: an *action* $G \times V \to V$ of G on a vector space V, a *representation* of G, a *homomorphism* $\rho\colon G \to \mathrm{GL}(V)$, a *matrix group* $G \subset \mathrm{GL}(n, K)$, a *$KG$-module* V. We will shortly have the result that a representation is determined up to isomorphism by its character, adding one more item to this list. I will try to be precise about which of these I mean, but I doubt if it possible for me to be consistent or for you to remember which of my conventions applies where. The material is very simple, and the ambiguities fairly harmless. Rather that worry about hair splitting precision, let's embrace diversity, and use whatever language is convenient.

Definition A *representation* of a group G on a vector space V over a field K is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$. For every $g \in G$, this specifies an invertible K-linear transformation $\rho(g) \in \mathrm{GL}(V)$, with the usual homomorphism axioms: $\rho(e) = \mathrm{Id}_V$ (the identity acts trivially) and $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ (multiplication in the abstract group G goes over to composition of maps in $\mathrm{GL}(V)$). When more than one representation is in play (say V, W, V_1 etc.), I write ρ_V, ρ_W, ρ_1 etc., to refer to their respective homomorphisms.

Almost always in this course, V is finite dimensional, and $\dim V$ is the dimension of the representation, also written $\dim \rho$. The choice of a basis makes $V = K^n$ and ρ is then a *matrix representation* $\rho\colon G \to \mathrm{GL}(n, K)$, taking each $g \in G$ to an $n \times n$ invertible matrix $\rho(g)$.

A *KG-module* is a K-vector space V together with a representation $\rho\colon G \to \mathrm{GL}(V)$. I often omit mention of ρ, writing simply $gv = \rho(g)(v)$ for the effect of the linear map $\rho(g)$ on vector $v \in V$.

Remark Faithful representations. I say that ρ or V is *faithful* if $\ker \rho = \{e\}$ or $G = \rho(G) \subset \mathrm{GL}(V)$. This seems fairly pointless at first sight, since $\ker \rho$ is a normal subgroup $H \lhd G$ and ρ induces an inclusion $\overline{G} = G/H \subset \mathrm{GL}(V)$.

However, at some points we are interested in the set of *all* representations of G (up to isomorphism), hoping to give it some structure. For example, the cyclic group $\boldsymbol{\mu}_r$ has 1-dimensional representations $\rho_a\colon \varepsilon \mapsto \varepsilon^a$ for $a \in \mathbb{Z}/r$. The faithful ones happen when a is coprime to r. But the set as a whole has the structure of cyclic group $\mathbb{Z}/r$.

Remark Group algebra KG. Let K be a field and G an abstract group. The *group algebra* KG is the K-vector space based by G (an element is a finite sum $A = \sum_{g\in G} a_g g$ with $a_g \in K$), made into an associative K-algebra by the K-bilinear multiplication $KG \times KG \to KG$ determined by $(g_1, g_2) \mapsto g_1 g_2$. This implies that the product of $A = \sum_{g\in G} a_g g$ and $B = \sum_{g\in G} b_g g$ is

$$AB = \sum_{g_1, g_2 \in G} a_{g_1} b_{g_2} g_1 g_2. \tag{2.11}$$

A module V over the algebra KG is a representation of G over K: the field K acts on V by scalar multiplication of vectors, and G acts by the given ρ. The term "KG-module" is an abbreviation used throughout the course, allowing me to omit mention of ρ. I have just explained its etymology, but the rest of the course avoids any further explicit reference to the algebra KG. Just remember that a KG-module is a set V that is a vector space over K with a specified action of G.

2.5 KG-submodules, homomorphisms, etc.

The immediate aim is to develop the theory of KG-modules, by analogy with vector spaces. We deal almost exclusively in finite dimensional representations. The *dimension* of a representation is the dimension of the vector space V (also called *rank* or *order*). Choosing a basis $\{e_1, \dots, e_n\}$ of a finite dimensional vector space V makes $V = K^n$, and expresses $\mathrm{GL}(V)$ as the group $\mathrm{GL}(n, K)$ of invertible $n \times n$ matrices (acting by left multiplication on column vectors $v = {}^t(x_1, \dots, x_n)$). If V is a KG-module, the map $\rho\colon G \to \mathrm{GL}(V)$ thus becomes a homomorphism $\rho\colon G \to \mathrm{GL}(n, K)$. It is a little exercise to see that two different bases of V lead to conjugate representations ρ (cf. Ex. 2.6).

Definition A *KG-homomorphism* $\alpha\colon V_1 \to V_2$ of two KG-modules V_1, V_2 is a K-linear map of vector spaces that *commutes* with the G-action, in the sense that

$$\alpha(gv) = g\alpha(v) \quad \text{for all } g \in G \text{ and } v \in V_1. \tag{2.12}$$

Write $\mathrm{Hom}_{KG}(V_1, V_2)$ for the set of KG-homomorphisms $\alpha\colon V_1 \to V_2$. I expand out the abbreviations: V_1, V_2 are KG-modules, so they have given homomorphisms $\rho_1\colon G \to \mathrm{GL}(V_1)$ and $\rho_2\colon G \to \mathrm{GL}(V_2)$ (implicit in the statement). On the left-hand side of (2.12), gv is $\rho_1(g)(v)$, the action of $g \in G$ on V_1 by ρ_1. The condition says that when I apply α to $\rho_1(g)(v)$, I get the same thing as first applying α to v, then making g act on the image $\alpha(v)$ by $\rho_2(g)$:

$$\begin{array}{ccc} V_1 & \xrightarrow{\alpha} & V_2 \\ \rho_1(g)\Big\downarrow & © & \Big\downarrow\rho_2(g) \\ V_1 & \xrightarrow{\alpha} & V_2 \end{array} \tag{2.13}$$

The condition (2.12) says that this diagram *commutes* (LaTeX `\copyright`). If you're seeing it for the first time, this means the composite $\alpha \circ \rho_1(g)$ going down then right equals the composite $\rho_2(g) \circ \alpha$ going right then down.

It is often useful to rewrite the condition as $\alpha(v) = \rho_2(g^{-1})\alpha\rho_1(g)v$, or even abbreviated to $\alpha = g^{-1}\alpha g$. In terms of the commutative diagram, this says that, for every $g \in G$, simply going across the top by α is the same K-linear map as going down by $\rho_1(g)$, across by α, and then back up by the inverse $\rho_2(g^{-1})$ of $\rho_2(g)$.

Definition A *KG-submodule* $U \subset V$ is a vector subspace taken to itself by every $\rho_V(g)$. That is $\rho_V(g)(u) \in U$ for every $u \in U$ and $g \in G$. We also refer to U as a *G-invariant subspace.*

Proposition *Given a KG-submodule $U \subset V$, the quotient vector space V/U has a unique structure of KG-module for which the quotient map $q\colon V \to V/U$ is a KG-homomorphism.*

Proof For $x \in V/U$, choose $v \in V$ for which $q(v) = x$. Since I require q to be a KG-homomorphism, $\rho_{V/U}(x) = q(\rho_V(v))$ is uniquely specified. It is well defined because if v' is a different choice then $q(v') = q(v)$, so $v - v' \in \ker q = U$ and then also $\rho_V(v) - \rho_V(v') \in U$, because U is invariant under the action of each $\rho_V(g)$, by the assumption that U is a KG-submodule. Then $\rho_{V/U}(g)$ is K-linear for each g, and $\rho_{V/U}$ is an action of G on V/U. Q.E.D.

Definition The direct sum of two KG-modules V_1 and V_2 is the vector space $V_1 \oplus V_2$ with the diagonal action of G. That is, g acts on $V_1 \oplus V_2$ by

$$\rho_{V_1 \oplus V_2}(g)(v_1, v_2) = \bigl(\rho_1(g)v_1, \rho_2(g)v_2\bigr). \tag{2.14}$$

As matrix representations, if $V_1 = K^{n_1}$ and $V_2 = K^{n_2}$ then $V_1 \oplus V_2 = K^{n_1+n_2}$ with the G-action $\rho_{V_1 \oplus V_2}\colon G \to \mathrm{GL}(n_1 + n_2, K)$ given in block matrix form by

$$\rho_{V_1 \oplus V_2}(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}. \tag{2.15}$$

The direct sum has inclusion maps $i_1\colon V_1 \hookrightarrow V_1 \oplus V_2$ and $i_2\colon V_2 \hookrightarrow V_1 \oplus V_2$ and projection maps $p_1\colon V_1 \oplus V_2 \twoheadrightarrow V_1$ and $p_2\colon V_1 \oplus V_2 \twoheadrightarrow V_2$, exactly as for vector spaces. These satisfy a string of identities of the type $p_1 \circ i_1 = \mathrm{Id}_{V_1}$ and $p_1 \circ i_2 = 0$.

Consider $E_1 = i_1 \circ p_1$. This project (v_1, v_2) to $v_1 \in V_1$ then includes v_1 back into $V_1 \hookrightarrow V_1 \oplus V_2$ as $(v_1, 0)$. In other works $E_1\colon V_1 \oplus V_2 \to V_1 \oplus V_2$ is the projection of the sum to the first factor. This is a *projector* or an *idempotent*: $E_1^2 = E_1$. If fact $E_1^2 = i_1 \circ p_1 \circ i_1 \circ p_1$ factors via $p_1 \circ i_1$, which is the identity of V_1. So we might as well cancel out the middle two factors, giving $E_1^2 = E_1$. Similarly for $E_2 = i_2 \circ p_2$; we have $\mathrm{Id}_V = E_1 + E_2$, where the two idempotent terms E_1, E_2 are the projections to two factors.

The main point of this discussion is Maschke's theorem 3.1. As a slogan: under appropriate conditions, a submodule is a direct summand. The next result provides the main step in the proof.

2.6 Average over G

Given KG-modules V_1 and V_2, the set of K-linear maps $\alpha\colon V_1 \to V_2$ is the vector space $\mathrm{Hom}_K(V_1, V_2)$. I rephrase what the commutativity condition (2.13) means for an element $\alpha \in \mathrm{Hom}_K(V_1, V_2)$: for each $g \in G$, the map $\rho_2(g)^{-1} \circ \alpha \circ \rho_1(g)$, abbreviated as $g^{-1}\alpha g$, goes down the left, across the bottom, then up the right of diagram (2.13). The condition is that $g^{-1}\alpha g = \alpha$ for all $g \in G$. There is no particular reason why it should hold for any-old $\alpha \in \mathrm{Hom}_K(V_1, V_2)$, but we can enforce it by *averaging over all* $g \in G$. The following key proposition assumes G is finite, and its order $|G|$ invertible in K.

Proposition *Let G be a finite group. For $\alpha \in \mathrm{Hom}_K(V_1, V_2)$, consider the* average

$$\beta = \frac{1}{|G|} \sum_{g \in G} g^{-1}\alpha g \in \mathrm{Hom}_K(V_1, V_2). \tag{2.16}$$

Then β is a KG-homomorphism, that is, $\beta \in \mathrm{Hom}_{KG}(V_1, V_2)$. In other words, $\beta(xv) = x\beta(v)$ holds for any $x \in G$ and any $v \in V_1$.

Proof This may look bewildering at first sight, but it is actually very simple once you see that the g in the sum (2.16) is just a dummy variable. Write out the value of $\beta(xv)$ as the sum

$$\beta(xv) = \frac{1}{|G|} \sum_{g \in G} g^{-1}\alpha g(xv). \tag{2.17}$$

Now $g(xv) = (gx)(v)$, and I can rewrite g^{-1} in terms of gx as $g^{-1} = x(gx)^{-1}$. This expresses each individual term in the sum as $x(gx)^{-1}\alpha(gx)v$. The sum over $g \in G$ is just the same thing as the sum over $gx \in G$, because g is a dummy variable, and gx runs through G as g runs through G. Therefore

$$\begin{aligned} \beta(xv) &= \frac{1}{|G|} \sum_{g \in G} x(gx)^{-1}\alpha(gx)(v) \\ &= \frac{1}{|G|} \sum_{g \in G} xg^{-1}\alpha g(v) \\ &= x\Big(\frac{1}{|G|} \sum_{g \in G} g^{-1}\alpha g(v)\Big) = x\beta(v). \quad \text{Q.E.D.} \end{aligned} \tag{2.18}$$

Addendum *Moreover, if α restricted to some KG-submodule $U \subset V_1$ is already a KG-module homomorphism then $\beta_{|U} = \alpha_{|U}$. In the same way, if $U \subset V_2$ is a KG-submodule with quotient $q\colon V_2 \to V_2/U$, and if $q \circ \alpha\colon V_1 \to V_2/U$ is already a KG-module homomorphism then $q \circ \beta = q \circ \alpha$.*

Proof For the addendum, in either case the average is the sum of $|G|$ equal terms, divided by $|G|$. Q.E.D.

2.7 Homework to Chapter 2

2.1. Jordan normal form Let $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right)$. If $e_1 = {}^t(1, 1)$ and $e_2 = {}^t(0, 1)$, calculate Me_1 and Me_2. Find the eigenvalues of M and its Jordan Normal Form.

Let $G = \mathbb{Z}^+ = \langle 1 \rangle$ be the infinite cyclic group and $\rho\colon G \to \mathrm{GL}(2,\mathbb{C})$ its representation defined by $1 \mapsto M$. Show that $\mathbb{C}e_1$ is a nontrivial submodule $\mathbb{C}^2$, but there is no G-invariant complementary subspace, to that $\mathbb{C}e_1$ is not a direct summand. Thus Maschke's theorem fails for infinite groups G.

2.2. Characteristic p Let K be a field of characteristic p, and $V = K^2$. Consider the map $\varphi\colon V \to V$ given by the matrix $M = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Show that $M^2 = \left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$ and $M^j = \left(\begin{smallmatrix}1&j\\0&1\end{smallmatrix}\right)$ for all j. In particular $M^p = \mathrm{Id}_V$ (because $p = 0$ in K), so that V gives a representation of $\mathbb{Z}/p$ on V.

As in Question 2.1, $\ker(M-1)$ is a 1-dimensional $\mathbb{Z}/p$-invariant subspace of V, but there is no $\mathbb{Z}/p$-invariant complementary subspace. Thus Maschke's theorem fails without the assumption $\frac{1}{|G|} \in K$.

2.3. Idempotent matrices Let $E_1 \in \mathrm{Mat}(n \times n, K)$ be a matrix that is idempotent, so satisfies $E_1^2 = E_1$. Determine the possible eigenvalues of E_1. Show that $E_2 = 1 - E_1$ is also idempotent. Show that $\ker E_1 = \operatorname{im} E_2$ and $\ker E_2 = \operatorname{im} E_1$, and that $V = V_1 \oplus V_2$ where $V_1 = \operatorname{im} E_1 = \ker E_2$ and $V_2 = \ker E_1 = \operatorname{im} E_2$. Write out the matrix representation of E_1 and E_2 in a basis made up of $e_1, \dots, e_m \in V_1$ and $e_{m+1}, \dots, e_n \in V_2$.

2.4. The binary dihedral groups BD_{4m} were defined in 2.2 as matrix groups in $\mathrm{GL}(2,\mathbb{C})$ generated by A, B of (2.1). For $m = 1$, show that $\mathrm{BD}_{4m} \cong \mathbb{Z}/4$.

Show that if m is even, a homomorphism $\mathrm{BD}_{4m} \to \mathbb{C}^\times$ takes each of A, B to $\{\pm 1\}$. Deduce that there are 4 different such homomorphisms, so that BD_{4m} has a surjective homomorphism to the 4-group V_4.

If m is odd, show that a homomorphism $\mathrm{BD}_{4m} \to \mathbb{C}^\times$ either does $A \mapsto 1$ and $B \mapsto \pm 1$ or $A \mapsto -1$ and $B \mapsto \pm i$ and that BD_{4m} has a surjective homomorphism to the cyclic group $\mathbb{Z}/4$.

2.5. Same abstract group BD_{4m}. For $j = 0, \dots, m$, show that there is a homomorphism $\rho_j\colon \mathrm{BD}_{4m} \to \mathrm{GL}(2,\mathbb{C})$ doing

$$A \mapsto \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix} \quad \text{and} \quad B \mapsto \begin{pmatrix} 0 & (-1)^j \\ 1 & 0 \end{pmatrix}. \tag{2.19}$$

[Hint. Check that these two matrices satisfy the relations of BD_{4m}.]

For even j, the centre $\{\pm\}$ of BD_{4m} is contained in $\ker \rho_j$, so that ρ_j factors via a representation of the dihedral group $D_{2m} = \mathrm{BD}_{4m}/\{\pm\}$.

2.6. Choice of basis and conjugacy A representation of G on V is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$. Let $e_1, \dots, e_n$ and $f_1, \dots, f_n$ be two bases of V, and T the matrix expressing the f_i in the basis $e_1, \dots, e_n$. Suppose that, as in 2.5, we use $e_1, \dots, e_n$ to express ρ as the matrix representation $\rho_e\colon G \to \mathrm{GL}(n, K)$, and likewise ρ_f in the basis $f_1, \dots, f_n$. Show that $\rho_f(g) = T\rho_e(g)T^{-1}$. [Hint: The e_i and f_i are column vectors. $\rho_e(g)$ is the matrix that writes $g(e_i)$ in the basis e_i, that is, $g(e_i) = \sum(\rho_e(g)_{ij})e_j$, etc. Now calculate how $T\rho_e(G)T^{-1}$ acts on $T(e_i)$.]

In other words, a change of basis in V leads to conjugate matrix representations. The tricky part here is to translate the abstract algebra into a meaningful calculation. Working with $\mathbb{C}G$-modules means that we can usually sweep this under the carpet.

2.7. Composition and inverse of KG-homorphisms If $\alpha\colon V_1 \to V_2$ and $\beta\colon V_2 \to V_3$ are KG-module homomorphisms, their composite $\beta \circ \alpha$ is again a KG-module homomorphism. [Hint: write out the commutative squares (2.13) for α and for β, and compose them to form a longer commutative rectangle for $\beta \circ \alpha$.]

If a KG-module homomorphism $\alpha\colon V_1 \to V_2$ is an isomorphism of vector spaces then its inverse α^{-1} is again a KG-module homomorphism.

2.8. In Proposition 2.6, if $\alpha\colon V_1 \to V_2$ was already $\mathbb{C}G$-linear, then $\beta = \alpha$.

2.9. Symmetries of the octahedron etc. as matrix groups Geometric considerations make it clear that the regular tetrahedron has S_4 symmetry permuting its 4 vertices. (Or we could take this to be the meaning of "regular".) When writing this out in coordinates in 3-space there are a couple of little traps to avoid, because the first coordinate systems you think of are not orthogonal.

The easy way is to start from the symmetries of the octahedron or the cube. First observe that the unit points $(\pm1, 0, 0)$, $(0, \pm1, 0)$, $(0, 0, \pm1)$ on the x, y, z coordinate axes are the 6 vertices of a regular octahedron. Permuting x, y, z gives an action of S_3 and multiplying any of x, y, z by ±1 gives an action of $(\mathbb{Z}/2)^3$. Putting these two together gives the group of order 48 containing the matrices

$$\begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm1 & 0 \\ \pm1 & 0 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \\ \pm1 & 0 & 0 \end{pmatrix}, \tag{2.20}$$

and similarly for the other permutation matrices $\left(\begin{smallmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{smallmatrix}\right)$. Each of these has determinant ±1. Restricting to the matrices with determinant $+1$ gives the subgroup $O_{24} \subset \mathrm{SO}(3)$ of order 24, that acts transitively on the 6 vertices and has 4 rotations fixing each vertex. This is the full group of rotational symmetries of the octahedron.

The regular cube $|x|, |y|, |z| \le 1$ in $\mathbb{R}^3$ has the 6 unit points as the centres of its faces. For example, it meets the plane $x = 1$ in the square with 4 vertices $1, \pm1, \pm1$ centred at $(1, 0, 0)$. The cube has the same symmetry group as the octahedron, so rotational symmetry $O_{24} \subset \mathrm{SO}(3)$.

2.10. The tetrahedron has rotational symmetry group $T_{12} \cong A_4$. Colour the vertices $\{\pm1, \pm1, \pm1\}$ of the cube alternately black and white, so that the black vertices (having evenly many minuses) are

$$P_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}. \tag{2.21}$$

These span a regular tetrahedron Δ. Show that the subgroup of $T_{12} \subset O_{24}$ that preserves the colours is the rotational group of symmetries of Δ, and acts on its vertices as the alternating group A_4.

[Hint: The matrices

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; N_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; N_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.22}$$

define an action of the Klein 4-group V_4 on Δ. Show that $T = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$ also acts on Δ, fixing P_0 and performing the 3-cycle $(1, 2, 3)$ on P_1, P_2, P_3. Also show that conjugacy by T performs a 3-cycle on the matrices (N_1, N_2, N_3).]

3 Irreducible representations

3.1 Maschke's theorem, first proofs

Let V be a KG-module. Recall from Chapter 2 that a KG-submodule $U \subset V$ is a K-vector subspace of V invariant under the action of G (that is, the action of any $g \in G$ takes U to itself).

Theorem (Maschke's theorem) *Let V be a KG-module (a representation of G over K). Assume V is finite dimensional over a field K, and G is a finite group whose order $|G|$ is invertible in K.*

Let $U \subset V$ be a KG-submodule of V. Then there exists a complementary KG-submodule $W \subset V$ such that $V = U \oplus W$.

Ex 2.1–2.2 gave counterexamples to the statement if G is infinite, or if $|G| = 0 \in K$ (that is, $\operatorname{char} K = p$ is a prime dividing $|G|$).

The corresponding result without a group action is a basic fact of linear algebra: given a subspace $U \subset V$ of a finite dimensional vector space V, there exists a complementary vector subspace $W_0 \subset V$, with $U \oplus W_0 = V$. The rules for $\oplus$ are that $V = U + W_0$ (that is, every vector $v \in V$ can be written $v = u + w$ with $u \in U$ and $w \in W_0$) and $U \cap W_0 = 0$, so that the $u \in U$ and $w \in W_0$ are unique. You remember the proof: first choose a basis $e_1, \dots, e_m$ for U, and complete it to a basis of V by adding $f_{m+1}, \dots, f_n$. Then $W_0 = \langle f_{m+1}, \dots, f_n \rangle$ is a complement.

First proof Start from a vector space complement W_0, so that $V = U \oplus W_0$ as vector space. I am not allowed to assume that the G action on V takes W_0 to itself, since that is exactly what I am trying to prove. The idea is to bully W_0 around a bit to make it G-invariant, by applying the averaging result Proposition 2.6 to appropriate inclusions and projections of the vector space direct sum $V = U \oplus W_0$. Roughly, "average out" W_0 to make it G-invariant (although that slogan doesn't actually mean anything yet).

The inclusion $i_1 \colon U \hookrightarrow V$ is a KG-module homomorphism, because $U \subset V$ is a KG-submodule. I use the chosen direct sum decomposition $V = U \oplus W_0$ to construct the first projection $p_1^0 \colon V \to U$. It does $(u, w) \mapsto u$, but it depends on the choice of W_0, and is not a priori a KG-module homomorphism. However, its restriction to U is a left inverse to i_1 (that is, it takes $(u, 0) \mapsto u$). Now the averaging procedure of 2.6 applied to p_1^0 gives

$$p_1 = \frac{1}{|G|} \sum_{g \in G} g^{-1} p_1^0 g \colon V \to U. \tag{3.1}$$

This is G-invariant by Proposition 2.6, so is a KG-homorphism. Moreover, each of the $|G|$ summands restricted to U is the identity. Thus p_1 is again a left inverse of i_1 by the Addendum to Proposition 2.6. Thus $W = \ker p_1$ is still a vector space complement of U, and is a KG-submodule of V such that $V = U \oplus W$. Q.E.D.

Second proof This is a rather similar variant, working with the quotient map $q\colon V \twoheadrightarrow V/U$ rather than the inclusion i_1. Given the vector space direct sum $V = U \oplus W_0$, the issue is to adjust $W_0 \subset V$ to be a KG-submodule. Now $U \subset V$ is a KG-submodule, so the quotient vector space V/U has a unique structure of KG-module for which the quotient map $q\colon V \twoheadrightarrow V/U$ is a KG-module homomorphism (see Proposition 2.4). Restricting q to W_0 gives an isomorphism of vector spaces $p\colon W_0 \to V/U$. Now consider the composite $j^0 = i_2 \circ p^{-1}\colon V/U \to V$ obtained as the inverse isomorphism p^{-1} followed by the inclusion of W_0 in the direct sum $U \oplus W_0 = V$. This j^0 is a K-linear map between the two KG-modules V/U and V, and is a right inverse of the quotient map $q\colon V \to V/U$ (meaning that $q \circ j^0 = \mathrm{Id}_{V/U}$), "lifting" V/U to $W_0 \subset V$.

The averaging procedure of 2.6 applied to j^0 now gives

$$j = \frac{1}{|G|} \sum_{g \in G} g^{-1} j^0 g\colon V/U \to V. \tag{3.2}$$

Proposition 2.6 ensures that j is G-invariant, so its image is a KG-submodule $W \subset V$. Moreover, each of the $|G|$ terms of the summand is a right inverse of the quotient map $q\colon V \to V/U$, so that j is again a right inverse of q by the Addendum to Proposition 2.6, and $V = U \oplus W$. Q.E.D.

3.2 Irreducible representations

Definition A KG-module V (or the corresponding representation $\rho\colon G \to \mathrm{GL}(V)$) is *irreducible* if $V \neq 0$ and V does not contain any nontrivial KG-submodule. Or in other words, if $U \subset V$ is a KG-submodule, then either $U = 0$ or $U = V$.

Corollary *Every finite dimensional KG-module V can be written as a direct sum of irreducible KG-modules, that is $V = \bigoplus V_i$ with $V_i \subset V$ irreducible.*

Proof Indeed, if V is irreducible, there is nothing to prove. Otherwise, it has a nontrivial KG-submodule $U \subset V$, so that by Maschke's theorem $V = U \oplus W$, with U and W of strictly smaller dimension. By induction, each of U and W is a direct sum of irreducible KG-submodules, and so is V. The induction starts with $\dim V = 1$, since a 1-dimensional K-vector space does not have any nontrivial vector subspace.

Think of the analogy with prime factorisation of integers. A prime p is defined by the condition that it is not a unit, and its only divisors are 1 and p itself. Then every integer is a product of primes. We will shortly also have the uniqueness in the expression $V = \bigoplus V_i$ of the corollary.

Remark A 1-dimensional representation V is automatically irreducible. If V is a 2-dimensional representation of G then the only way that can V fail to be irreducible is that the operators $\rho_V(g)$ for all $g \in G$ have a common eigenvector in V. For a nontrivial KG-submodule $U \subset V$ can only be 1-dimensional, and then each $\rho_V(g)$ is a linear map taking U to itself, so any $v \in U$ is an eigenvector of $\rho_V(g)$. A similar argument works if V is 3-dimensional, because any splitting $V = V_1 \oplus V_2$ must have one of $\dim V_1$ or $\dim V_2 = 1$.

3.3 Schur's lemma

I defined homomorphism of KG-module in 2.4. To say that a KG-module homomorphism $\varphi\colon V_1 \to V_2$ is injective, surjective or bijective just means that the corresponding map of sets is. However, these are K-linear maps of finite dimensional vector spaces, and we know from linear algebra that a K-linear map $\varphi\colon V_1 \to V_2$ is injective if and only if it has a K-linear left inverse $p\colon V_2 \to V_1$ with $p \circ \varphi = \mathrm{Id}_{V_1}$; Maschke's theorem guarantees the same with p a KG-module homomorphism. In the same way, φ is surjective if and only if it has a K-linear right inverse $s\colon V_2 \to V_1$ such that $\varphi \circ s = \mathrm{Id}_{V_2}$. (This restates the set theoretic properties "injective" and "surjective" in terms of KG-module homomorphisms, and thus as properties of φ in the category of KG-modules. I am using the word *category* here in the purely informal sense of "all KG-modules and KG-module homomorphisms"; this course does not need category theory.)

Finally $\varphi\colon V_1 \to V_2$ is bijective if and only if it is an *isomorphism of KG-modules*: that is, there exists a KG-module homomorphism $\psi\colon V_2 \to V_1$ that is a 2-sided inverse $\psi = \varphi^{-1}$ with $\psi \circ \varphi = \mathrm{Id}_{V_1}$ and $\varphi \circ \psi = \mathrm{Id}_{V_2}$.

The small points involved in proving these assertions were discussed in Ex. 2.7.

Theorem (Schur's lemma) *(I) Any KG-homomorphism $\varphi\colon V_1 \to V_2$ of irreducible KG-modules V_1 and V_2 is either 0 or an isomorphism.*

(II) Suppose that K is algebraically closed, and let V be a finite dimensional irreducible KG-module. Then any KG-module homomorphism $\varphi\colon V \to V$ is a scalar multiple of the identity.

Proof (I) Set $U = \ker\varphi \subset V_1$. This is a KG-submodule, so by the irreducible assumption, either $U = V_1$ and $\varphi = 0$, or $U = 0$ and φ is injective. Arguing in the same way on $W = \operatorname{im}\varphi \subset V_2$ gives that either $\varphi = 0$ or $W = V_2$ so that φ is surjective. This proves (I).

(II) Now suppose $V_1 = V_2 = V$. Write M for the $n \times n$ matrix representing φ in some basis of V. The characteristic polynomial $P_M(t)$ of M is defined by $P_M(t) = \det(M - t\,\mathrm{Id}_n)$; clearly $(-1)^n P_M(t) \in K[t]$ is a monic polynomial of degree $n \geq 1$.

I assume that K is algebraically closed, so P_M has at least one root $\lambda \in K$, for which $\det(M - \lambda\,\mathrm{Id}_n) = 0$. Now $\varphi - \lambda\,\mathrm{Id}_V\colon V \to V$ is a KG-module homomorphism, but it cannot be an isomorphism because it is represented by the singular matrix $M - \lambda\,\mathrm{Id}_n$. Thus (I) implies $\varphi = \lambda\,\mathrm{Id}_V$. Q.E.D.

On the one hand, the result is a tautology. On the other hand, it is a subtle and extremely powerful result that is a vital ingredient in the whole of the rest of the course. This is the glory of abstract algebra.

Corollary *(a) Assume that K is algebraically closed. Suppose that a KG-module V has two different expressions*

$$V \cong \bigoplus_{i=1}^{n} U_i \quad \textit{and} \quad V \cong \bigoplus_{j=1}^{m} W_j \tag{3.3}$$

as a direct sum of irreducible KG-modules. Then there is bijection between the indices $i \mapsto i' = j$ so that $U_i \cong W_{i'}$.

(b) Quite generally, let $U_1, \dots, U_k$ be irreducible KG-modules with no two isomorphic, and suppose

$$V_1 = \bigoplus a_i U_i \quad \textit{and} \quad V_2 = \bigoplus b_i U_i \tag{3.4}$$

are KG-modules that are obtained as direct sums of the U_i. (I write $aU = U^{\oplus a} = \bigoplus_{i=1}^{a} U$ for the direct sum of a copies of U.) Then a KG-module homomorphism $\varphi\colon V_1 \to V_2$ takes the summand $a_i U_i \subset V_1$ to the summand $b_i U_i \subset V_2$ for each i. Between each summand $U_i \subset V_1$ and $U_i \subset V_2$ it is a scalar multiple of the identity, so between $a_i U_i \subset V_1$ and $b_j U_i \subset V_2$ it is given by an $a_i \times b_i$ matrix. The whole of φ is given by a matrix with these as diagonal blocks.

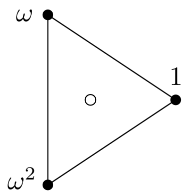
Proof (a) Write $\varphi\colon \bigoplus U_i \cong V \cong \bigoplus W_j$ for the composite of the isomorphisms in (3.3). If $V = 0$ there is nothing to prove. Otherwise there is at least one irreducible factor U_1 (say). Renumber the U_i so that $U_1, \dots, U_a$ are isomorphic, and $U_i \not\cong U_1$ for $i > a$. By Schur's Lemma (I), these first a summands only have nonzero maps to the irreducible summand W_j isomorphic to the same U_1. However, the restriction of the isomorphism φ to their sum $\bigoplus_{i=1}^{a} U_i$ must be injective, and so by dimension, the decomposition on the right has at least a such summands. The same applies to all the summands U_i, which proves (a).

(b) Restrict φ to each summand $a_i U_i \subset V_1$. By Schur's lemma (I), φ can only be nonzero when it takes each copy of U_i to the corresponding summand $\bigoplus b_i U_i \subset V_2$. It is a scalar multiple of the identity between each pair of summands U_i in V_1 and V_2, and we can represent this as an $a_i \times b_i$ matrix. Q.E.D.

3.4 Some matrix groups and irreducible representations

I describe some irreducible representations for S_3, then some for all the nice groups introduced in 2.2. At present I only have a few ad hoc methods to work with, and I am not in a position to prove that the lists of irreducible representations I give are complete. I write out some of the character tables without explanation. The character table lists the conjugacy classes in G, and the trace of each conjugacy class on the different irreducible representations. It will be discussed in detail in what follows.

The symmetric group S_3 I view S_3 as the symmetry group of the regular triangle in $\mathbb{C}$ with vertices $\boldsymbol{\mu}_3 = \{1, \omega, \omega^2\}$, generated by rotation through $\frac{2\pi}{3}$ and reflection $(x, y) \mapsto (x, -y)$:



$$a = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(cf. Ex. 1.2).

By this construction $S_3 = \langle a, b \rangle$ is a matrix group in $\mathrm{GL}(2, \mathbb{R}) \subset \mathrm{GL}(2, \mathbb{C})$, so it has a *given representation* on $V = \mathbb{C}^2$. I claim that *this representation is irreducible over* $\mathbb{C}$. In fact, a nontrivial $\mathbb{C}G$-submodule $U \subset V$ must be 1-dimensional, so a simultaneous eigenspace of a and b. But the eigenspaces of b are the x and y-axes, whereas by Ex. 1.1 those of a are $\mathbb{C} \cdot (1, \omega)$ and $\mathbb{C} \cdot (1, \omega^2)$.

Next, S_3 has two 1-dimensional representations $S_3 \to \mathbb{C}^\times$, namely the trivial 1-dimensional representation $S_3 \to 1$ and the sign representation $S_3 \to \{\pm 1\}$ (that does $a \mapsto 1$, $b \mapsto -1$). The character table of S_3 is

S_3	e	a	b
L_1	1	1	1
L_{-1}	1	1	-1
V	2	-1	0

For example, $V = \mathbb{C}^2$. Its identity map has trace 2; the matrix a has trace $2\cos\frac{2\pi}{3} = \omega + \omega^2 = -1$; and b has trace 0.

The binary dihedral group BD_{4m} See 2.2 for $\mathrm{BD}_{4m} = \langle A, B \rangle$. In Ex. 2.4 you have already studied its four 1-dimensional representations: for even m, these are given by $(A, B) \mapsto \{\pm 1\}$. Ex. 2.5 constructed 2-dimensional representations ρ_j for $j = 0, \dots, m$. For $j = 0$ or $j = m$, A acts by a diagonal scalar matrix ± 1, and it follows that ρ_0 and ρ_m split into 1-dimensional representations. On the other hand, for $j = 1, \dots, m-1$ the representations ρ_j are irreducible: a 1-dimensional invariant subspace $U \subset V_j$ would be an eigenspace of both $\rho_j(A)$ and $\rho_j(B)$, which is impossible: $\rho_j(A)$ has eigenvectors ${}^t(1, 0)$ and ${}^t(0, 1)$, where $\rho_j(B)$ has eigenvectors ${}^t(1, \pm 1)$ or ${}^t(1, \pm i)$, according as j is even or odd. The character table for BD_{12} is

BD_{12}	1	-1	A	A^4	B	B^3
L_1	1	1	1	1	1	1
L_2	1	1	1	1	-1	-1
L_3	1	-1	-1	1	i	$-i$
L_4	1	-1	-1	1	$-i$	i
V_1	2	-2	1	-1	0	0
V_2	2	2	-1	-1	0	0

The alternating group A_4 We saw in 2.2 that A_4 has a surjective homomorphism to the cyclic group $\mathbb{Z}/3$ with kernel $V_4 \lhd A_4$, arising from its permutation action on the 3 pairings $[12, 34]$ etc. This gives 3 different 1-dimensional representations that take (say) $(234) \mapsto 1$ or ω or ω^2.

On the other hand, the description of A_4 as the rotations of the regular tetrahedron in Ex. 2.8 expresses A_4 as a matrix group in $\mathrm{GL}(3, \mathbb{R}) \subset \mathrm{GL}(3, \mathbb{C})$. Write V_3 for this. The subgroup V_4 acts by the matrices N_1, N_2, N_3 of Ex. 2.8, whereas the 3-cycles (234) and (243) act by the permutation matrices T and T^2.

I claim that V_3 is an irreducible representation of A_4 on $\mathbb{C}^3$. For if it had a nontrivial A_4-invariant subspace, it would have split as a direct sum by Maschke's theorem. One of the summands would have dimension 1, and would be a simultaneous eigenspace of every $g \in A_4$. But the matrices N_i only have eigenspaces contained in coordinate hyperplanes $x_j = 0$, whereas the matrix T has the 3 eigenvectors ${}^t(1, 1, 1)$, ${}^t(1, \omega, \omega^2)$ and ${}^t(1, \omega^2, \omega)$. Therefore V_3 cannot be split, so is irreducible.

The character table for A_4 is

A_4	e	$(12)(34)$	(234)	(243)
L_1	1	1	1	1
L_ω	1	1	ω	ω^2
L_ω^2	1	1	ω^2	ω
V_3	3	-1	0	0

The binary tetrahedral group BT_{24} As I said in 2.3, BT_{24} can be viewed as the matrix group in $\mathrm{GL}(2, \mathbb{C})$ consisting of the 8 matrices of H_8 together with the 16 matrices of the form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with $a, b \in \{\frac{\pm 1 \pm i}{2}\}$. Notice that $\frac{\pm 1 + \pm i}{\sqrt{2}}$ runs through $\exp(\theta i)$ with $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}]$ so the product of any two entries of these matrices is $\pm\frac{1}{2}$ or $\pm\frac{i}{2}$, and the set of 24 matrices is closed under multiplication.

I use a couple of short-cuts to describe the irreducible representations of BT_{24}. First, from 2.2 we know that $\mathrm{BT}_{24} / \{\pm 1\} = A_4$ (the group of rotations of the tetrahedron). Therefore BT_{24} inherits the representations of A_4, giving one 3-dimensional irreducible representation and three 1-dimensional representations.

The three one dimensional representations are $\rho_1, \rho_\omega, \rho_{\omega^2} \colon A_4 \to \mathbb{C}^\times$, and take (234) and its conjugates to the *scalars* $1, \omega, \omega^2$. These are also the key to the remaining representations of BT_{24}. Namely, when we view BT_{24} as a matrix group in $\mathrm{GL}(2, \mathbb{C})$, it has a *given* representation $V = \mathbb{C}^2$. We can multiply V by the scalar representations $\rho_1, \rho_\omega, \rho_{\omega^2}$ to get $V = V_1$, $V_\omega = \rho_\omega \cdot \rho_V$ and $V_{\omega^2} = \rho_{\omega^2} \cdot \rho_V$ We will see later that this gives all the irreducible representations of BT_{24} up to isomorphism.

Since we know these irreducible representations of BT_{24} via explicit matrices, it is easy enough to calculate their characters evaluated on the conjugacy classes. This gives the character table of BT_{24} (of course, so far without proof, and without the correct setup). Or see Ex. 3.10 for how to do it by computer algebra. I will treat this all later in more detail once all the definitions and properties of characters and character tables are in place.

3.5 Homework to Chapter 3

3.1. Determinant Let $\rho\colon G \to \mathrm{GL}(2,\mathbb{C})$ be a 2-dimensional representation of a finite group G. Prove that $g \mapsto \det(g)$ is a 1-dimensional representation.

3.2. Square Let $\rho\colon G \to \mathrm{GL}(2,\mathbb{C})$ be a 2-dimensional representation of a finite Abelian group G. Prove that $g \mapsto (\rho(g))^2$ (square of matrix $\rho(g)$) is again a representation of G.

3.3. Symmetric square Let $\rho\colon G \to \mathrm{GL}(2,\mathbb{C})$ be a 2-dimensional representation of a finite group, and write $\rho(g) = \left(\begin{smallmatrix} a_g & b_g \\ c_g & d_g \end{smallmatrix}\right)$. Prove that $\sigma = \mathrm{Sym}^2\rho\colon G \to \mathrm{GL}(3,\mathbb{C})$ defined by

$$g \mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

is a 3-dimensional representation of G. [Hint. You can do this by brute force. Alternatively, consider G acting as $(x,y) \mapsto (ax+by, cx+dy)$, and figure out how G then acts on $\mathrm{Sym}^2(x,y) = (x^2, xy, y^2)$.]

3.4 Find a counterexample to the assertion of Ex. 3.2 for a representation of a non-Abelian group.

3.5. KG-homorphism and G-invariant subspaces Show that the image of an injective KG-homorphism is a G-invariant subspace $U \subset V$ and vice versa.

3.6. Schur's lemma (II) and algebraic closure Give an example of a 2-dimension irreducible representation V_1 of $\mathbb{Z}/r$ over $\mathbb{R}$. [Hint: Redo Ex. 1.1.] Give an example of a group G and a homomorphism $\varphi\colon V_1 \to V_2$ between irreducible $\mathbb{R}G$-modules that is not a scalar multiple of the identity. Thus Schur's lemma (II) fails without the assumption that K is algebraically closed.

3.7. 1-dimensional representations of S_n and A_n Show that a group homomorphism $S_n \to \mathbb{C}^\times$ has image contained in ± 1. For $n \geq 2$, describe the two 1-dimension representations of S_n. For $n \geq 5$, show that A_n only has the trivial 1-dimensional representation.

3.8. Character table of A_4 You either know, or can quickly ascertain, that the alternating group A_4 has 4 conjugacy classes represented by e, $(12)(34)$ (with 3 elements), (234) and (243) (with 4 elements each). In 3.4, I gave you the three 1-dimensional representations coming from $A_4 \to \mathbb{Z}/3$; each element of A_4 acts there by multiplying by one of $1, \omega, \omega^2$. Check the top 3 lines of the character table given in 3.4.

In the 3-dimensional irreducible representation oof A_4 (rotations of the tetrahedron), (12)(34) and his buddies act by the matrices N_i, whereas (234) and (243) act by the permutation matrices T and T^2 (see Ex. 2.8 for these matrices). Calculate their traces and check the final line.

3.9. BD_{48} **by computer** Go to the Magma calculator website at

`http://magma.maths.usyd.edu.au/calc`

You can get a specification of each command by clicking Documentation + Handbook and searching for it.

Type out (or copy and paste) a few of the following lines in the input window and press Submit.

```
K<ep> := CyclotomicField(24);
GL := GeneralLinearGroup(2,K);
A := elt< GL | ep, 0, 0, ep^-1 >;
B := elt< GL | 0,1,-1,0 >;
// I check that A and B satisfy the relations of BD48.
Order(A); Order(B); A^12 eq B^2; A*B*A eq B;
BD := sub< GL | [A,B] >; // matrix group gen by A and B
Order(BD);
BD;
ConjugacyClasses(BD); // There are 14 conjugacy classes.
// two of them have 12 elements. Ask for all g in BD conjugate
// to specimen elt of ConjugacyClass(BD)[5].
[g : g in BD | IsConjugate(BD, g, $1[5,3])];
CharacterTable(BD);
```

3.10. More of the same Try running the Magma routines of

`https://homepages.warwick.ac.uk/staff/Miles.Reid/MA3E1`

4 Preparing for characters

4.1 Inner products and 3rd proof of Maschke's theorem

You learn in first year linear algebra that you can find a complement to a vector subspace $U \subset V$ by choosing a complementary basis. Meanwhile, in the applied courses, given a plane in $\mathbb{R}^3$ you take the orthogonal line, and vice versa, so that the complement is given to you for free, or more precisely, by the dot product of vectors.

Here I work over $\mathbb{C}$. Let $V = \mathbb{C}^n$ be a finite dimensional $\mathbb{C}$-vector space with a given basis. A *Hermitian inner product* on V is a map $\Phi\colon V \times V \to \mathbb{C}$ with the properties:

(i) Φ is $\mathbb{C}$-linear in the second factor: for all $v, w_i \in V$ and $\lambda_i \in \mathbb{C}$,

$$\Phi(v, \lambda_1 w_1 + \lambda_2 w_2) = \Phi(v, \lambda_1 w_1) + \Phi(v, \lambda_2 w_2).$$

(ii) Φ is Hermitian symmetric: $\Phi(v, w) = \overline{\Phi(w, v)}$ (complex conjugate).

(iii) Φ is positive definite: $\Phi(v, v) > 0$ for all $v \in V$.

The conjugate symmetry assumption (ii) implies that $\Phi(v, v) \in \mathbb{R}$, whereas the associated quadratic form of a symmetric $\mathbb{C}$-bilinear form is never real-valued, so positive definite would not make sense. (ii) breaks up as the two statements that the real part of Φ is symmetric and the imaginary part is skew. (ii) implies of course that Φ is complex conjugate linear in the first factor: $\Phi(\lambda_1 v_1 + \lambda_2 v_2, w) = \Phi(\overline{\lambda}_1 v_1, w) + \Phi(\overline{\lambda}_2 v_2, w)$. A basis $e_1, \dots, e_n$ of V over $\mathbb{C}$ makes $V = \mathbb{C}^n$; if I write $z_1, \dots, z_n$ for the coordinates then

$$\Phi((y_1, \dots, y_n), (z_1, \dots, z_n)) = \sum_i \overline{y}_i z_i \tag{4.1}$$

is a Hermitian inner product, positive because $\Phi(z, z) = \sum_i \overline{z}_i z_i = \sum |z_i|^2$.

Let V be a finite dimensional $\mathbb{C}$-vector space with Hermitian inner product Φ. The *orthogonal complement* $U^\perp$ of a $\mathbb{C}$-subspace $U \subset V$ is defined by

$$U^\perp = \bigl\{ w \in V \bigm| \Phi(u, w) = 0 \text{ for all } u \in U \bigr\}. \tag{4.2}$$

Lemma *$U^\perp$ is a $\mathbb{C}$-vector space complementary to U, that is $V = U \oplus U^\perp$.*

First, it is a $\mathbb{C}$-vector subspace, because the conditions on w in (4.2) are $\mathbb{C}$-linear. Next, $U \cap U^\perp = 0$ because any u in the intersection must have $\Phi(u, u) = 0$, and Φ is positive definite. Finally the traditional linear algebra argument: for a basis $e_1, \dots, e_m \in U$ and for any $v \in V$ set

$$a_i = \frac{\Phi(e_i, v)}{\Phi(e_i, e_i)} \quad \text{for } i = 1, \dots, m. \tag{4.3}$$

Then setting $u = \sum a_i e_i \in U$ and $w = v - u$ gives $\Phi(e_i, w) = 0$ for all i, so that $w \in U^\perp$, and $v = u + w$. Q.E.D.

Proposition *A finite dimensional $\mathbb{C}G$-module V has a G-*invariant *Hermitian inner product Φ, with $\Phi(gv, gw) = \Phi(v, w)$ for all $g \in W$.*

Proof Start with any Hermitian inner product Φ_0 as just described. Define its *average* over G to be $\Phi(v,w) = \frac{1}{|G|}\sum_{g\in G}\Phi_0(gv,gw)$. Then Φ is G-invariant, because for any $x\in G$

$$\begin{aligned}\Phi(xv,xw) &= \frac{1}{|G|}\sum_{g\in G}\Phi_0(gxv,gxw)\\ &= \frac{1}{|G|}\sum_{gx\in G}\Phi_0(gxv,gxw) = \frac{1}{|G|}\sum_{g\in G}\Phi_0(gv,gw)\end{aligned} \tag{4.4}$$

(changing the dummy index in the sum from gx to g). Each term in the sum is positive definite. Q.E.D.

Proposition (Third proof of Maschke's theorem) *Let V be a finite dimensional $\mathbb{C}G$-module with a G-invariant Hermitian product Φ. If $U\subset V$ is a $\mathbb{C}G$-submodule then its orthogonal complement $U^\perp$ is also a $\mathbb{C}G$-submodule.*

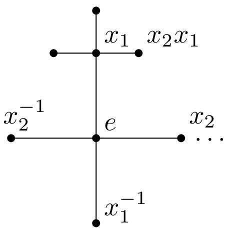
The only point is that U is a G-invariant subset of V, so the conditions defining $U^\perp$ are G-invariant: $\Phi(u,v)=0$ for all $u\in U$ gives also $\Phi(gu,gv)=0$, so that $gv\in U^\perp$. Q.E.D.

4.2 Finitely presented group

This section and the next discuss some basic results from group theory. Up to now, we have mainly had a few examples of matrix groups. A finitely presented group is something of the form $F(m)/R$ where $F(m)$ is the free group on m generators and R is the normal subgroup generated by a number of relations (see below for more details). For example, the dihedral group $D_{2m} = \left\langle a,b \;\middle|\; a^m = b^2 = e, bab^{-1} = a^{-1}\right\rangle$. The good thing in this case, as we have seen, is that the presentation by generators and relations gives us in short order a list of the $2m$ elements of D_{2m} and a recipe to multiply any two of them. So the presentation is *really useful*.

The *free group* $F(m)$ on m generators $x_1,\dots,x_m$ is made as follows: consider the alphabet of $2m$ letters $\{x_i, x_i^{-1} \mid i=1,\dots,m\}$ and all possible words in them. Here a *word* w of length n is a concatenation $z_1z_2\cdots z_n$ of the given alphabet. We say that w is *reduced* if no two consecutive letters are inverse, that is, w does not contain any occurrence of $x_ix_i^{-1}$ or $x_i^{-1}x_i$. Given any word, we can reduce it by cancellation, and this reduction is unique, as can be checked. In this set up, the identity element is the empty word e, the inverse of $w=z_1z_2\cdots z_n$ is $z_n^{-1}\cdots z_2^{-1}z_1^{-1}$, and the group multiplication is concatenation of words followed by reduction. We draw the elements of $F(m)$ as the infinite graph:

Γ_{2m}:



(4.5)

(with $2m-1$ new edges out of every vertex, that is, $2m(2m-1)^{n-1}$ reduced words of length n for $n \geq 1$).

Given a list $\{r_1, \dots, r_k\}$ of words, we write R for the normal subgroup generated by $r_1, \dots, r_k$, that is, all words in the r_i, and all their conjugates by elements of $F(m)$. This is a normal subgroup, and the quotient group by R is

$$F(m)/R = \left\langle x_1, \dots, x_m \mid r_1, \dots, r_k \right\rangle. \tag{4.6}$$

This is called a *finitely presented group.*

Think of this as an artificial and in general a pretty wretched way of defining a finite group. Even for an easy well known group, using the given generators and relations to list its elements and determine the group law on them may be enormously complicated and unwieldy. Every finite group and many infinite ones can be given in this way, but one can make very bad choices of generators and relations. By a kind of translation of Gödel's incompleteness theorem, it is known that in general "the word problem is insoluble", that is, for general R there does not exist an algorithm that can establish whether two words in the generators are equal in $F(m)/R$, or whether $F(m)/R$ is finite. We will see by example that this can be a big headache. There are a limited number of cases where the method is extremely useful, a small number of boundary cases that are famous as difficult puzzles, but beyond that, the method just leads to intractables computations. (Those interested should try the 4th year module MA467 Presentations of Groups.)

4.3 Conjugacy and centraliser

Conjugacy classes Elements $g_1, g_2 \in G$ are *conjugate* in G if $g_2 = xg_1x^{-1}$ for some $x \in G$. Conjugacy is an equivalence relation; the equivalence classes are called *conjugacy classes.* These play a starring role in character theory: the character table of G has columns labelled by its conjugacy classes.

Conjugacy classes are orbits of the action of G on itself by conjugacy: in other words, let $x \in G$ act on the set G by $(x, g) \mapsto xgx^{-1}$. The orbit of g under this action is the conjugacy class $C(g) = C(g, G)$ of g, the set of its conjugates; the set of elements $x \in G$ that fix g (the *stabiliser* of g) is the *centraliser* $Z(g) = Z(g, G)$ subgroup of g in G. That is, $Z(g) = \{x \in G \mid xg = gx\}$. By the orbit-stabiliser theorem, the conjugacy class of g is the set of cosets $Z(g)\backslash G$.

When G is contained in a bigger group (say a finite group contained in a matrix group $G \subset \mathrm{GL}(V)$) we must distinguish conjugacy inside G from conjugacy in the bigger group. For example, the 3-cycles (234) and (243) are conjugate in S_4, but not in A_4.

Example: $\mathrm{GL}(n, \mathbb{C})$ **and JNF** Two elements in $\mathrm{GL}(n, K)$ are conjugate (essentially by definition) if and if they are given by the same matrix in suitable bases of K^n. In more detail, view $\mathrm{GL}(n, K)$ as the nonsingular linear transformations of K^n to itself. Here we map the space *to itself*, so we are not free to change basis in the domain and in the target any-old-how: to get meaningful expressions, we have to change basis in the domain and then change back in the target.

Given $A, T \in \mathrm{GL}(n, K)$, we can view T as a change of basis in K^n (taking the standard basis e_i to Te_i, the columns of T). Then

$$TAT^{-1}(Tv) = TA(v), \quad \text{that is,} \quad TAT^{-1}\colon Tv \mapsto TAv,$$

which is just A written in the new basis Te_i.

Over $\mathbb{C}$, every matrix M viewed as an endomorphism $\mathbb{C}^n \to \mathbb{C}^n$ has a Jordan Normal Form. This means that there is a change of basis so that M in the new basis, that is, TMT^{-1} is the map given by that standard shaped matrix. Therefore every matrix in $\mathrm{GL}(n, \mathbb{C})$ is conjugate in $\mathrm{GL}(n, \mathbb{C})$ to its JNF. Moreover two elements are conjugate in $\mathrm{GL}(n, \mathbb{C})$ if and only if they have the same JNF (that is, the same eigenvalues and same size Jordan blocks).

Example: Euclidean motions For those of you who did M243 Geometry: the classification of Euclidean motions of the plane is mostly about conjugacy classes in Eucl(2). For example, all rotations by $\pm\theta$ (about any point P) are conjugate, and two translations by vectors v_1 and v_2 are conjugate if and only if $|v_1| = |v_2|$.

Example: permutations Work in $\mathrm{Sym}(\Sigma)$, the permutations of a set Σ of n elements. Choosing an ordering of Σ as $\{\sigma_1, \ldots, \sigma_n\}$, or simply as $\{1, \ldots, n\}$ makes $g \in \mathrm{Sym}(\Sigma)$ into an element of S_n. If we choose a different ordering, we get two different expressions for one and the same g as permutations of $\{1, \ldots, n\}$. These two expressions are not identical, but they are conjugate.

Any permutation is a product of disjoint cycles. This means something like $(\sigma_1, \ldots, \sigma_k)(\sigma_{k+1}, \ldots, \sigma_l) \ldots$ (etc.), where all the σ_i are distinct. Two disjoint cycles are conjugate if and only if the cycles have the same number of elements.

For example, for any $n \geq 5$, the symmetric group S_n has a single conjugacy class $(\sigma_1, \sigma_2)(\sigma_3, \sigma_4, \sigma_5)$ having 2 disjoint cycles of length 2 and 3. There are $\binom{n}{2} \times \binom{n-2}{3} \times 2$ permutations in this class. The $\times 2$ is because $(\sigma_3, \sigma_4, \sigma_5)$ and $(\sigma_3, \sigma_5, \sigma_4)$ are conjugate; each subset of 3 elements has 2 cyclic orderings.

The conjugacy classes for S_6 are

$$\Big\{e,\ (12),\ (123),\ (1234),\ (12345),\ (123456), (12)(34),\ (12)(345),\\ (12)(3456),\ (123)(456),\ (12)(34)(56)\Big\}. \qquad (4.7)$$

The number of elements in each conjugacy class are $e = 1$, $(12) = 15$, $(123) = 40 = 2 \times \binom{6}{3}$, $(1234) = 90 = 6 \times \binom{6}{4}$, $(12345) = 144$ and so on, adding to 720. See Ex. 4.10–12.

4.4 Commutators $[g_1, g_2]$ and Abelianisation G^{Ab}

The *commutator* of two elements $g_1, g_2 \in G$ is $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. The set of all commutators generate a normal subgroup $[G, G] \lhd G$, the *commutator subgroup*, sometimes called *derived group* or G' in group theory textbooks.

If $\varphi\colon G \to A$ is a group homomorphism to an Abelian group A, it must take every commutator to e_A, so $[G, G] \subset \ker \varphi$. Conversely, the quotient $G/[G, G] = G^{\mathrm{Ab}}$ by the commutator subgroup is an Abelian group, because any two elements commute modulo $[G, G]$. This quotient G^{Ab} is the *Abelianisation* of G. Every homomorphism $\varphi\colon G \to A$ to an Abelian group A factors via G^{Ab} in a unique way; that is, φ equals the composite of $G \to G^{\mathrm{Ab}}$ followed by a group homomorphism $\varphi^{\mathrm{Ab}}\colon G^{\mathrm{Ab}} \to A$.

Examples In S_n, every 3-cycle is a commutator: $(12)(13)(12)^{-1}(13)^{-1} = (123)$. One can see that all 3-cycles generate the alternating group $A_n \subset S_n$. We know that A_n is a normal subgroup of index 2, the kernel of the sign map $S_n \twoheadrightarrow \{\pm 1\}$, so that $[S_n, S_n] = A_n$ and $G^{\mathrm{Ab}} \cong \mathbb{Z}/2$.

We know that D_{2m} contains the normal subgroup $\mathbb{Z}/m$ generated by a. From $bab^{-1} = a^{-1}$ we find that $a^2 = aba^{-1}b^{-1}$, so that a^2 is a commutator. Consider the commutator subgroup $[D_{2m}, D_{2m}]$. It contains a^2, and also $a^m = e$, so if m is odd, it follows that $[D_{2m}, D_{2m}] = \langle a \rangle$ and $D_{2m}^{\mathrm{Ab}} = \mathbb{Z}/2$.

By contrast, when m is even $[D_{2m}, D_{2m}] = \left\langle a^2 \right\rangle$ is a normal subgroup of index 4 and $D_{2m}^{\mathrm{Ab}} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The quotient by $\left\langle a^2 \right\rangle$ is a surjective homomorphism $D_{2m} \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ that takes $a \mapsto \overline{a} = (1, 0)$, $b \mapsto \overline{b} = (0, 1)$.

A geometer thinks of D_{2m} as the rotations and reflections of a regular m-gon. If m is even, these form a *bipartite graph*: you can colour them alternately white and black, and every edge connects a white and black vertex. Then b reverses the orientation, and a swaps black and white.

One dimensional representations and linear characters Consider a representation $\rho\colon G \to \mathrm{GL}(n, \mathbb{C})$ in the case $n = 1$. It is quite special, because the image $\rho(g) \in \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$ is a 1×1 invertible matrix, so simply a nonzero number, necessarily a root of 1 if G is finite; there is no distinction between the matrix $\rho(g)$ and its trace $\chi_\rho(g)$. There is also no dependence on a choice of basis in $\mathbb{C}$. We call ρ a 1-*dimensional representation* or a *linear character*. Because the image $\mathbb{C}^\times$ is Abelian, a linear character factors via the Abelianisation G^{Ab}.

4.5 Finite Abelian groups

I recall a result treated in MA251 Algebra I.

Theorem *Any finite Abelian group A is isomorphic to a sum of cyclic groups, $A \cong \bigoplus_{i=1}^{d} \mathbb{Z}/n_i$. Moreover, we can choose the orders so that*

$$n_1 \mid n_2 \mid \cdots \mid n_d,$$

and then the sequence $\{n_1, n_2, \ldots, n_d\}$ is unique. (However, the isomorphism $A \cong \bigoplus_{i=1}^{d} \mathbb{Z}/n_i$ of course depends on the choice of generators.)

Characters of a finite Abelian group We know the representation theory of the cyclic group $\mathbb{Z}/r$ from Chapter 1: there are r different 1-dimensional representations ρ_k for $k = 0, 1, \ldots, r-1$, where ρ_k takes the generator 1 of $\mathbb{Z}/r$ to ε^k, so takes $j \in \mathbb{Z}/r$ to ε^{jk} (here, as in 1.4, $\varepsilon = \varepsilon_r = \exp \frac{2\pi i}{r}$ is my favourite choice of primitive rth root of 1).

Lemma *Let $A_1, A_2, \ldots, A_m \in \mathrm{GL}(n, \mathbb{C})$; suppose each of the A_i is of finite order, and any two commute. Then there is a basis of $\mathbb{C}^n$ consisting of common eigenvalues of all the A_i.*

Proof First, suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of A_1, say with multiplicity $a_1, a_2, \ldots, a_k$, and write $\mathbb{C}^n$ as a direct sum of the eigenspaces of A_1, that is,

$$\bigoplus U_i \quad \text{where} \quad U_i = \ker(A_1 - \lambda_i) = \mathbb{C}^{a_i}. \tag{4.8}$$

Now a matrix M commutes with A_1 if and only if it preserves the decomposition (4.8). In fact if some $v \in U_1$ is mapped to a vector Mv with nonzero component $u \in U_2$ then MA_1v equals $\lambda_1 Mv$ so has component $\lambda_1 u \in U_2$, whereas A_1V has the different component $\lambda_2 u \in U_2$. (And similarly for $1, 2$ replaced by any i, j.)

In coordinates, the same argument translates as: in a basis subordinate to (4.8), A_1 has the form of $a_i \times a_i$ diagonal blocks equal to λ_i times the identity. Now M must also consist of $a_i \times a_i$ matrix blocks down the diagonal, and be zero outside these blocks. In fact A_1M multiplies the rows of M corresponding to U_i by λ_i, whereas MA_1 multiplies the corresponding colums of M, and any nonzero entries of M outside these diagaonal blocks contradict the assumption $A_1M = MA_1$.

Now by induction on m, I can assume that $A_2, \ldots, A_m$ restricted to U_i are diagonalisable, and putting it all together in the direct sum (4.8) proves the lemma. Q.E.D.

Corollary *A finite dimensional representation of a finite Abelian group A is a direct sum of* 1*-dimensional representation.*

The special point to notice is that the character of a 1-dimensional representation is the trace of a 1×1 matrix, so there is really no distinction between the trace and the representation itself, so that the trace is itself a group homomorphism. (This is completely different for higher dimensional representations.) Thus the set of 1-dimensional representations is $\mathrm{Hom}_{\mathrm{Groups}}(\mathbb{Z}/r, \mathbb{C}^\times)$, and is itself a group isomorphic to $\mathbb{Z}/r$.

This isomorphism is noncanonical, that is, not uniquely defined by $\mathbb{Z}/r$ as abstract group – we chose the generator $1 \in \mathbb{Z}/r$, and the generator $\varepsilon_r \in \boldsymbol{\mu}_r$.

Theorem 1 *Let A be a finite Abelian group. The set of characters is the group $\widehat{A} = \mathrm{Hom}(A, \mathbb{C}^\times)$, and is isomorphic to A (noncanonically, of course).*

The choices here are similar to the choice of bases in a f.d. vector space V, and choice of isomorphism $V \cong V^\vee$.

Theorem 2 *For any finite group G, the 1-dimensional characters of G are the same thing as 1-dimensional characters of the Abelianisation G^{Ab}. Thus if $G^{\mathrm{Ab}} = A$, the 1-dimensional characters of G form a group isomorphic to A.*

4.6 Homework to Chapter 4

4.1 Presentations of groups that you already know. Determine what they are.

1. $\langle x, y \mid xyx = yxy,\ xyxy = yxyx \rangle$.
2. $\langle a, b \mid a^4,\ a^2b^2,\ b^{-1}aba \rangle$.
3. $\langle x, y \mid x^2,\ y^2,\ (xy)^3 \rangle$.
4. $\langle a, b, c \mid a^2 = b^5,\ b^2 = c^3,\ c^2 = a^7 \rangle$.

[Hint: Don't give up.]

4.2 (Part of past exam question.) Let H_8 be the group

$$\langle a, b \mid a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^{-1} \rangle . \tag{4.9}$$

Prove that $ba = a^3b$, and deduce that every element of H_8 can be written in the form a^i or a^ib for $0 \le i \le 3$. Assume as given that H_8 has five conjugacy classes; list them.

4.3 Let $G = \langle a, b \mid a^7, b^3, ba = a^2b \rangle$. Show that G is a group of order 21, with a normal subgroup $H = \langle a \rangle \lhd G$ and quotient $G/H = \mathbb{Z}/3$. Calculate enough commutators of a and b to determine the commutator subgroup $[G, G]$. Calculate enough conjugate elements to determine the 5 conjugacy classes of G.

Construct three different 1-dimensional irreducible representation of G.

4.4 The two permutations $x = (2,5,3)(4,6,7)$ and $y = (1,4,2)(3,5,6)$ are the perfect riffle shuffles of a pack of 7 cards. (In other words, x interleaves 1234 and 567 to give the permutation $\left(\begin{smallmatrix}1&2&3&4&5&6&7\\1&5&2&6&3&7&4\end{smallmatrix}\right) = (2,5,3)(4,6,7)$, and sim. for y.) Show that $z = y^{-1}x$ is a 7-cycle and that $yz = z^2y$. Deduce that they generate a subgroup of S_7 isomorphic to G of Ex. 4.3. [Here I write the product of permutations as the composite function, so that p, q have product $pq\colon i \mapsto p(q(i))$.]

4.5. The elements of D_{2m} are $\{a^i, ba^i \mid i \in [0,\dots,m-1]\}$. I calculated the centraliser of D_{2m} in 4.4, making a fuss about the parity of m. Calculate the 4 conjugacy classes of D_{10} and the 6 conjugacy classes of D_{12}.

4.6 A famous presentation of the symmetric group S_n take the transpositions $s_i = (i, i+1)$ for $i = 1, \dots, n-1$ as generators, and the relations

$$s_i^2 = e, \quad (s_i s_{i+1})^3 = e \quad \text{and} \quad s_i s_j = s_j s_i \quad \text{for } |i-j| \geq 2 \tag{4.10}$$

as defining relations. However, there are many more possible presentations, some of them arbitrarily complicated.

Suppose given the transposition $a = s_1 = (12)$ and the n-cycle $b = (12\dots n)$. Calculate the conjugates $b^{-1}ab$, $b^{-2}ab^2$, and so on, and deduce that a, b also generate S_n.

Let $a = (1,2)$ and $b = (1,2,3,4) \in S_4$, so that a^2 and b^4 are obvious relations. Calculate ab, and deduce that $(ab)^3$ is also a relation them. [Not required for assessment: These are defining relations, so that $S_4 = \left\langle a, b \mid a^2, b^4, (ab)^3 \right\rangle$.]

4.7 Show that every finite group can be given by a finite presentation. (Take all the elements as generators and all the products as relations, and reread the definitions.)

4.8. Conjugacy class $C(g)$ and left coset of $Z(g)$ In 4.3, I defined the conjugacy action of G on itself by $(x, g) \mapsto xgx^{-1}$. It is an action because $x_1(x_2g) = (x_1x_2)g$ and similar for $gx_2^{-1}x_1^{-1}$. Spell out the statement that the orbit of $g \in G$ under the action equals the set of left cosets $C(g) = Z(g)\backslash G$. [Hint: This is an extended headache: what elements $x_1, x_2 \in G$ take $g \in G$ into the same conjugate element $x_1gx_1^{-1} = x_2gx_2^{-1} \in C(g)$?]

4.9. In any group G, prove that

$$[xg_1x^{-1}, xg_2x^{-1}] = x[g_1, g_2]x^{-1}. \tag{4.11}$$

Deduce that the conjugate of a product of commutators is a product of commutators, so the subgroup generated by commutators is normal.

4.10. Conjugacy classes in S_n Write out the conjugacy classes of S_5 (that is, cycle types) and the number of elements in each. Check they add to 120.

4.11. Same for S_6 Complete the calculation started in (4.7). How many elements in each conjugacy class? Check they add to $6! = 720$. For example, why is the number of 5-cycles in S_6 equal to 144?

4.12. Same for A_6 List the conjugacy classes in A_6 and the number of elements in each. Check that they add to 360. You will need to determine which cycle types are conjugate in A_6, which is almost but not quite the same as the question for S_6.

4.13 Run the following Magma routine to prove that the two 4-cycles $a = (1234)$ and $b = (4567)$ generate S_7. Deduce from the output what the two 5-cycles $c = (12345)$ and $d = (34567)$ generate.

```
S7 := SymmetricGroup(7); A := S7!(1,2,3,4); B := S7!(4,5,6,7);
G := sub<S7 | A,B>; G eq S7; Order(G);
C := S7!(1,2,3,4,5); D := S7!(3,4,5,6,7);
H := sub<S7 | C,D>; Index(G,H);
```

S_7 has order 5040, so it would take some time to figure out the defining relations for a, b by hand.

4.14. Consider $A = \mathbb{Z}/p \oplus \mathbb{Z}/pq$ for distinct prime numbers p, q. Find formulas for the number of elements of A of order $1, p, q, pq$, and the number of subgroups of A of order $1, p, q, pq$. Show that A can be written as $A = \mathbb{Z}/p \oplus \mathbb{Z}/pq$ in many different ways.

Imagine doing the same for $A = \mathbb{Z}/a \oplus \mathbb{Z}/ab$ for other values of a, b. (To be concrete, take $(a, b) = (4, 5)$ or $(10, 2)$, or whatever you like.)

4.15. Recall that linear characters or 1-dimensional representations are also homomorphisms $\rho\colon A \to \mathbb{C}^\times$. For distinct primes p, q, set $\varepsilon = \varepsilon_{pq} = \exp \frac{2\pi i}{pq}$. Show how to write out the p^2q characters of the group $A = \mathbb{Z}/p \oplus \mathbb{Z}/pq$. Show that they form a group $\widehat{A}$ isomorphic to A.

4.16. Finitely presented groups Section 4.2 described the elements of $F(m)$ as the graph Γ of (4.5). If w is a reduced word of length $n \geq 0$ not starting with x_i^{-1}, prove that $x_i w$ is a reduced word of length $n + 1$. Deduce the formula given under (4.5) for the number of reduced word of length $n \geq 1$.

Deduce also that Γ_m has no loops, so is simply connected as a metric space. Prove that the translations $w \mapsto x_i w$ define a simply transitive action of $F(m)$ on Γ_m. (You might like to know that the graph Γ_m can be drawn in the hyperbolic plane, with $F(m)$ acting by isometries.)

Difficulty of using finite presentations The next questions illustrate the difficulty of finite presentation as a method of defining a group. Even when we know everything about the group, finite presentations can be pretty cumbersome. Even in simple cases, deducing what the group is from a presentation may be difficult or impossible (there is a theorem that it is algorithmically undecidable in some cases).

4.17 Prove that $\left\langle x, y \mid x^3, y^3, yx^2y = xyx^2 \right\rangle$ is isomorphic to the group G of Ex. 4.3. [Hint: I know how to do this by writing out all the short words in x and y and left multiplying them successively by x and y modulo the relations until the system closes up, but that may be too hard a question for assessed homework.]

4.18 Consider

$$H := \left\langle x, y, z \mid x^2 = y^2 = z^2 = xyz \right\rangle. \tag{4.12}$$

Prove that $c = x^2$ is in the centre of H (that is, commutes with all the generators). Prove that $x = yz$, $y = zx$, $z = xy$. So far, so trivial. Consider the quotient of H by $\left\langle c^2 \right\rangle$, that is

$$\overline{H} := H / \left\langle c^2 \right\rangle = \left\langle x, y, z \mid x^2 = y^2 = z^2 = xyz, x^4 = e \right\rangle. \tag{4.13}$$

Since $x^4 = e$ in $\overline{H}$, we have $x^{-1} = cx = xc$ and similar for y, z. Find formulas for the products zy, xz, yx in $\overline{H}$, and deduce who $\overline{H}$ is.

Now it is a fact that the relations for H imply that $c^2 = e$, but it takes at least 4 applications of the relations to prove it. You have to really care about it to get it out. Please think about it; I'll give another hint in the homework to Chapter 5

4.19 The presentation $G = \left\langle a, b \mid a^2 = bab, b^2 = aba \right\rangle$ is a famous puzzle. Without any hints, I guess that most mathematicians would take days to get it out, or give up.

First hint: (1) Prove that the element $c = abab$ is central. (2) Study the quotient by $\langle c \rangle$, that is, the group $\overline{G} = \left\langle a, b \mid a^2 = bab, b^2 = aba, abab \right\rangle$. Prove that it has order 12. I'll give a second hint in the homework to Chapter 5.

5 Characters and the Main Theorem

5.1 Introduction

In Chapters 2–3 we worked with groups like BD_{4m}, writing out their representations in terms of 2×2 or 3×3 matrices. We specially chose those groups to be "very close" to an Abelian group. For a bigger group, this procedure is unworkable: it would involve many $n \times n$ matrices, and there would be no easy way of manipulating them all together.

The point of character theory is that it replaces all of these matrices with functions $\chi\colon G/(\text{conjugacy}) \to \mathbb{C}$, together with rules for manipulating them, that are much easier to handle. Here I state the Main Theorem, with the proofs deferred to later. You can think of this as just a list of rules, or User's Guide. As an illustration, I work out the character table of the octahedral group O_{24}, and repeat what I said in 3.4 about the binary tetrahedral group BT_{24}.

5.2 The trace of an $n \times n$ matrix

The *trace* of a square matrix $A \in \mathrm{Mat}(n \times n, K)$ is the sum of its diagonal entries: $\mathrm{Tr}\, A = \sum_{i=1}^{n} a_{ii}$. I recall some easy results from Linear Algebra.

Proposition *(i) For any two matrices $A, B \in \mathrm{Mat}(n \times n, K)$ we have*

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA). \tag{5.1}$$

(ii) For $A \in \mathrm{Mat}(n \times n, K)$ and $T \in \mathrm{GL}(n, K)$ we have

$$\mathrm{Tr}(TAT^{-1}) = \mathrm{Tr}\, A. \tag{5.2}$$

I give an easy mechanical proof first, then a more insightful treatment based on the characteristic polynomial. Write $A = (a_{ij})$ and $B = (b_{ij})$ with $i, j = 1, \dots, n$. The product AB then has entries $\sum_j a_{ij} b_{jk}$, with the j summed out. To get its trace, set $k = i$ and sum out the i, arriving at the double sum

$$\mathrm{Tr}(AB) = \sum_i \sum_j a_{ij} b_{ji}. \tag{5.3}$$

In the same way, the trace of BA comes to $\sum_i \sum_j b_{ij} a_{ji}$, which is the same thing: because $a_{ij} b_{ji} = b_{ji} a_{ij}$ in each term, and we can exchange the order of summation in the double sum.

(ii) follows from (i) applied to (TA) and T^{-1}:

$$\mathrm{Tr}(TAT^{-1}) = \mathrm{Tr}((TA)T^{-1}) = \mathrm{Tr}(T^{-1}(TA)) = \mathrm{Tr}\, A. \quad \text{QED} \tag{5.4}$$

5.3 The characteristic polynomial is conjugacy invariant

Write $\Delta_A(t) = \det(t\,\mathrm{Id} - A)$ for the *characteristic polynomial* of a square matrix $A \in \mathrm{Mat}(n \times n, K)$. Here $\Delta_A(t) \in K[t]$ where t is a new polynomial variable.

The characteristic polynomial of a linear transformation $\varphi\colon V \to V$ is defined in terms of a basis of V and the resulting identification $V = K^n$. The results discussed below make it independent of the choice of basis and provide a second proof of Proposition 5.2.

Proposition *For two square matrices $A, B \in \mathrm{Mat}(n \times n, K)$, the product AB has determinant equal to the product of those of A and B:*

$$\det(AB) = \det A \det B. \tag{5.5}$$

I discuss this concisely for completeness. As you know, the determinant of an $n \times n$ matrix $M = m_{ij}$ is defined as the sum over the symmetric group S_n

$$\det M = \sum_{\sigma \in S_n} \operatorname{sign} \sigma \prod_{i=1}^{n} m_{i\sigma(i)}. \tag{5.6}$$

For each permutation σ of $1, \dots, n$, the term $m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{n\sigma(n)}$ is the product of one entry from each row and each column, with sign $+1$ if σ is even and -1 if odd. The familiar invariance properties of $\det M$ under row operations follow tautologically from the definition:

(a) swapping two rows i, j multiplies $\det M$ by -1;

(b) adding a multiple of row i to row j leaves M unchanged.

For the proposition, consider A and B and their product AB. If either $\det A$ or $\det B$ is zero, the corresponding linear transformation is degenerate, so also $\det(AB) = 0$.

As you know, successive row operations reduce any nonsingular matrix to make it diagonal. For a diagonal matrix, the sum in (5.6) is just the product $\prod m_{ii}$ of the diagonal entries. We can use column operations to exactly the same effect.

Now row operations applied to A reduce it to diagonal form A', and column operations reduce B to diagonal form B'. The same row and column operations reduce the product AB to the diagonal form $A'B'$. Now (up to a common sign, that I ignore)

$$\det A = \det A', \quad \det B = \det B' \quad \text{and} \quad \det(AB) = \det(A'B'). \tag{5.7}$$

For diagonal matrices $A'B'$ the equality $\det(A'B') = \det A' \det B'$ is obvious. Q.E.D.

Proposition *The characteristic polynomial $\Delta_A(t)$ is invariant under conjugacy. That is,*

$$\Delta_{TAT^{-1}}(t) = \Delta_A(t) \quad \text{for } A \in \mathrm{Mat}(n \times n, K) \text{ and } T \in \mathrm{GL}(n, K). \tag{5.8}$$

The characteristic polynomial has the form

$$\Delta_A(t) = t^n - (\operatorname{Tr} A) t^{n-1} + (\text{intermediate terms}) + (-1)^n \det A. \tag{5.9}$$

Therefore $\operatorname{Tr}(TAT^{-1}) = \operatorname{Tr} A$.

Proof For any $A, B \in \mathrm{Mat}(n \times n, K)$, we have $\det(AB) = \det A \det B$. Therefore

$$\det(T(t\,\mathrm{Id}_n - A)T^{-1}) = \det T \det(t\,\mathrm{Id}_n - A)(\det T)^{-1} = \det(t\,\mathrm{Id}_n - A). \quad (5.10)$$

the left-hand side of (5.10) is $\Delta_{TAT^{-1}}(t)$, and the right-hand side is $\Delta_A(t)$.

The coefficient of t^{n-1} in $\det(t\,\mathrm{Id}_n - A)$ is a calculation: t only appears in the diagonal entries of $t\,\mathrm{Id}_n - A$, so to get t^{n-1} in a summand of the determinant, we must choose the t in $n-1$ diagonal entries times the constant term in the single complementary diagonal entry.

5.4 Definition of characters and immediate properties

Definition Let G be a finite group and $\rho\colon G \to \mathrm{GL}(V)$ a representation of G over $\mathbb{C}$. The *character* of ρ (or of the $\mathbb{C}G$ module V) is the function

$$\chi_\rho = \chi_V : G \to \mathbb{C} \quad \text{given by} \quad \chi_V(g) = \mathrm{Tr}(\rho(g). \quad (5.11)$$

We sometimes say *irreducible character* for the character of an irreducible representation, and call $\dim V$ the *dimension* of the character χ_V (etc.). (Compare the linguistic disclaimer in 2.4)

Proposition *(i) χ_V only depends on V up to isomorphism of $\mathbb{C}G$-modules.*

(ii) $\chi_V(g)$ only depends on g up to conjugacy.

(iii) $\chi_V(e_G) = \dim V$.

(iv) Moreover, $\chi_V(g) = \dim V$ if and only if $g \in \ker \rho$.

(v) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ (complex conjugate).

(vi) A direct sum of $\mathbb{C}G$-modules has character the sum of the characters of the summands. That is, if $\rho_i\colon G \to \mathrm{GL}(V_i)$ are $\mathbb{C}G$-modules and $\rho = \oplus\rho_i\colon G \to \mathrm{GL}(\bigoplus V_i)$ their direct sum then $\chi_\rho = \sum \chi_{\rho_i}$.

(i) and (ii) follow at once from Proposition 5.2 (conjugate matrices have equal trace). (iii) says that the identity of V has trace $\dim V$, because it is represented by the identity $n \times n$ matrix. For (iv), note that $\rho(g)$ is diagonalisable, with eigenvalues λ_i, so has trace $\sum \lambda_i$. The λ_i are roots of unity, with $|\lambda_i| = 1$. Thus the real part of λ_i is ≤ 1, and equals 1 if and only if $\lambda_i = 1$. Therefore the only way for $\sum \lambda_i$ to score $\dim V$ is if all $\lambda_i = 1$, so $\rho(g) = \mathrm{Id}_V$. (v) follows because $\rho(g^{-1}) = \rho(g)^{-1}$ has eigenvalues $\lambda_i^{-1} = \overline{\lambda_i}$. Finally (vi) follows since $\oplus\rho_i$ maps G to diagonal block matrices, whose trace just adds up the sum of that of its blocks.

5.5 Space of class functions and its Hermitian pairing

Definition A *class function* on G is a map $f\colon G \to \mathbb{C}$ that is invariant under conjugacy, in the sense that $f(xgx^{-1}) = f(g)$ for all $x \in G$. We can view a class function as a function on conjugacy classes

$$G/(\mathrm{conj}) \to \mathbb{C}. \tag{5.12}$$

The set of all class functions $\mathcal{C} = \mathcal{C}(G)$ is a vector space with basis in 1-to-1 correspondence with the conjugacy classes of G; it has a Hermitian inner product $\langle \varphi, \psi \rangle$ defined by

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi(g). \tag{5.13}$$

The sum has duplicate summands for conjugate elements. That is, if $C(g)$ is the conjugacy class of g, its elements all repeat the same summand. It often simplifies things to choose a representative of each conjugacy class, say $g_1, \dots, g_r$, and rewrite the repeated summands in (5.13) as $|C(g_i)| \times \overline{\varphi(g_i)} \psi(g_i)$. Moreover, each conjugacy class $C(g_i)$ can be identified with the cosets $Z(g_i)\backslash G$ of G by the centraliser $Z(g_i)$, so that (5.13) takes the form

$$\begin{aligned} \langle \varphi, \psi \rangle &= \frac{1}{|G|} \sum_{i=1}^{r} |C(g_i)| \overline{\varphi(g_i)} \psi(g_i) \\ &= \sum_{i=1}^{r} \frac{1}{|Z(g_i)|} \overline{\varphi(g_i)} \psi(g_i). \end{aligned} \tag{5.14}$$

5.6 Main Theorem

I now state without proof the main result of the course. First, choose a set $\{U_i\}_{i=1}^{k}$ of irreducible $\mathbb{C}G$-modules that are pairwise nonisomorphic, and such that every irreducible $\mathbb{C}G$-module is isomorphic to some U_i. This is called a *complete set of nonisomorphic irreducible representations.* Write $d_i = \dim U_i$ for their dimensions and $\chi_i = \chi_{U_i}$ for their characters.

Theorem *(I) The χ_i are orthonormal with respect to $\langle\ ,\ \rangle$. That is*

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases} \tag{5.15}$$

(II) The number k of the U_i equals the number r of conjugacy classes. It follows that the χ_i form an orthonormal basis of $\mathcal{C}$.

(III) $\sum_{i=1}^{k} d_i^2 = |G|$.

Corollary *(i) Let V be a $\mathbb{C}G$-module with character χ_V. Write $V \cong \bigoplus a_i U_i$ as a direct sum of irreducibles $\mathbb{C}G$-modules. (This decomposition exists and is unique by Maschke's Theorem ss!Mas and Schur's Lemma 3.3.) Then $a_i = \langle \chi_i, \chi_V \rangle$.*

(ii) The character χ_V determines V up to isomorphism.

(iii) χ_V has norm-squared the positive integer $\langle \chi_V, \chi_V \rangle = \sum a_i^2$.

(iv) In particular, $\langle \chi_V, \chi_V \rangle = 1$ if and only if V is irreducible.

Proof (i) If $V = \sum a_i U_i$ then $\chi_V = \sum a_i \chi_i$ by Proposition 5.4, and then we can read off $\langle \chi_i, \chi_V \rangle = a_i$ from (I). (ii) χ_V determines the values of a_i such that $V \cong \bigoplus a_i U_i$, and so determines V up to isomorphism. (iii) is equally easy; (iv) involves analysing the solutions in positive integers of $\sum a_i^2 = 1$. Q.E.D.

5.7 Character table of O_{24}

What we need to know about $\mathbb{C}G$-modules is contained in the character table of G. This is a square array formed by the values of the irreducible characters χ_i evaluated on the conjugacy classes g_j.

I work out the example of the group O_{24} of rotational symmetries of the cube in $\mathbb{R}^3$ to illustrate the meaning of the orthonormality relations of (I). Figure 1 is the cube of side 2 with vertices $(\pm 1, \pm 1, \pm 1)$.

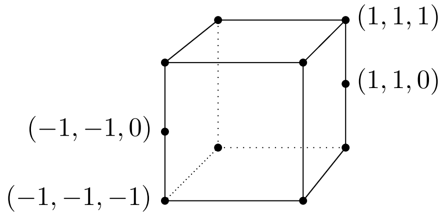


Figure 2: Symmetry group of the cube.

Please imagine drawn the following axes of rotational symmetry: (a,d) The z-axis, an axis of 2-fold and 4-fold rotation. (b) The median line through $(-1,-1,0)$ and $(1,1,0)$, an axis of 2-fold rotation. (c) the main diagonal from $(-1,-1,-1)$ to $(1,1,1)$ (an axis of 3-fold rotation).

The group has order 24. Indeed, there are 8 vertices, and exactly 3 rotational symmetries fix a given vertex, acting as 3-cycle on the 3 edges out of it. The elements that we can see at once are: the identity e (1 element); (d) the 4-fold rotation by $\pi/2$ around a directed coordinate axis (6 elements); (a) the 2-fold rotation around a coordinate axis (3 elements); (c) the 3-fold rotations around any of the 4 main diagonals (8 elements); (b) the 2-fold rotation around the median lines (6 elements), with each of the 3 coordinate planes containing 2 such lines. Since $1 + 6 + 3 + 8 + 6 = 24$, this accounts for every element of O_{24}. Morever, all the elements in each of the enumerated types are conjugate. (For example, rotations about the main diagonal through $2\pi/3$ and $4\pi/3$ are conjugate by any rotation that takes $(1,1,1)$ to $(-1,-1,-1)$, such $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.)

Matrices representing the non-identity conjugacy classes are (for example)

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To describe all the irreducible representations of O_{24}, the remaining fact we need is that it has the 4-group V_4 as normal subgroup with quotient group S_3.

$$V_4 \lhd O_{24} \twoheadrightarrow S_3. \tag{5.16}$$

Here S_3 is the permutation group on the unordered coordinate axes, and V_4 acts by ± 1 on evenly many of the coordinates (like the matrix A).

Now the character table of O_{24} is as follows:

		e	a	b	c	d
$\#C(g)$		1	3	6	8	6
order g		1	2	2	3	4
trivial rep	χ_1	1	1	1	1	1
sign of S_3	χ_2	1	1	-1	1	-1
2-dim rep of S_3	χ_3	2	2	0	-1	0
given rep	χ_4	3	-1	-1	0	1
$\chi_2 \cdot \chi_4$	χ_5	3	-1	1	0	-1

(5.17)

Here χ_2 is the 1-dimensional representation of S_3. $a \in V_4$ is in the kernel of $O_{24} \twoheadrightarrow S_3$, b and d map to transpositions such as (12), c maps to 3-cycles such as (123). In its 2-dimensional representation S_3 is the dihedral group D_6, so $a \in V_4$ acts by the identity, b and d act with eigenvalues ± 1, and c is a 3-fold rotation, so has eigenvalue ω and ω^2 adding to -1.

For χ_4, just calculate the trace of the 4 matrices A, B, C, D. Finally χ_5 is the product of χ_4 with the 1-dimensional character χ_2.

Let's use this to illustrate the Main Theorem. The dimensions of the 5 characters $1, 1, 2, 3, 3$ give $1^2 + 1^2 + 2^2 + 3^3 + 3^2 = 24$, which is (III). This guarantees that we have all the irreducible representations. In each row, the squares of the entries times $\#C(g)$ from the header add to 24. For example, for the χ_3 row, $2^2 + 2^2 \times 3 + (-1)^2 \times 8 = 24$. Then for each pairs of rows, the product of the entry and $\#C(g)$ add to zero. There are 10 calculations to do here, but they are all harmless fun.

5.8 Homework to Chapter 5

Further hint for Ex. 4.18 As I said, you need at least 4 applications of the relations to prove that $c^2 = e$. The hint is to use relations like $x = yz$ etc. to make *longer and longer* words. For example, $y^2 = xyz$ is given, and you have already deduced that $x = yz$, $y = zx$ and so on. So $y^2 = (yz)(zx)z$ and so on. You eventually get to something that you recognise. There are lots of ways to the truth, but all of them are long.

Different approach to the group of Ex. 4.19 In the finitely presented group of Ex. 4.19, set $i = ab$, $j = ba = a^{-1}ia$, $k = a^{-1}ja$. Prove that $i^2 = j^2 = k^2 = c$. Given in addition that $c^2 = e$, get out the result of Ex. 4.19.

Further hint for Ex. 4.19 We are given $a^2 = bab$ and $b^2 = aba$, and we deduced easily that $c = abab$ is central. You also determined that the quotient $G/\langle c\rangle$ is A_4. In a similar vein to Ex. 4.18, it takes at least 6 applications of the defining relations to prove that $c^2 = 1$. You could for example start with $c = abab = baba$ so that $c^2 = abab^2aba$, then apply the rule $b^2 \mapsto aba$ to make the word on the right *longer*, and continue likewise. I'll give my solution in Ex. 6.1, but finding your own would be so much more rewarding.

In these questions, assume the material of 5.4–5.6 around the Main Theorem.

5.1 If A is diagonalisable, conjugate to $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$, the coefficients of its characteristic polynomial $\Delta_A(t)$ are the elementary symmetric polynomials in the λ_i, that is $\pm\sigma_k(\lambda_1, \dots, \lambda_n)$. Therefore they are also invariant under conjugacy, in the same way as the trace.

Prove that $\operatorname{Tr}(A^2) = \sum \lambda_i^2$ and deduce a formula for the coefficient of t^{n-2} in $\Delta_A(t)$ as a linear combination of $(\operatorname{Tr} A)^2$ and $\operatorname{Tr}(A^2)$.

Remark There are similar formulas (called Newton's formulas) that express all the elementary symmetric functions σ_k in terms of the sums of powers Σ_k. The character of a representation includes $\chi_V(g^k)$ for all k, so it actually knows the characteristic polynomial of $\Delta_{\rho(g)}(t)$, which determines $\rho(g)$ up to conjugacy. So it is possibly not so very surprising that the character of a representation determines the representation up to isomorphism.

5.2 (v) of Main Theorem gave a necessary condition on $\rho\colon G \to \mathrm{GL}(V)$ and an element $g \in G$ for $\chi_\rho(g) = \dim V$. State and prove the condition for $|\chi_\rho(g)| = \dim V$

5.3 Verify the row orthogonality relations in the character table of O_{24} in 5.7. Do the same for the character table of BT_{24} treated below.

5.4 As I said in 1.4, the case of a cyclic group (and roots of unity) are foundational for the course, and you could reasonably expect an exam question on them. Compare the orthogonality relations of Main Theorem (I) with what we already know for cyclic groups from 1.4.

5.5 We know that the matrix group H_8 of 2.3 has a 2-dimensional irreducible representation. Deduce that H_8 must have 5 conjugacy classes. Complete the character table of H_8.

5.6 G is a finite group, and its character table includes the 3 rows:

$$\begin{array}{r|ccccc} & e & g_2 & g_3 & g_4 & g_5 \\ \#C(g) & 1 & 3 & 3 & 7 & 7 \\ \hline \chi_2 & 1 & 1 & 1 & \omega & \omega^2 \\ \chi_4 & 3 & \omega & \omega^2 & 0 & 0 \\ \chi_5 & 3 & \omega^2 & \omega & 0 & 0 \end{array} \tag{5.18}$$

Determine the order of G, and the number and dimensions of its remaining irreducible representations. Finish the character table. Show how to write out the group by generators and relations.

5.7 An element $g \in G$ is in the commutator subgroup $[G, G]$ if and only if it is in the kernel of every 1-dimensional representation. If we have complete information about a conjugacy class in G, state and prove a method of determining from the character table of G which conjugacy classes are contained in $[G, G]$.

5.8 Most easy groups that turn up in introductory courses have 1-dimensional representations (compare 4.4). You can read directly from the character table which characters are 1-dimensional. Prove that if χ_1 is a 1-dimensional character and χ_2 some other irreducible character, then the product $\chi_1 \cdot \chi_2$ must again be an irreducible character. This is useful if you need to treat the character table as a crossword puzzle.

5.9 Appendix. Character table of BT_{24}

The Main Theorem is also illustrated by the character table of BT_{24}. This repeats what I said in 3.4. It is all easy hand calculation, but Magma can handle much bigger cases effortlessly. I ran the following code in the online Magma calculator

`http://magma.maths.usyd.edu.au/calc`

```
K4<i> := CyclotomicField(4);  // define i as primitive 4th root
GL2 := GeneralLinearGroup(2,K4);  // allows 2x2 matrices
I := elt< GL2 | i,0,0,-i >;
J := elt< GL2 | 0,1,-1,0 >;
K := I*J;  // 2x2 matrices corresponding to quaternions i,j,k
A := 1/2*Matrix(2, [1+i, 1+i, -1+i, 1-i]);
B := Transpose(A);
I*J*I^-1 eq -J; A*B eq I; A^2 eq B*A*B; B^2 eq A*B*A;
// play with these generators as sanity check
BT24 := sub< GL2 | [A,I] >;
Order(BT24);
// ConjugacyClasses(BT24);
CharacterTable(BT24);
```

As discussed in Chapters 2–3, the binary tetrahedral group BT_{24} contains the subgroup $H_8 = \{\pm 1, \pm I, \pm J, \pm K\}$ made up by the standard unit quaternions and the 16 matrices $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$ with $a, b \in \{\frac{\pm 1 \pm i}{2}\}$. It is generated by I and $A = \frac{1}{2}\left(\begin{smallmatrix} 1+i & 1+i \\ -1+i & 1-i \end{smallmatrix}\right)$ and contains the central element $c = -1 = I^2 = A^3 = ABAB$. This is its character table:

$$\begin{array}{r|ccccccc} & e & c & A^2 & B^2 & I & A & B \\ \text{size} & 1 & 1 & 4 & 4 & 6 & 4 & 4 \\ \text{order} & 1 & 2 & 3 & 3 & 4 & 6 & 6 \\ \hline \chi_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \chi_2 & 1 & 1 & \omega & \omega^2 & 1 & \omega^2 & \omega \\ \chi_3 & 1 & 1 & \omega^2 & \omega & 1 & \omega & \omega^2 \\ \chi_4 & 2 & -2 & -1 & -1 & 0 & 1 & 1 \\ \chi_5 & 2 & -2 & -\omega^2 & -\omega & 0 & \omega & \omega^2 \\ \chi_6 & 2 & -2 & -\omega & -\omega^2 & 0 & \omega^2 & 1 \\ \chi_7 & 3 & 3 & 0 & 0 & -1 & 0 & 0 \end{array} \tag{5.19}$$

The conjugacy classes are $\{e\}$, $\{c\}$, then $\{A^2, IA^2, JA^2, KA^2\}$ of order 3 and similarly for B^2, then the six elements $\{\pm I, \pm J, \pm K\}$ and finally the elements $\{A, I^3A, J^3A, K^3A\}$ and similarly for B. You need to do some little checks (by hand or by computer) to verify that the stated elements are conjugate, and that they give all 24 elements of the group.

Next, the characters are the rows of the table: χ_1 is the character of the trivial 1-dimensional representation. Obviously each element of the group scores 1.

The next two lines are 1-dimensional characters (or linear characters). This means they are characters of the Abelian quotient $\mathrm{BT}_{24}^{\mathrm{Ab}} = \mathrm{BT}_{24}/H_8 = \mathbb{Z}/3$. In this quotient group, A acts by conjugacy $I \mapsto J \mapsto K \mapsto I$; on the two eigenspaces of this action, A acts by ω and ω^2, and A^2, B, B^2 follow suit in the only possible way.

The next 3 lines are 2-dimensional characters. The first χ_4 corresponds to the *given representation*: BT_{24} is given as a 2×2 matrix group, so acts naturally on $\mathbb{C}^2$ (column vectors). Then χ_4 just records the trace of the group elements e, c, A^2, B^2, I, J, K, A, B. The next two lines are products of the given representation by the two linear characters: quite generally if ρ is a 1-dimensional representation then $\rho(g)$ is just a scalar, so that for any representation σ it makes sense to take the product $\rho\sigma$, which takes any $g \in G$ to $\rho(g)\sigma(g)$ (with $\rho(g)$ a scalar multiplying matrix $\sigma(g)$). Taking trace of the product $\rho(g)\sigma(g)$ just gives

$$\chi_6 = \chi_2 \times \chi_4 \quad \text{and} \quad \chi_5 = \chi_3 \times \chi_4. \tag{5.20}$$

The final line is a 3-dimensional character. The literal minded description of the corresponding 3-dimensional representation in terms of matrices is

$$A \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad I \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad K \mapsto \text{sim.} \tag{5.21}$$

This can be argued in several ways: for example, one sees in group theory that the quotient of BT_{24} by its centre $\langle c\rangle$ is the alternating group A_4, with $A \mapsto (123)$ and $I \mapsto (12)(34)$. Or the 3-dimensional representation is the symmetric square Sym^2 of the given representation as described in Ex. 3.3 (the group acting on quadratic forms u^2, uv, v^2, giving $\chi_7 = \chi_4^2 - \chi_1$). Or the representation is the 3-dimensional space generated by I, J, K

$$\mathbb{R}^3 = \mathbb{R}I \oplus \mathbb{R}J \oplus \mathbb{R}K \subset \mathrm{Mat}(2 \times 2, \mathbb{C}). \tag{5.22}$$

on which BT_{24} acts by conjugacy.

6 Proof of Main Theorem

6.1 Irreducible characters are mutually orthogonal

Proposition *If U_1 and U_2 are nonisomorphic irreducible $\mathbb{C}G$-modules, their characters χ_1, χ_2 are orthogonal w.r.t. the Hermitian inner product of 5.5:*

$$\langle \chi_1, \chi_2 \rangle = 0. \tag{6.1}$$

First $\overline{\chi_1(g)} = \chi_1(g^{-1})$ (by 5.4, Proposition, (iv)), so I need to prove that $\sum_{g \in G} \chi_1(g^{-1})\chi_2(g) = 0$. This follows by a calculation based on the averaging trick together with Schur's lemma (I).

Recall that Proposition 2.6 averages out any $\mathbb{C}$-linear map $\varphi\colon V_2 \to V_1$ between $\mathbb{C}G$-modules (with representations ρ_i) to give a $\mathbb{C}G$-module homomorphism

$$\psi = \frac{1}{|G|} \sum_{g \in G} \rho_1(g^{-1}) \circ \varphi \circ \rho_2(g) \in \mathrm{Hom}_{\mathbb{C}G}(V_2, V_1). \tag{6.2}$$

If U_2 and U_1 are nonisomorphic irreducible $\mathbb{C}G$-modules, Schur's lemma (I) says that the average value ψ in (6.2) is zero for any $\mathbb{C}$-linear map $\varphi\colon U_2 \to U_1$.

Fix bases to make $U_1 = \mathbb{C}^{n_1}$ and $U_2 = \mathbb{C}^{n_2}$, and write $\rho_1(g) \in \mathrm{GL}(U_1)$ as the $n_1 \times n_1$ matrix $(\rho_1(g))_{ij}$ for $i, j = 1, \dots, n_1$, and similarly for ρ_2. Before going to traces, I prove the following stronger statement on matrix entries.

Lemma *For all $i, a = 1, \dots, n_1$ and $b, j = 1, \dots, n_2$,*

$$\frac{1}{|G|} \sum_{g \in G} (\rho_1(g^{-1}))_{ia} (\rho_2(g))_{bj} = 0. \tag{6.3}$$

Here you should appreciate the full power of Schur's lemma. (6.2) applies to any $\mathbb{C}$-linear map $\varphi\colon U_2 \to U_1$ (given by any $n_1 \times n_2$ matrix), and for every such φ, the statement (6.2) is an equality of $n_1 \times n_2$ matrices. Thus the bland-looking statement in abstract algebra actually gives us $(n_1 n_2)^2$ identities.

Proof Write M_{ab} for the elementary $n_1 \times n_2$ matrix with only nonzero entry $m_{ab} = 1$ for the given a, b. For any $n_1 \times n_1$ matrix R and $n_2 \times n_2$ matrix S, the product $R M_{ab} S$ is the rank 1 matrix obtained as the matrix product of the ath column of R multiplied by the bth row of S.

$$R M_{ab} S = \begin{pmatrix} R_{1a} \\ R_{2a} \\ \vdots \\ R_{n_1 a} \end{pmatrix} \begin{pmatrix} S_{b1} & S_{b2} & \dots & S_{bn_2} \end{pmatrix} = (R_{ia} S_{bj})_{ij}. \tag{6.4}$$

The matrix M_{ab} corresponds to a homomorphism $\varphi\colon U_2 \to U_1$. Set $R = \rho_1(g^{-1})$ and $S = \rho_2(g)$ for each $g \in G$, then average over $g \in G$. Then (6.2) together with Schur's lemma (I) gives that the average of these is 0, which proves the Lemma.

The statement (6.1) follows from (6.3) on substituting $a = i$ and $b = j$, then summing over i and j. Q.E.D.

6.2 An irreducible character has norm 1

The remaining orthonormality statement is that the character $\chi = \chi_U$ of an irreducible representation U satisfies

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})\chi(g) = 1. \tag{6.5}$$

It follows by a similar argument, based on averaging and Schur's lemma (II). As above, by Proposition 2.6, the average over G of any $\mathbb{C}$-linear map $\varphi \colon U \to U$ is a $\mathbb{C}G$-module homomorphism

$$\psi = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) \circ \varphi \circ \rho(g) \in \mathrm{Hom}_{\mathbb{C}G}(U, U). \tag{6.6}$$

Since U is irreducible, Schur's lemma (II) says that ψ is a scalar times Id_U. Do we know what scalar multiple? The little advantage we have here is that φ is a map from U to itself, as are all the $\rho(g)$, so that each of the $|G|$ summands in (6.6) is conjugate to φ, so that they all have the same trace by 5.2. Therefore

$$\psi = \lambda \, \mathrm{Id}_U, \quad \text{where} \quad \lambda = \tfrac{1}{n} \operatorname{Tr} \varphi \tag{6.7}$$

(where $n = \operatorname{Tr}(\mathrm{Id}_U) = \dim U$, so that $\operatorname{Tr} \psi = \lambda$).

From this point on, the argument works as before: write $(\rho(g))_{ij}$ for the matrix of $\rho(g)$ is any basis. Then argue on the elementary $n \times n$ matrix M_{ab} as in the above lemma. The conclusion is that

$$\frac{1}{|G|} \sum_{g \in G} (\rho(g^{-1}))_{ia} (\rho(g))_{bj} = \tfrac{1}{n} \operatorname{Tr} M_{ab} \quad \text{for any } i, j, a, b = 1, \dots, n. \tag{6.8}$$

As before, set $i = a$ and $j = b$ and sum over i and j. On the right-hand side, the sum of $\operatorname{Tr} M_{ab}$ over a, b equals n, which cancels the denominator. On the left, the sum over i and j gives $\frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})\chi(g)$, and equating the two gives $\langle \chi, \chi \rangle = 1$. Q.E.D.

This completes the proof of (I) of Main Theorem 5.6. I next introduce the regular representation $\mathbb{C}G$, and prove (III) in Corollary 6.4 as an application. I leave (II) to the start of Chapter 7.

6.3 The regular representation $\mathbb{C}G$

The *regular representation* $\mathbb{C}G$ is the vector space with basis $x \in G$. That is, an element of $\mathbb{C}G$ is a formal sum $\sum_{x \in G} \lambda_x x$, with coefficients $\lambda_x \in \mathbb{C}$; the G-action is left multiplication

$$\Big(g, \sum \lambda_x x\Big) \mapsto \sum \lambda_x g x. \tag{6.9}$$

In other words, $\mathbb{C}G$ is the permutation representation corresponding to G acting on itself by left multiplication: $g \in G$ acts by $x \mapsto gx$. Write $x \in \mathbb{C}G$ for the basis vector $1 \cdot x$ (that is $\sum \lambda_x x$ with only one nonzero coefficient $\lambda_x = 1$).

As a baby example, number the elements of S_3 as $x_1 = e$, $x_2 = (12)$, $x_3 = (13)$, $x_4 = (23)$, $x_5 = (123)$, $x_6 = (132)$. Then $g = (12)$ acts on these by $gx_1 = x_2$, $gx_2 = x_1$, $gx_3 = (12)(13) = (132) = x_6$, $gx_4 = x_5$, so that the action of $g = (12)$ on $\mathbb{C}S_3$ is given by the permutation $(12)(36)(45)$, or the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{6.10}$$

The regular representation is key to the theory because, as I now explain, it contains every irreducible representation as a direct summand.

6.4 Homming from $\mathbb{C}G$ and the proof of (III)

The point of the following proposition is that $\mathbb{C}G$ has a *preferred element* $1_{\mathbb{C}G} = 1 \cdot e_G$, that behaves like the unit element of a ring or $\mathbb{C}$-algebra.

Proposition *Let V be a $\mathbb{C}G$-module with representation $\rho\colon G \to \mathrm{GL}(V)$. Then for every $v \in V$ there is a uniquely defined $\mathbb{C}G$-module homomorphism $\varphi_v\colon \mathbb{C}G \to V$ such that $\varphi(1_{\mathbb{C}G}) = v$. In other words, there is a natural identification of $\mathbb{C}$-vector spaces (without G-action)*

$$\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, V) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V) = V. \tag{6.11}$$

Proof For any $v \in V$, define $\varphi_v\colon \mathbb{C}G \to V$ by

$$\sum \lambda_g g \mapsto \sum \lambda_g \rho_V(g) v. \tag{6.12}$$

This is a homomorphism of $\mathbb{C}G$-modules, because it is $\mathbb{C}$-linear, and for any $x \in G$, it does

$$x \sum \lambda_g g \mapsto \sum \lambda_g \rho_V(xg) v = \rho_V(x) \left(\sum \lambda_g \rho_V(g) v \right). \tag{6.13}$$

This φ_v is uniquely specified as a $\mathbb{C}G$-module homomorphism by the condition that it sends $1_{\mathbb{C}G} \mapsto v$: each basis element $g = 1 \cdot g \in \mathbb{C}G$ goes to $\rho(g)v$, and then φ_v extends to the whole of $\mathbb{C}G$ by $\mathbb{C}$-linearity.

According to this definition, $\varphi_v(1_{\mathbb{C}G})$ maps to $1 \cdot \rho(e)v = v$. Moreover, any homomorphism of $\mathbb{C}G$-modules $\varphi\colon \mathbb{C}G \to U$ must take $1_{\mathbb{C}G}$ to some element v, and then $\varphi = \varphi_v$. Q.E.D.

Corollary *(a) If C is any $\mathbb{C}G$-module then* $\dim \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, V) = \dim V$.

(b) Decompose the regular representation $\mathbb{C}G = \bigoplus_{i=1}^k d_i U_i$ as a direct sum of irreducibles (as in Corollary ss!Irr of Maschke's theorem). Then $d_i = \dim U_i$. In words, $\mathbb{C}G$ contains exactly d_i isomorphic copies of every irreducible representation U_i, so that the summand $d_i U_i$ has dimension d_i^2.

(c) $\sum d_i^2 = |G|$.

(d) Every irreducible $\mathbb{C}G$-module U is isomorphic to one of the U_i appearing in the decomposition of (b). In particular, there are only finitely many irreducible $\mathbb{C}G$-modules up to isomorphism.

Proof The Proposition gives $\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, V) = V$, so the Hom space has dimension $\dim V$. Now suppose $V = U$ is irreducible (the definition includes $U \neq 0$). Then by Corollary 3.3, (b) of Schur's lemma, a $\mathbb{C}G$-homomorphism $\varphi\colon \bigoplus a_j U_j \to U$ is zero on all the factors U_j not isomorphic to U, so the only way that Hom space can be nonzero, is that one of the summands U_i in $\mathbb{C}G$ is isomorphic to U as $\mathbb{C}G$-module. Identify $U = U_i$. Then φ is a scalar multiple of the identity on each factor U_i, so that the same Hom space has dimension a_i. This implies $a_i = d_i$.

This proves that up to isomorphism, every irreducible $\mathbb{C}G$-module is one of the U_i, so in particular there are only finitely many. And $|G| = \dim \mathbb{C}G = \sum d_i^2$, proving (III). Q.E.D.

Example The dihedral group D_8 has elements

$$\{e, a, a^2, a^3, b, ba, ba^2, ba^3\}, \tag{6.14}$$

with the multiplication rules $a^4 = b^2 = e$ and $ba^j = a^{4-j}b$. Write $i = \sqrt{-1}$ and consider the elements

$$\begin{aligned} u_1 &= e + i^3a + i^2a^2 + ia^3 \\ u_2 &= b + iab + i^2a^2b + i^3a^3b \end{aligned} \quad \text{and} \quad \begin{aligned} v_1 &= b + i^3ab + i^2a^2b + ia^3b \\ v_2 &= e + ia + i^2a^2 + i^3a^3 \end{aligned} \tag{6.15}$$

in $\mathbb{C}D_8$. One sees by direct calculation that a and $b \in D_8$ act on these by

$$a : \begin{matrix} u_1 \mapsto iu_1 \\ u_2 \mapsto i^3u_1 \end{matrix}, \quad b : u_1 \leftrightarrow u_2 \tag{6.16}$$

and the same for v_1, v_2. Therefore u_1, u_2 and v_1, v_2 base 2 copies of the given representation of D_8 inside $\mathbb{C}D_8$. We certainly don't ever intend to carry out similar explicit hand calculations for groups of order bigger than about 10. Character theory is a miraculously neat substitute.

6.5 Proof of Main Theorem 5.6 (II)

For Main Theorem 5.6, I still need prove that the number k of irreducible representations equals the number r of conjugacy classes. The vector space $\mathcal{C}(G)$ of class functions has dimension r. As in 6.4, Corollary (b) let $\mathbb{C}G = \bigoplus_{i=1}^{k} d_i U_i$ be the decomposition of the regular representation into nonisomorphic irreducibles U_i, and write $\chi_i \in \mathcal{C}(G)$ for the character of U_i. Then the χ_i are orthonormal, so linearly independent, and $k \le r$.

Proposition *If $f \in \mathcal{C}(G)$ is a nonzero class function, then $\langle f, \chi_i \rangle \neq 0$ for one of the χ_i.*

The definition of Hermitian inner product gives $\langle \chi_i, f \rangle = \overline{\langle f, \chi_i \rangle}$. The proposition then gives that one of the $\mathbb{C}$-linear forms $\langle \chi_i, - \rangle \in \mathcal{C}^\vee$ is nonzero on every nonzero $f \in \mathcal{C}(G)$. Thus the χ_i all together define an injective map $\mathcal{C}(G) \hookrightarrow \mathbb{C}^k$, so $r \le k$, which proves Theorem 5.6 (II).

Step 1 Let V be a $\mathbb{C}G$-module with representation $\rho \colon G \to \mathrm{GL}(V)$. For a class function f on G, define the linear map

$$T = T_{f,V} \colon V \to V \quad \text{by} \quad T = \sum_{g \in G} \overline{f(g)} \rho(g). \tag{6.17}$$

Here $g \in G$ is summed out, so a priori this is only $\mathbb{C}$-linear. Nevertheless, the assumption that f is a class function makes T into a $\mathbb{C}G$-module homomorphism: Indeed, for $x \in G$,

$$\begin{aligned} \rho(x^{-1}) T \rho(x) &= \rho(x^{-1}) \sum_{g \in G} \overline{f(g)} \rho(g) \rho(x) \\ &= \sum_{g \in G} \overline{f(g)} \rho(x^{-1}) \rho(g) \rho(x) = \sum_{g \in G} \overline{f(g)} \rho(x^{-1} g x). \end{aligned} \tag{6.18}$$

Now $f(g) = f(x^{-1}gx)$ because $f \in \mathcal{C}(G)$, and summing over $x^{-1}gx$ instead of over $g \in G$ gives $\rho(x^{-1}) T \rho(x) = T$, that is $\rho(x) T = T \rho(x)$.

Step 2 The trace of $T_{f,V}$ equals $|G| \langle f, \chi_V \rangle$: in fact $\chi_V(g) = \operatorname{Tr} \rho(g)$ for any $g \in G$, and summing gives

$$\operatorname{Tr} T = \sum_{g \in G} \overline{f(g)} \chi_V(g) = |G| \langle f, \chi_V \rangle. \tag{6.19}$$

Step 3 Applying Step 1 to the regular representation $V^{\mathrm{reg}} = \mathbb{C}G$ gives a $\mathbb{C}G$-module homomorphism $T = T_{f,V^{\mathrm{reg}}}$. Now T sends the preferred element $1 \cdot e_G \in \mathbb{C}G$ to

$$T(1 \cdot e_G) = \sum_{g \in G} \overline{f(g)} \rho(g) e_G. \tag{6.20}$$

However $\rho(g) e_G$ is just g. The right hand side is thus $\sum \overline{f(g)} g$, and, since the g form a basis of $\mathbb{C}G$, this is nonzero if f is. Therefore $T = T_{f,V^{\mathrm{reg}}} \neq 0$.

Step 4 If $\mathbb{C}G = V^{\text{reg}} = \bigoplus a_i U_i$ is an irreducible decomposition, it follows that $T_{f,V^{\text{reg}}}$ must be nonzero on some summand U_i. By Schur's lemma $T_{f,V^{\text{reg}}}$ can only map U_i to another summand isomorphic to U_i by a multiple of the identity, and one of these multiples must be nonzero.

Now $T_{f,V^{\text{reg}}}$ restricted to the $\mathbb{C}G$-submodule U_i is the same thing as $T_{f,\chi_{U_i}}$. Thus at least one of the U_i has $T_{f,\chi_{U_i}} \neq 0$, hence $\langle f, \chi_i \rangle \neq 0$. Q.E.D.

Remark We can't apply V^{reg} directly. Indeed, although $T_{f,V^{\text{reg}}}$ is never zero, $\chi_{V^{\text{reg}}}$ is zero evaluated on every nonidentity $g \in \mathcal{C}(g)$.

Remark It follows a posteori that every class function $f \in \mathcal{C}(G)$ satisfies $f(g^{-1}) = \overline{f(g)}$, because this holds for a character χ_V (Proposition 5.4, (v)), and characters span $\mathcal{C}(G)$ by what we have just proved. However, a priori it is not at all obvious.

6.6 Column orthonormality

Main Theorem 5.6 (I) corresponds to row orthonormality in the character table of G:

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi_j(g) = \delta_{ij}. \tag{6.21}$$

This should by now be familiar.

The character table has a similar-looking column orthonormality property. Write $\{g_a\}_{a=1,\dots,r}$ for representatives of the conjugacy classes of G. Then

$$\sum_{i=1}^{k} \overline{\chi_i(g_a)} \chi_i(g_b) = |Z_G(g_a)| \delta_{ab} \quad \text{for } a, b \in 1, \dots, r, \tag{6.22}$$

where $Z_G(g_a)$ is the centraliser of g_a. Try it out in a few cases (e.g., those of 3.4 or Ex. 3.9).

This has no new content. I leave it to you to prove as an exercise based on row orthogonality, following the hints: The irreducible characters $\{\chi_i\}_{i=1,\dots k}$ are an orthonormal basis for the class functions $\mathcal{C}(G)$. A more obvious basis consists of the functions $f_a \in \mathcal{C}(G)$ for $a = 1, \dots, r$ given by $f_a(g_b) = \delta_{ab}$; in other words, score 1 on $g \in G$ for g conjugate to g_a, else 0.

Write the basis $\{f_a\}$ in terms of the χ_i: $f_a = \sum \lambda_{ai} \chi_i$. The coefficients λ_{ai} are of course determined by the Hermitian inner product

$$\lambda_{ai} = \langle \chi_i, f_a \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} f_a(g). \tag{6.23}$$

By what I said above, the sum in (6.23) is simply $\chi_i(g_a) \times \# C_G(g_a)$, where $C_G(g_a)$ is the conjugacy class of g_a, that is $\# C_G(g_a) = |G|/|Z_G(g_a)|$.

Now evaluate $f_a(g_b) = \delta_{ab}$ as $\sum \lambda_{ai} \chi_i(g_b)$ with $\lambda_{ai} = \chi_i(g_a) \times \# C_G(g_a)$ to deduce (6.22).

6.7 Character table of A_5 and the icosahedron

It is known, and discussed in detail later, that $A_5 \cong I_{60} \cong \mathrm{PSL}(2, \mathbb{F}_5)$, where I_{60} is the symmetry group of the icosahedron, and $\mathrm{PSL}(2, \mathbb{F}_5)$ is the projective special linear group. Here I work out its character table as an illustration of the methods developed so far. The same result is worked out by a completely different method in the homework, see Ex. 6.4–6.8. Amazingly, in either derivation, the whole character table follows from seemingly more-or-less trivial input!

The conjugacy classes in A_5 have representatives e (1 element), $a = (12)(34)$ (15 elements), $b = (123)$ (20 elements), $c = (12345)$ (12 elements), and $c^2 = (13524)$ (also 12 elements). The two 5-cycles c and c^2 are of course conjugate in S_5: consider the permutation

$$\begin{pmatrix} 12345 \\ 13524 \end{pmatrix} = (2354). \tag{6.24}$$

Then by the general principle, renumbering $1 \mapsto 1$, $2 \mapsto 3$ etc. transforms (12345) into (13524). Or in other words, composing permutations gives

$$(2354)c(2354)^{-1} = c^2. \tag{6.25}$$

However, (2354) is odd, and c, c^2 are not conjugate in A_5.

Recall that the icosahedron Δ has 20 triangular faces, 12 vertices and 30 edges. The conjugacy classes of I_{60} are the types of its symmetries. They are: the identity e (1 element), rotation a through π about the median through the midpoint of opposite edges (15 elements), rotation b through $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ about an axis through a pair of opposite faces (20 elements), rotation c through $\frac{2\pi}{5}$ or $\frac{8\pi}{5}$ about an axis through two opposite vertices (12 elements), and rotation c^2 through $\frac{4\pi}{5}$ or $\frac{6\pi}{5}$ (also 12 elements). The latter two types are obviously not conjugate in SO(3), because they have a completely different effect on the topology of the faces.

We know two representations of I_{60}: the 1-dimensional trivial representation, and the 3-dimensional representation on $\Delta \subset \mathbb{R}^3$. Working in $\mathbb{R}^3$, we see that a has diagonal form $\mathrm{diag}(1, -1, -1)$ so $\mathrm{Tr}\, a = -1$. The rotation b has diagonal form $\mathrm{diag}(1, \omega, \omega^2)$ where ω is the cube root of 1, so $\mathrm{Tr}\, b = 0$. The rotation c has diagonal form $\mathrm{diag}(1, \varepsilon, \varepsilon^4)$ where $\varepsilon = (\cos + i \sin)(\frac{2\pi}{5})$, and thus has trace the Golden Number $\alpha = 1 + 2\cos\frac{2\pi}{5} = \frac{1+\sqrt{5}}{2}$. (Please rework the material of Chapter 1 if you have forgotten this, esp. 1.3.) c^2 has diagonal form $\mathrm{diag}(1, \varepsilon^2, \varepsilon^3)$, so its trace is $1 - \alpha = 1 + 2\cos\frac{4\pi}{5} = 1 - 2\cos\frac{2\pi}{5} = \frac{1-\sqrt{5}}{2}$.

This gives the first two lines of the character table (6.39). Reading the conclusions of the main theorem, we know that there must be 3 more irreducible representations, of dimension n_3, n_4, n_5 satisfying

$$1^2 + 3^2 + n_3^2 + n_4^2 + n_5^2 = 60, \quad \text{so that} \quad n_3^2 + n_4^2 + n_5^2 = 50. \tag{6.26}$$

This equation only has one solution in positive integers, namely $3, 4, 5$. For example, the 3 squares have average value $16\frac{2}{3}$, so the biggest of them has to be > 4. Neither $n_5 = 7$ nor $n_5 = 6$ work, so $n_5 = 5$, leaving $n_3^2 + n_4^2 = 25$.

To find the other 3-dimensional irreducible representation, observe that the abstract group A_5 has a symmetry that interchanges c and c^2. For consider as before the 4-cycle $s = (2354)$ of (6.25). Conjugacy by s defines an inner automorphism $i\colon S_5 \to S_5$, that restricts to an outer automorphism of A_5 taking c to c^2. If $\rho_2\colon A_5 \to I_{60} \subset \mathrm{SO}(3) \subset \mathrm{GL}(3,\mathbb{C})$ is the representation of A_5 corresponding to the isomorphism with the icosahedral group, the map $\rho_3 = \rho_2 \circ i$ is a different representation that interchanges the roles of c and c^2. This gives χ_3 in the third line of the character table (6.39).

We can derive either of χ_4 and χ_5 separately from the orthonormality relations together with an entertaining calculation.

χ_4: suppose the entries are $4, x, y, z, t$. Then orthogonality gives

$$\langle \chi_1, \chi_4 \rangle = 0 \implies 4 + 15x + 20y + 12z + 12t = 0 \tag{6.27}$$
$$\langle \chi_2, \chi_4 \rangle = 0 \implies 12 - 15x + 12\alpha z + 12\beta t = 0 \tag{6.28}$$
$$\langle \chi_3, \chi_4 \rangle = 0 \implies 12 - 15x + 12\beta z + 12\alpha t = 0 \tag{6.29}$$
$$\langle \chi_4, \chi_4 \rangle = 60 \implies 16 + 15x^2 + 20y^2 + 12z^2 + 12t^2 = 60 \tag{6.30}$$

subtracting (6.28) $-$ (6.29) gives $z = t$. Now note that $\alpha + \beta = 1$, and treat (6.28) and (6.27) $+$ (6.28) as equations for x and y:

$$5x = 4 + 4z \quad \text{and} \quad 5y = -4 - 9z \tag{6.31}$$

Substituting for x, y, t in $5 \times$ (6.30) gives a quadratic equation

$$80+3(4+4z)^2+4(-4-9z)^2+120z^2 = 300 \implies (1+z)(108-492z) = 0, \tag{6.32}$$

that is, $z = -1$, giving χ_4. (The solution $z = 9/41$ is eliminated because the value of a character is a sum of roots of 1, so an algebraic integer.)

χ_5: suppose the entries are $5, x, y, z, t$. Then orthogonality gives

$$\langle \chi_1, \chi_5 \rangle = 0 \implies 5 + 15x + 20y + 12z + 12t = 0 \tag{6.33}$$
$$\langle \chi_2, \chi_5 \rangle = 0 \implies 15 - 15x + 12\alpha z + 12\beta t = 0 \tag{6.34}$$
$$\langle \chi_3, \chi_5 \rangle = 0 \implies 15 - 15x + 12\beta z + 12\alpha t = 0 \tag{6.35}$$
$$\langle \chi_5, \chi_5 \rangle = 60 \implies 25 + 15x^2 + 20y^2 + 12z^2 + 12t^2 = 60 \tag{6.36}$$

subtracting (6.34) $-$ (6.35) gives $z = t$. As before treating (6.34) and (6.33) $+$ (6.34) as equations for x and y gives

$$5x = 5 + 4z \quad \text{and} \quad 5y = -5 - 9z \tag{6.37}$$

Substituting for x, y, t in $5 \times$ (6.30) gives a quadratic equation

$$125 + 3(5+4z)^2 + 4(-5-9z)^2 + 120z^2 = 300 \implies (492z + 480)z = 0 \tag{6.38}$$

that is, $z = 0$, giving χ_5. (Since $z = -40/41$ is impossible.)

name	e	a	b	c	c^2
size	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	β
χ_3	3	-1	0	β	α
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

(6.39)

Figure 3: Character table of A_5. Here $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

6.8 Homework to Chapter 6

6.1. More hints for Ex. 4.19 I proposed the challenge problem of the presentation $G = \langle a, b \mid a^2 = bab, b^2 = aba \rangle$. As we saw, if the central element $c = abab = a^3 = b^3$ has order 2 then the group has order 24 and is a central extension of A_4 isomorphic to BT_{24}. I'm sure that you've all got out the challenge.

My deceptively elegant solution hides 20 pages of preliminary scribbles: start from ab^2a and successively replace the underlined bits to make *longer* words:

$$a\underline{b^2}a = \underline{a^2}ba^2 = ba\underline{b^2}a^2 = (ba^2b)a^3$$
$$\text{and symmetrically} \quad b\underline{a^2}b = \underline{b^2}ab^2 = ab\underline{a^2}b^2 = (ab^2a)b^3. \tag{6.40}$$

(These involved six applications of the relations.) Putting these together gives

$$ab^2a = (ba^2b)a^3 = (ab^2a)b^3a^3 \implies b^3a^3 = e \quad \text{so that} \quad c^2 = e. \tag{6.41}$$

6.2. Product with a linear character Three names for the same thing:

(1) A homomorphism $\alpha\colon G \to \mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$.

(2) A linear character of G.

(3) A 1-dimensional representation of G.

If V is a $\mathbb{C}G$-module with representation $\rho\colon G \to \mathrm{GL}(V)$ and α a linear character, show the product $\alpha\rho\colon G \to \mathrm{GL}(V)$ defined by $g \mapsto \alpha(g)\rho(g)$ is another representation; here $\alpha(g)\rho(g)$ is multiplication of matrix $\rho(g)$ by the scalar $\alpha(g) \in \mathbb{C}^\times$. Determine the character $\chi_{\alpha\rho}$ in terms of α and χ_ρ.

6.3. The conclusion from Ex. 6.2 is that multiplication by 1-dimensional characters is a symmetry of the representation theory of G. Recall that the binary tetrahedral group $\mathrm{BT}_{24} \subset \mathrm{GL}(2, \mathbb{C})$ has

$$H_8 \triangleleft \mathrm{BT}_{24} \twoheadrightarrow \mathbb{Z}/3, \tag{6.42}$$

and has 3 linear characters corresponding to the 3 possible homomorphisms $\mathbb{Z}/3 \to \mathbb{C}^\times$. Deduce that BT_{24} has three 2-dim representations. Calculate their characters. Compare Ex. 5.8.

6.4. Permutation representation of G A homomorphism $\sigma\colon G \to S_n$ takes each $g \in G$ to a permutation $\sigma(g)$ of $\{1,\dots,n\}$. The associated linear representation is $\rho_\sigma\colon G \to \mathrm{GL}(n,K)$, where $\rho_\sigma(g)$ permutes the basis of K^n by $e_i \mapsto e_{\sigma(i)}$. Each $\rho_\sigma(g)$ is then a permutation matrix. Show how to calculate its trace.

6.5. 4-dimensional character of A_5 Calculate the character χ of the natural permutation action of A_5 on K^5 (that is, calculate its value on $a = (12)(34)$, $b = (123)$, $c = (12345)$ and $c^2 = (13524)$). By calculating $\langle\chi,\chi\rangle$ and using the Main Theorem, show that this representation splits as a direct sum of two irreducible representations. Deduce from this that A_5 has a 4-dimensional irreducible representation, and calculate its character. Use the conjugacy classes

	e	$a=(12)(34)$	$b=(123)$	$c=(12345)$	$c^2(13524)$
size	1	15	20	12	12

(6.43)

6.6. Permutation action of $A_5 \to S_6$ The 5-cycles of A_5 fall into 6 cyclic subgroups $\mathbb{Z}/5$. Conjugacy by A_5 defines a permutation action of A_5 on the 6 subgroups $\mathbb{Z}/5$. Calculate the character of this action. [Hint. You need to determine when conjugacy by $(25)(34)$ takes a 5-cycle $\sigma = (12abc)$ into an element of the same subgroup $\langle\sigma\rangle$, and similarly for (123).]

6.7 For the permutation representation of Ex. 6.6, calculate $\langle\chi,\chi\rangle$ and deduce that this representation also splits as a direct sum of two irreducible representations. Show that A_5 has a 5-dimensional irreducible representation and calculate its character.

6.8 A_5 has the trivial character and the two characters χ_4 and χ_5 calculated in Ex. 6.5 and Ex. 6.7. Deduce from the Main Theorem that it must also have two 3-dimensional characters. If $3,p,q,r,s$ is one of these, write out the orthonormality relations provided by the Main Theorem, and determine the values of p,q, and the values of r,s as the roots of a quadratic equation.

This gives an alternative derivation of the Character Table of A_5, also based on seemingly trivial input.

6.9. We know the conjugacy classes of S_5:

	e	(12)	$(12)(34)$	(123)	(1234)	(12345)	$(12)(345)$
size	1	10	15	20	30	24	20

(6.44)

Use earlier results of the course to prove that S_5 has exactly two 1-dimensional representations.

Calculate the character of the natural permutation representation of S_5 on $\mathbb{C}^5$. Prove that it is the direct sum of two irreducible representations and deduce that S_5 has an irreducible representation on $\mathbb{C}^4$. Write out its character.

Write out the character of the sign representation of S_5. Prove that S_5 has a second nonisomorphic irreducible 4-dimensional representation. Use the sum of squares formula plus the above to deduce that the remaining irreducible representations of S_5 have dimension $5, 5, 6$.

6.10. Past exam question

(a) Let G be a group and U a $\mathbb{C}G$-module. Define what it means for U to be irreducible. [**1**]

If U_1 and U_2 are irreducible $\mathbb{C}G$-modules and $\alpha\colon U_1 \to U_2$ is a $\mathbb{C}G$-module homomorphism, prove that α is either zero or an isomorphism. [**2**]

If U is an irreducible $\mathbb{C}G$-module, describe $\operatorname{Hom}_{\mathbb{C}G}(U, U)$. [**2**]

(b) The symmetric group S_3 has generators $s_1 = (12)$ and $s_2 = (23)$ with defining relations $s_1^2 = s_2^2 = (s_1s_2)^3 = e$. Prove that S_3 has an irreducible 2-dim representation U defined by $s_1 \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $s_2 \mapsto \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$. [**2**]

Assume the corollary of Maschke's theorem that every $\mathbb{C}G$-module is a direct sum of irreducibles.

(c) Let $U_1, \dots, U_k$ be nonisomorphic irreducible $\mathbb{C}G$-modules; suppose that

$$V = \bigoplus_{i=1}^{k} n_i U_i \quad \text{and} \quad W = \bigoplus_{j=1}^{k} m_j U_j$$

(here $n_iU_i = U_i^{\oplus n_i}$ denotes the direct sum of n_i copies of U_i). Calculate $\dim \operatorname{Hom}_{\mathbb{C}G}(V, W)$. [**2**]

(d) Define the regular representation $\mathbb{C}G$ of G. [**2**]

Assume as known that for any $\mathbb{C}G$-module V there is a canonically defined isomorphism of $\mathbb{C}$-vector spaces $\operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G, V) \cong \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, V) \cong V$.

(e) Prove that there are finitely many nonisomorphic irreducible $\mathbb{C}G$-modules U_i up to isomorphism, and that their dimensions $d_i = \dim U_i$ satisfy $\sum d_i^2 = |G|$. [**4**]

(f) The regular representation of S_3 contains 2 copies of the irreducible representation U discussed in (b). Find them explicitly following these indications (or otherwise): Order the elements of S_3 as

$$\{x_1, x_2, x_3, x_4, x_5, x_6\} = \{e, (12), (13), (23), (123), (132)\}.$$

Left multiplication by the generators s_1 and s_2 correspond to certain permutations. Search for vectors f_1, f_2 in the form $x_i + x_j - x_k - x_l$ such that (a) f_1 is invariant under s_2; (b) s_1 acts by $f_1 \leftrightarrow f_2$; and (c) s_2 acts by $f_2 \leftrightarrow -f_1 - f_2$. **[10]**

6.11. Puzzle corner A partial character table of G:

	e	u	v	w	x	y
size	1	1	2	2	3	$*$
χ_1	1	1	1	1	1	$*$
χ_2	1	1	1	1	-1	$*$
χ_3	1	-1	1	-1	i	$*$
χ_4	1	-1	1	-1	$-i$	$*$
χ_5	2	-2	-1	-1	0	$*$
χ_6	$*$	$*$	$*$	$*$	$*$	$*$

(6.45)

(i) Recover the missing information.

(ii) Prove that order $u = 2$, order $x = 4$, order $w = 6$, order $v = 3$, and find order y.

(iii) Show that the subgroup $\langle v \rangle$ is normal.

(iv) Determine G.

7 Finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO}(3,\mathbb{R})$, the icosahedron

The dihedral group, the rotational symmetry group of the regular tetrahedron, the cube (or octahedron) and icosahedron and their binary covers appear as standard examples throughout elementary group theory (see 2.2–2.3, Ex. 2.7, 3.4, 5.7, 6.7, etc.). It is interesting to write out the finite subgroup of $\mathrm{SO}(3,\mathbb{R})$ and their binary coverings in $\mathrm{SL}(2,\mathbb{C})$, insofar as possible as one integrated list. The treatment should include abstract presentations, representations as linear groups over finite fields, representations as permutation groups, and as matrix groups over $\mathbb{R}$ and $\mathbb{C}$.

7.1 Subgroups of $\mathrm{SO}(3)$

The groups are

$$\begin{array}{ccccccccccc} \mathrm{BD}_{4m} & & H_8 & \subset & \mathrm{BT}_{24} & \subset & \mathrm{BO}_{48} & & \mathrm{BT}_{24} & \subset & \mathrm{BI}_{120} \\ \downarrow & \text{and} & \downarrow & & \downarrow & & \downarrow & \text{and} & \downarrow & & \downarrow \\ D_{2m} & & V_4 & \subset & A_4 & \subset & S_4 & & A_4 & \subset & A_5 \end{array} \tag{7.1}$$

The groups in the top line have the central element c with $c^2=e$, and those in the bottom line are the quotients by $\langle c\rangle$. When I treat them as matrix groups in $\mathrm{SL}(2,\mathbb{C})$, the central element c will be $-I_2\in\mathrm{SL}(2,\mathbb{C})$. The dihedral cases $\mathrm{BD}_{4m}\to D_{2m}$ are already treated in 2.2. The binary icosahedral group BI_{120} and its quotient A_5 is more complicated than the others, mainly because A_5 is simple.

The finite subgroups of $\mathrm{SO}(3,\mathbb{R})$ are the groups of rotations of regular solids: the cyclic groups, the dihedral groups, and the groups of rotation of the regular tetrahedron, the regular octahedron and the regular icosahedron. (The cube and the regular octahedron have the same symmetry group, and likewise for the dodecahedron and the icosahedron.) Each of these groups has a double cover in $\mathrm{SU}(2)$ or $\mathrm{SL}(2,\mathbb{C})$. Here a *double cover*[1] means a central extension by the centre $\langle c\rangle$ or $\mathbb{Z}/2=\{\pm I_2\}\subset\mathrm{SL}(2,\mathbb{C})$.

Starting with the bottom line, the groups are the following:

[1]We don't especially need this, but for the curious, the binary covers come from a general property of $\mathrm{SO}(3)$. A general $A\in\mathrm{SU}(2)$ has the form $A=\left(\begin{smallmatrix} a+ib & c+id \\ -c+id & a-ib\end{smallmatrix}\right)$ with $a,b,c,d\in\mathbb{R}$ such that $a^2+b^2+c^2+d^2=1$. There is a 2-to-1 cover $\mathrm{SU}(2)\to\mathrm{SO}(3,\mathbb{R})$ defined by

$$A\mapsto\begin{pmatrix} a^2+b^2-c^2-d^2 & -2ad+2bc & 2ac+2bd \\ 2ad+2bc & a^2-b^2+c^2-d^2 & -2ab+2cd \\ -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2\end{pmatrix} \tag{7.2}$$

This comes by making A act by $Q\mapsto AQA^{-1}$ on the vector space $\mathbb{R}^3$ made up of matrices $Q=\left(\begin{smallmatrix} ix & y+iz \\ -y+iz & -ix\end{smallmatrix}\right)$. (Or if you prefer, a unit quaternion $A\in\mathrm{SU}(2)$ acts on the imaginary quaternions $Q\in\mathbb{R}^3$ by AQA^{-1}.) It is clear that $-1\in\mathrm{SU}(2,\mathbb{C})$ goes to the identity. One checks that the image is in $\mathrm{SO}(3,\mathbb{R})$, and that this defines a homomorphism $\mathrm{SU}(2)\to\mathrm{SO}(3,\mathbb{R})$ with kernel -1 that is a 2-to-1 cover.

- $V_4 =$ Klein 4-group $= \mathbb{Z}/2 \oplus \mathbb{Z}/2$;
- $A_4 = V_4 \rtimes \mathbb{Z}/3 \cong$ Alternating group(4) $\cong$ rotations of regular tetrahedron $\cong \left\langle a, b \,\middle|\, a^2, b^3, (ab)^3 \right\rangle \cong \mathrm{PSL}(2, \mathbb{F}_3)$;
- $S_4 = V_4 \rtimes S_3 =$ Symmetric group(4) $\cong$ rotations of octahedron (or cube) $\cong \left\langle a, b \,\middle|\, a^2, b^3, (ab)^4 \right\rangle \cong \mathrm{PGL}(2, \mathbb{F}_3)$;
- $A_5 =$ Alternating group(5) $\cong$ rotations of icosahedron (or dodecahedron) $\cong \left\langle a, b \,\middle|\, a^2, b^3, (ab)^5 \right\rangle \cong \mathrm{PSL}(2, \mathbb{F}_5)$.

Each has many possible presentations as abstract group. For example, everyone knows how to generate S_4 by 3 transpositions s_1, s_2, s_3 satisfying the "Coxeter relations" $s_1^2 = s_2^2 = s_3^2 = e$, $(s_1 s_2)^3 = (s_2 s_3)^3 = e$. However, for my purpose, the effective method of argument is to build up from the smallest group.

The 4-group V_4 has 3 nonzero elements x, y, z satisfying $x^2 = y^2 = z^2 = xyz = e$; it is a little exercise to check that these imply $xy = yx$, so that $V_4 = \left\langle x, y, z \,\middle|\, x^2, y^2, z^2, xyz \right\rangle$. Setting $z = y^{-1}x^{-1}$ would shorten the list of generators, but spoil the symmetry of the presentation.

As abstract groups, A_4 and S_4 appear naturally from the symmetry of V_4. In brief, they are semidirect products $A_4 = V_4 \rtimes \mathbb{Z}/3$ and $S_4 = V_4 \rtimes S_3$ where $\mathbb{Z}/3$ is the 3-fold cyclic symmetry (x, y, z) and S_3 the full symmetric group on (x, y, z).

In more detail, a *semidirect product* $H \rtimes F$ occurs when F, H are groups and $\alpha \colon F \to \operatorname{Aut} H$ a homomorphism, so that F acts as symmetries of H. Then $H \rtimes F$ is the direct product of sets $G = H \times F$, with binary operation $G \times G \to G$

$$(h_1, f_1) \cdot (h_2, f_2) = (\alpha(f_2)(h_1) h_2, f_1 f_2). \tag{7.3}$$

In other words, rather than just the direct product multiplication, we make f_2 act on h_1 before calculating the first component. It follows that G has the structure

$$H \lhd G \twoheadrightarrow F, \tag{7.4}$$

with the quotient F acting by conjugacy on the normal subgroup H by the stated action α. A number of little exercises identify the semidirect products with the above groups A_4 and S_4.

As illustration, consider the 3-cycle (x, y, z) as a symmetry of V_4. It allows us to construct a bigger group $G = \langle V_4, a \rangle$ with a new generator a satisfying $a^3 = e$, so that G has 3 cosets V_4, aV_4, a^2V_4, and conjugacy by a act by the 3-cycle (x, y, z). Thus $G = \langle V_4, a \rangle = V_4 \rtimes \mathbb{Z}/3$. Spelling this out in full gives the presentation

$$G = \left\langle x, y, z, a \,\middle|\, x^2, y^2, z^2, xyz, a^3, axa^{-1} = y, aya^{-1} = z, aza^{-1} = x \right\rangle \tag{7.5}$$

(long-winded, but symmetric).

Exercise 1 Prove that this group is isomorphic to A_4. [Hint: Take $x = (12)(34)$ and $a = (132)$. Invent suitable y, z so that x, y, z, a generate G and satisfy the same relations.]

Exercise 2 Show from first principles that the group G with presentation (7.5) has exactly 4 subgroups H_1, H_2, H_3, H_4 of order 3. Show that they are not normal, and that conjugacy by G permutes the $\{H_i\}$ as the alternating group A_4.

Exercise 3 Do the same for S_4 as the semidirect product $S_4 = V_4 \rtimes S_3$.

There are lots of interesting puzzles about making this compatible with other presentations of the groups, esp. the $\mathrm{PSL}(2,\mathbb{F}_3)$ and $\mathrm{PSL}(2,\mathbb{F}_3)$ models in the projective group over the finite field $\mathbb{F}_3$ and the $(2,3,3)$ and $(2,3,4)$ triangle groups of 7.2 of the notes. The icosahedral group relates in the same way to the alternating group A_5, the matrix group $\mathrm{PSL}(2,\mathbb{F}_5)$ over $\mathbb{F}_5$ and the $(2,3,5)$ triangle group.

7.2 The triangle groups $G_{2,3,r}$

Consider the finitely presented groups

$$G_r = G_{2,3,r} = \langle x, y \mid x^2 = y^3 = (xy)^r = e \rangle \quad \text{for } r \geq 2. \tag{7.6}$$

These are finite for $r \leq 5$, and infinite for $r \geq 6$. In fact

G_2 is the dihedral group $D_6 = S_3 \cong \mathrm{PSL}(2,\mathbb{F}_2)$.

G_3 is the tetrahedral group $T_{12} = A_4 \cong \mathrm{PSL}(2,\mathbb{F}_3)$.

G_4 is the octahedral group $O_{24} \cong \mathrm{PGL}(2,\mathbb{F}_3)$.

G_5 is the icosahedral group $I_{60} = A_5 \cong \mathrm{PSL}(2,\mathbb{F}_5)$.

G_6 is the rotational symmetry group of the tesselation of $\mathbb{R}^2$ by regular triangles, or the honeycomb lattice in $\mathbb{R}^2$. It contains $\mathbb{Z}^2$ as translational subgroup, and has the structure of semidirect product $G_6 = \mathbb{Z}^2 \rtimes \mathbb{Z}/6$.

G_7 is the rotational symmetry group of the tesselation of hyperbolic space $\mathcal{H}^2$ by hyperbolic triangles with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{7}$, or by regular heptagons with angle $\frac{2\pi}{3}$. It contains a normal subgroup Π of index 168 acting freely on $\mathcal{H}^2$, with quotient group the simple group G_{168} that we discuss in detail later. (In fact Π is the topological fundamental group of a Riemann surface of genus 3.)

For $r \geq 7$, the group G_r has a similar description in terms of the hyperbolic triangle with angles $\pi/2$, $\pi/3$, π/r. I don't have anything more to say about them (but compare Ex. 7.12).

Working with these presentations is fairly labour-intensive. Compare for example Ex. 7.12 for a pedestrian proof that $G_{2,3,r}$ has the right order when $r = 3$ or 4 or 5. (This continues my obsession with the material surrounding Ex. 4.19.)

7.3 Case $r = 6$: the regular triangular lattice

I start with the case $G_{2,3,6}$, in many ways the most instructive: Work with the

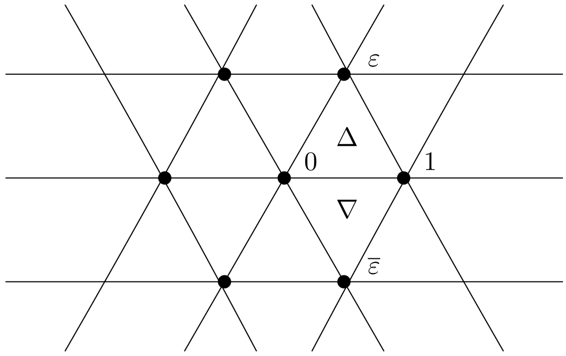


Figure 4: The regular triangular lattice $\mathbb{Z} \oplus \mathbb{Z}\varepsilon$

regular triangular lattice in $\mathbb{C} = \mathbb{R}^2$, based by 1 and $\varepsilon = \exp \frac{2\pi i}{6}$. Note that $\varepsilon = \frac{1+\sqrt{-3}}{2}$ and $\overline{\varepsilon} = \varepsilon^{-1} = \frac{1-\sqrt{-3}}{2} = 1 - \varepsilon$. Write Δ for the first triangle with vertices $0, 1, \varepsilon$ and ∇ for its reflection in the x-axis, the triangle with vertices $0, 1, \overline{\varepsilon}$. Then set

$$x = \mathrm{Rot}((\tfrac{1}{2}, 0), \pi)\colon z \mapsto 1 - z \tag{7.7}$$

for the half turn about the midpoint of the interval $[0, 1]$, and

$$r = \mathrm{Rot}((0, 0), -\tfrac{2\pi}{6})\colon z \mapsto \varepsilon^{-1} z = (1 - \varepsilon) z \tag{7.8}$$

for the clockwise rotation by 60^{o} of the plane around 0.

Clearly r rotates Δ down to ∇ around the vertex 0, whereas x turns Δ and ∇ upside down and interchanges them. The composite xr thus takes Δ to itself. In fact if we set $y = xr$, then $y\colon z \mapsto 1 - \varepsilon^{-1} z$ rotates Δ anticlockwise by $\frac{2\pi}{3}$ about its centroid $\frac{1+\varepsilon}{3}$:

$$\begin{array}{ccccc} & r & & x & \\ 0 & \mapsto & 0 & \mapsto & 1 \\ 1 & \mapsto & \overline{\varepsilon} & \mapsto & \varepsilon \\ \varepsilon & \mapsto & 1 & \mapsto & 0 \end{array} \tag{7.9}$$

Thus $x^2 = \mathrm{Id}_{\mathbb{R}^2}$, $y^3 = \mathrm{Id}_{\mathbb{R}^2}$ and $r = xy$ satisfies $r^6 = \mathrm{Id}_{\mathbb{R}^2}$.

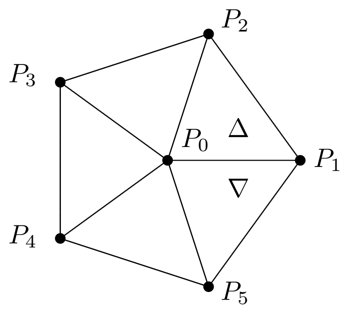
The group generated by x, y contains the translation lattice $\mathbb{Z} \oplus \mathbb{Z}\varepsilon$ as a subgroup: indeed, $r^3\colon z \mapsto -z$ is half-turn around 0, so that xr^3 is translation by 1, and its conjugate $r^{-1}xr^4$ is translation by ε (please check). One checks that the translation lattice $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}\varepsilon$ is a normal subgroup of index 6 with

$$\mathbb{Z}^2 \lhd G_6 \twoheadrightarrow \mathbb{Z}/6, \tag{7.10}$$

and that $G_{2,3,6}$ has the structure of semidirect product $G_6 = \mathbb{Z}^2 \rtimes \mathbb{Z}/6$. The quotient $\mathbb{Z}/6$ corresponds to the rotational symmetries around the origin, and is generated by r.

7.4 G_5 is the icosahedral group

The same argument gives the relation between G_5 and the icosahedral group I_{60}. Indeed, suppose that we put together 5 regular triangles of angle $\pi/3$ around one vertex. That doesn't fit in Euclidean plane geometry, but it is exactly what happens at each vertex of the icosahedron.



(7.11)

The picture is the projection to the plane of 5 regular triangles

$$\Delta = P_0P_1P_2, \quad P_0P_2P_3, \quad \dots, \quad \nabla = P_0P_5P_1 \tag{7.12}$$

in $\mathbb{R}^3$ around the vertex P_0 of the icosahedron. The triangles are regular in $\mathbb{R}^3$, but of course not in the plane, because in the plane figure, the angles at P_0 are $\frac{2\pi}{5} > \frac{2\pi}{6}$. (Alternatively, we could draw this by radial projection outwards onto the unit sphere S^2, which would make the triangles into regular spherical triangles with 3 angles equal to $\frac{2\pi}{5}$. By the angular defect formula, the area of each triangle must be $\frac{\pi}{5}$, which is $\frac{1}{20}$th the area of the sphere.)

The axis through the midpoint of P_0P_1 and its antipodal point is a median line. Set x to be the rotation through π about this axis. As before, it inverts the edge P_0P_1 and swivels Δ into ∇ and vice versa.

Set r to be the clockwise rotation through $\frac{2\pi}{5}$ about the axis through the vertex P_0. It rotates Δ down to ∇ around the vertex 0. It follows exactly as before that $y = xr$ is a symmetry of the icosahedron that takes triangle Δ to itself, permuting the vertices by (P_0, P_1, P_2), so it is the anticlockwise rotation through $\frac{2\pi}{3}$ around the axis through the midpoint of Δ

Thus x, y and $r = xy$ satisfy the relations $x^2 = e$, $y^3 = e$, $r^5 = e$ of G_5.

Why is $I_{60} \cong A_5$? The key point for the isomorphism $I_{60} = A_5$ is to see that *the* 30 *edges of the icosahedron make up* 15 *parallel pairs, that break up as* 5 *orthogonal frames of* $\mathbb{R}^3$. The group I_{60} permutes these 5 frames, and this permutation action defines an isomorphism of I_{60} and A_5. I give a somewhat extravagant treatment of this in coordinates, with the 5 frames given in (7.19). If you take $I_{60} = A_5$ on trust, this material is not needed for the rest of the course. I don't have time to express it more concisely.

7.5 The generators of I_{60} as matrices

I write out I_{60} as a matrix group. Choose coordinates with P_0 on the x-axis, making r the standard rotation matrix in the plane $\mathbb{R}^2_{\langle y,z\rangle}$. Choose the scale of the icosahedron so that its vertices lie on the sphere S^2 of radius $\sqrt{5}$, which puts the North Pole at $P_0 = (\sqrt{5}, 0, 0)$. It turns out (see below) that the neighbouring vertices $P_1, \dots, P_5$ are then on the plane $x = 1$, and I choose $P_1 = (1, 2, 0)$ on the plane $z = 0$.

Proposition *With the above conventions, the clockwise rotation matrix R and the matrix rotating $\mathbb{R}^3$ through π about the midpoint of P_0P_1 are given by*

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2\pi}{5} & \sin\frac{2\pi}{5} \\ 0 & -\sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{pmatrix}, \quad X = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -\sqrt{5} \end{pmatrix} \tag{7.13}$$

Then X, R and $Y = XR$ satisfy the relations $X^2 = e$, $Y^3 = e$, $R^5 = e$ of G_5.

Proof This is fairly easy. R is the standard rotation matrix that fixes the north-south axis through P_0 (see (7.11)), and rotates the 5 vertices $P_{1\dots5}$ as a 5-cycle. One checks that X interchanges the two column vectors $P_0 = (\sqrt{5}, 0, 0)$ and $P_1 = (1, 2, 0)$, and that $X^2 = \mathrm{Id}_3$. Arguing as in 7.3 one sees that the composite map defined by $Y = XR$ is a 3-fold rotation doing $N \mapsto P_1 \mapsto P_2 \mapsto N$, so that $Y^3 = \mathrm{Id}_3$. Q.E.D.

7.6 Triangle group presentations for the binary groups

The binary groups forming the top line of (7.1) are obtained as abstract groups from the triangle group presentations by a uniform trick: in each case, we replace the relations $x^2 = y^3 = (xy)^r = e$ for G_r by the relations $x^4 = e$ and $y^3 = (xy)^r = x^2$ for the binary group BG_{2r}. Then x^2 is a central element (it commutes with x and y because it is a power of either) of order 2, and $\mathrm{BG}_{2r} \,/\, \langle x^2 \rangle = G_r$.

Compare the Magma code

```
F2<x,y> := FreeGroup(2);
for r in [2..5] do
G := quo< F2 | x^2, y^3, (x*y)^r >; Order(G);
BG := quo< F2 | x^4, y^3 = x^2, (x*y)^r = x^2 >; Order(BG);
end for;
```

7.7 The binary group BO_{48} as a matrix group in $\mathrm{SU}(2)$

2.3 discussed the quaternion group H_8 and the binary tetrahedral group BT_{24} as matrix groups. Recall that BT_{24} consists of the 8 quaternion elements $\{\pm1, \pm I, \pm J, \pm K\}$ together with the 24 matrices $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with $a, b = \frac{\pm1\pm i}{2}$.

For BO_{48} we just need to add the diagonal matrix $Z = \left(\begin{smallmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{smallmatrix}\right)$, where $\zeta_8 = \frac{1+i}{\sqrt{2}} = \exp\frac{2\pi}{8}$ is the standard primitive 8th root of 1 with $\zeta_8^2 = i$. The statement is that BO_{48} has BT_{24} as a normal subgroup of index 2, and is the union of two cosets $\mathrm{BT}_{24} \sqcup Z\,\mathrm{BT}_{24}$. Conjugacy by Z is the automorphism that interchanges the two generators $A = \frac{1}{2}\left(\begin{smallmatrix} 1+i & 1+i \\ -1+i & 1-i \end{smallmatrix}\right)$ and $B = \frac{1}{2}\left(\begin{smallmatrix} 1+i & -1+i \\ 1+i & 1-i \end{smallmatrix}\right)$.

7.8 The binary icosahedral group BI_{120} as a matrix group

As a matrix group, BI_{120} is generated by the two matrices

$$Z = \begin{pmatrix} -\varepsilon^3 & 0 \\ 0 & -\varepsilon^2 \end{pmatrix} = \begin{pmatrix} \exp\frac{\pi i}{5} & 0 \\ 0 & \exp -\frac{\pi i}{5} \end{pmatrix} \quad \text{and} \tag{7.14}$$

$$X = \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon + \varepsilon^4 & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix} = \frac{2i}{\sqrt{5}} \begin{pmatrix} -\sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & \sin\frac{2\pi}{5}. \end{pmatrix} \tag{7.15}$$

They satisfy $X^2 = Z^5 = (X * Z)^3 = -\mathrm{Id}$

7.9 Appendix. Coordinates for the icosahedron

Everything we need for the icosahedron can be expressed in terms of the single surd quantity $s = \sqrt{\frac{5-\sqrt{5}}{8}} = \sin\frac{\pi}{5} \approx 0.5878$, that has minimal polynomial $(8s^2-5)^2 - 5$:

$$\begin{gathered} \sqrt{5} = 5 - 8s^2, \quad \cos\tfrac{\pi}{5} = \tfrac{3}{2} - 2s^2 = \tfrac{1+\sqrt{5}}{4}, \quad \sin\tfrac{\pi}{5} = s, \\ \cos\tfrac{2\pi}{5} = 1 - 2s^2 = \tfrac{-1+\sqrt{5}}{4}, \quad \sin\tfrac{2\pi}{5} = 3s - 4s^3. \end{gathered} \tag{7.16}$$

Quite a number of bizarre identities turn up in manipulating these. For example, you might enjoy the exercise of checking that $\sin^2\frac{2\pi}{5} + \sin^2\frac{\pi}{5} = \frac{5}{4}$.

For the icosahedron in explicit coordinates, start from the antiprism formed by the regular pentagon in the circle of radius 2 at height 1, and its negative (see Figure 5). The antiprism is inscribed in the sphere of radius $\sqrt{5}$ and has obvious 5-fold rotational symmetry.

To make the icosahedron, just add the North and South poles, giving the 12 vertices

$$\begin{array}{cccc} & P_1 = (1,2,0), & Q_1 = (-1,-2,0), & \\ & P_2 = (1, 2\cos\frac{2\pi}{5}, 2\sin\frac{2\pi}{5}), & Q_2 = -P_2, & \\ N = (\sqrt{5},0,0), & P_3 = (1, -2\cos\frac{\pi}{5}, 2\sin\frac{\pi}{5}), & Q_3 = -P_3, & S = (-\sqrt{5},0,0). \\ & P_4 = (1, -2\cos\frac{\pi}{5}, -2\sin\frac{\pi}{5}), & Q_4 = -P_4, & \\ & P_5 = (1, 2\cos\frac{2\pi}{5}, -2\sin\frac{2\pi}{5}), & Q_5 = -P_5, & \end{array}$$

I check that all the edges of the icosahedron have length $4s = \sqrt{10 - 2\sqrt{5}}$, so its faces are regular triangles in $\mathbb{R}^3$: for P_3P_4 this is more-or-less the definition

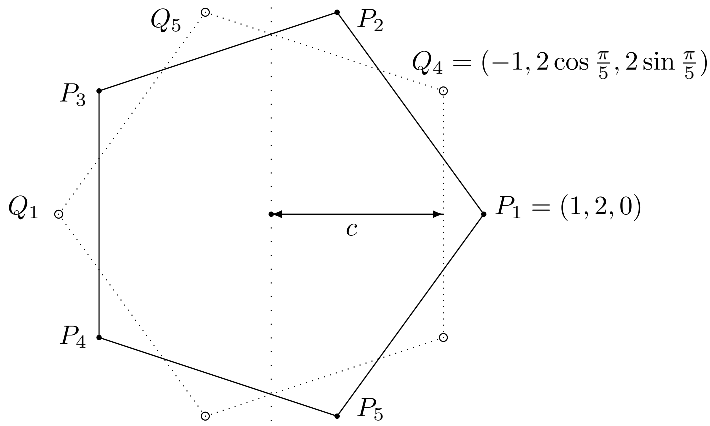


Figure 5: The regular pentagon in the unit circle

of $s = \sin\frac{\pi}{5}$, and it follows by symmetry for all the sides in the two planes $x = \pm 1$. Then calculate $(NP_1)^2 = (\sqrt{5}-1)^2 + 2^2 = 10 - 2\sqrt{5} = (4s)^2$ and

$$(P_1Q_4)^2 \;=\; 4\big(1 + (1-\cos\tfrac{\pi}{5})^2 + \sin^2\tfrac{\pi}{5}\big) \;=\; 4(3 - 2\cos\tfrac{\pi}{5}) \;=\; (4s)^2. \quad (7.17)$$

This gives 3-fold rotational symmetry, for example the element y that rotates the regular triangle $\Delta = \triangle NP_1P_2$ by $N \mapsto P_1 \mapsto P_2 \mapsto N$ as in 7.2. The three faces adjacent to Δ are NP_1P_5, $P_1P_2Q_4$ and NP_2P_3; so the rotation y must take $P_3 \mapsto P_5 \mapsto Q_4 \mapsto P_3$.

Clearly r permutes the 12 vertices by (12345), that is, it fixes N and S and does $(P_1,P_2,P_3,P_4,P_5)(Q_1,Q_2,Q_3,Q_4,Q_5)$. One checks using adjacency of vertices that x does $(N,P_1)(P_2,P_5)(P_3,Q_3)(P_4,Q_4)(Q_1,S)(Q_2,Q_5)$ and the composite $y = xr$ does $(P_1,N,P_2)(P_3,Q_4,P_5)(P_4,Q_5,Q_3)(Q_1,S,Q_2)$. You will notice from these formulas that G acts on the set of 6 axes P_iQ_i, NS joining antipodal vertices, and in this action $x = (1,6)(2,5)$, $r = (1,2,3,4,5)$ and $y = xr = (1,6,2)(3,4,5)$.

One can also make I_{60} act on the 10 pairs of opposite faces, but we still don't have a direct link with A_5. Can we think up a set of 5 objects naturally associated with the icosahedron on which I_{60} acts as the alternating group?

7.10 Key point: 5 orthogonal frames

The sides of the icosahedron provide 5 orthogonal frames. For this, consider first the 3 pairs of antipodal parallel edges:

$$\begin{array}{rclcl}
\overrightarrow{P_1N} &=& \overrightarrow{SQ_1} &=& (\sqrt{5}-1,\ -2,\ 0),\\
\overrightarrow{Q_5P_2} &=& \overrightarrow{Q_2P_5} &=& (2,\ \sqrt{5}-1,\ 0),\\
\overrightarrow{P_4P_3} &=& \overrightarrow{Q_3Q_4} &=& (0,\ 0,\ 4s)
\end{array} \qquad (7.18)$$

In Figure 1, $\overrightarrow{P_4P_3}$ points in the vertical z direction, so is perpendicular to the horizontal plane $z = 0$ that contains the first two vectors $\overrightarrow{P_1N}$ and $\overrightarrow{Q_5P_2}$. The formulas just given prove that $\overrightarrow{Q_5P_2} \perp \overrightarrow{P_1N}$. Alternatively we can argue on the rotation by π around the dotted vertical axis in Figure 1, the z-axis (that is $(x, y, z) \mapsto (-x, -y, z)$), that interchanges $Q_5 \leftrightarrow P_2$ and $N \leftrightarrow S$. Thus the 3 vectors (7.18) are pairwise orthogonal, so form an orthogonal frame.

The 5 orthogonal frames are

$$\Sigma_k : P_kN \,||\, SQ_k, \quad Q_{k-1}P_{k+1} \,||\, Q_{k+1}P_{k-1}, \quad P_{k-2}P_{k+2} \,||\, Q_{k+2}Q_{k-2} \tag{7.19}$$

for $k = 1, 2, 3, 4, 5$. The group I_{60} of rotational symmetries of the icosahedron acts on these 5 objects by alternating permutations, providing the isomorphism $I_{60} \cong A_5$.

In fact each of these 5 orthogonal frame e_0, e_1, e_2 of $\mathbb{R}^3$ is taken to itself by the group of order 12 generated (in that basis) by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{7.20}$$

Thus I_{60} as an abstract group contains 5 subgroups isomorphic to A_4, and its conjugacy action on these defines the same isomorphism $I_{60} \cong A_5$.

The icosahedron has six axes of 5-fold rotation, namely P_iQ_i for $i = 0, \dots, 4$ and the x-axis NS, and it is the permutation action of I_{60} on these that defines the embedding $A_5 \subset A_6$ used in deriving the irreducible representation V_5. In fact 5-fold rotation around NS permutes these as (12345), whereas one sees that the 3-fold rotation of the regular triangle $\triangle P_1P_2P_N$ does $(015)(243)$, the a and b used in (9.14) and (9.20).

7.11 Homework to Chapter 7

7.1. Revision of roots of 1 (cf. Ex. 1.3.)

1. Describe the set of complex 6th roots of 1. Write down an irreducible polynomial whose roots are the primitive 6th roots of 1.
2. Find an expression in radicals for $\cos \frac{2\pi}{6}$.
3. Determine the irreducible representations of $\mathbb{Z}/6$ over $\mathbb{C}$.
4. Let $r\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation through an angle of $60^o = \frac{2\pi}{6}$ about an axis and write $\rho\colon \mathbb{Z}/6 \to \mathrm{GL}(3, \mathbb{R})$ for the representations sending $1 \in \mathbb{Z}/6$ to r. Find all $\mathbb{R}$-vector subspaces invariant under this action.
5. Write $\sigma\colon \mathbb{Z}/6 \to \mathrm{GL}(3, \mathbb{R}) \hookrightarrow \mathrm{GL}(3, \mathbb{C})$ for the same representation over $\mathbb{C}$. Determine its decomposition into irreducible representations.

7.2. Tetrahedral group T_{12} (cf. Ex. 2.8.) Show that the 4 points

$$P_1 = (1,1,1), \quad P_2 = (1,-1,-1), \quad P_3 = (-1,1,-1), \quad P_4 = (-1,-1,1) \tag{7.21}$$

are the vertices of a regular tetrahedron Δ. Write down in coordinates the rotation matrix A that permutes the P_i as $(1,2)(3,4)$ and B giving $(2,3,4)$. Prove that $A^2 = B^3 = (AB)^3$. Deduce that $T_{12} \cong A_4$.

The group $\mathrm{PSL}(2,\mathbb{F}_3)$ has order 12 and permutes the 4 points of the projective line $\mathbb{P}^1_{\mathbb{F}_3}$. Choose some ordering of these points, and write down 2×2 matrices with determinant 1 that permute them as $(1,2)(3,4)$ and $(2,3,4)$.

7.3. Character Table of A_4 (cf. Ex. 3.8.) Prove that the alternating group A_4 has 4 irreducible representations, writing them down as explicit homomorphisms $\rho\colon T_{12} \to \mathrm{GL}(n,\mathbb{C})$. Write out the conjugacy classes of A_4, its irreducible representations and the character table. Illustrate the row orthonormality relations by calculating $\langle \chi_i, \chi_j \rangle$.

7.4. Characters of an Abelian group (cf. Ex. 4.15.) Recall that linear characters or 1-dimensional representations are also homomorphisms $\rho\colon A \to \mathbb{C}^\times$. For distinct primes p, q, set $\varepsilon = \varepsilon_{pq} = \exp \frac{2\pi i}{pq}$. Show how to write out the p^2q characters of the group $A = \mathbb{Z}/p \oplus \mathbb{Z}/pq$. Show that they form a group $\widehat{A}$ isomorphic to A.

7.5. Reading information from character tables (cf. Ex. 5.2 and 5.7.)

(i) For V a representation of G, we know that $\chi_V(g) = \dim V$ if and only if $\rho(g) = \mathrm{Id}_V$. State and prove necessary conditions on $\rho(g)$ for $|\chi_V(g)| = \dim V$.

(ii) State and prove a method of determining from the character table of G which conjugacy classes are contained in the commutator subgroup $[G,G]$. [Hint: Dimensions and kernels are in the character table. $[G,G]$ is determined as the kernel of 1-dimensional representations.]

7.6. Past exam paper, slightly reworded Someone spilt gravy on a character table of a finite group G, leaving the following information:

	e	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	$*$	$*$	$*$	$*$	$*$	$*$
χ_2	1	-1	1	1	-1	1	-1
χ_3	1	1	1	1	1	-1	-1
χ_4	1	$*$	$*$	$*$	$*$	$*$	$*$
χ_5	2	$i\sqrt{2}$	0	-2	$-i\sqrt{2}$	0	0
χ_6	2	$*$	$*$	$*$	$*$	$*$	$*$
χ_7	2	$*$	$*$	$*$	$*$	$*$	$*$

(7.22)

(a) Fill in the missing information, stating briefly the results from the course that you use. [7]

(b) Compute the order of G and the size of its conjugacy classes. [6]

(c) Determine the order of the centre of G. Justify. [4]

(d) What is the order of the commutator subgroup $[G, G]$? [4]

(e) Is G isomorphic to a subgroup of $\mathrm{GL}(2, \mathbb{C})$. Justify. [4]

This puzzle is fun, and each piece of it involves remembering some useful points from the course.

7.12 Exercise on $G_{2,3,p}$

Add Ex. $\mathrm{SL}(2, \mathbb{F}_p)$ has generators

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{with} \quad xy = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

They satisfy $x^2 = \mathrm{Id}_2$ is central and $x^4 = y^3 = (xy)^p$ (and other relations). Therefore $\mathrm{PSL}(2, \mathbb{F}_p)$ is a quotient of the triangle group $G_{2,3,p}$.

7.13 Exercise on presentation $G_{2,3,3}$ gives A_4

Consider the abstract group $G = \langle x, y \mid x^2, y^3, (xy)^3 \rangle$. Show that G is a group of order 12 isomorphic to A_4.

Step 1 Working in the abstract group G, use the relations $x^2 = y^3 = (xy)^3 = e$ to deduce the following:

$$xyxy = y^2x, \quad xy^2x = yxy, \quad yxyx = xy^2,$$
$$y^2xy^2 = xyx \quad \text{and} \quad yxyxy^2 = xy.$$

Easy: start from $xyxyxy = e$ and multiply both sides successively by x and y.

Step 2 Using these, check that the list of 12 elements

$$L = \{e, x, y, y^2, xy, yx, xy^2, y^2x, xyx, yxy, yxy^2, y^2xy\}$$

is closed under multiplication on left and on the right by x and y.

Therefore these 12 elements form a group and G has at most 12 elements. We only need to be really sure that there are not any more equalities between them implied by the relations $x^2 = y^3 = (xy)^3 = e$ that we may have missed.

Step 3 At the same time as checking closure under left and right multiplication, you probably noticed a couple of cases where $xg = gx$; write these out and check that they are conjugates of x in the abstract group. If we assume for the moment that the 12 elements are distinct, then x and its conjugates generate a normal subgroup $V_4 \lhd G$.

Step 4 Show that G divided by V_4 is $\mathbb{Z}/3$, and conclude that $G \cong A_4$.

Step 5 I don't really trust the assumption that the 12 elements of L are distinct. However if we take $x = (12)(34)$ and $y = (123)$ in A_4, we can check that they are generators, satisfy the relations, and the 12 elements of L are all the elements of A_4. Therefore there is a surjective group homomorphism $G \to A_4$, so that $|G| \geq 12$. In Magma:

```
A4 := Alt(4);
x := A4!((1,2)(3,4)); y := A4!(1,2,3); x*y;
L := [Id(A4), x, y, y^2, x*y, x*y^2, y*x, y^2*x, x*y*x,
   x*y^2*x, y*x*y^2, x*y*x*y^2]; #L; #SequenceToSet(L);
```

The same method can be used (in principle) to prove that the abstract triangle groups $\langle x, y \mid x^2, y^3, (xy)^n \rangle$ for $n = 4, 5$ are isomorphic respectively to S_4 and A_5. But the calculations become quite a lot bigger, and there is more and more excuse to entrust them to the computer.

7.14 Exercise on the triangle groups $G_{2,3,r}$ and their Cayley graphs

The Cayley graph of a group G with a set of generators x_i is the graph having $g \in G$ as its nodes and edges g—$x_i g$ for each generator x_i. (Compare (4.5), the Cayley graph of the free group on 2 generators.) For the triangle groups $G_{2,3,n}$, the Cayley graph can be identified with the 3-fold central subdivision of the tesselation of S^2, E^2 or $\mathcal{H}^2$ by regular triangles.

An element of $G_{2,3,r}$ acts on a regular triangular tesselation of the sphere S^2 (for $r \leq 5$) or the Euclidean plane $\mathbb{E}_2$ (for $r = 6$) or the hyperbolic plane $\mathcal{H}^2$ (for $r \geq 7$). Thus in Figure 7.3, one can make a frame of reference out of a triangle marked with an edge. For example, with the triangle Δ and the line $[0, 1]$, draw the inward perpendicular to $[0, 1]$ at the midpoint $1/2$. The group acts simply transitively on the set of all such frames.

8 New representations for old

8.1 Representations from a quotient group G/H

Definition A representation $\rho\colon G \to \mathrm{GL}(V)$ is *faithful* if no element of G acts on V by the identity. Since ρ is a group homomorphism, its kernel is a normal subgroup $H = \ker\rho \lhd G$, and any ρ arises from a faithful representation of the quotient G/H.

If $H \lhd G$ is a normal subgroup and $\pi\colon G \to G/H$ the quotient group by H then a representation of G/H gives rise to a representation of G with kernel H. We have already seen, for example, that A_4 and S_4 act as the full symmetric group S_3 on the set of 3 pairings $\{(1,2;3,4),(1,3;2,4),(1,4;3,4)\}$, and that this defines surjective homomorphisms $A_4 \twoheadrightarrow \mathbb{Z}/3$ and $S_4 \twoheadrightarrow S_3$ with kernel the 4-group V_4. Thus the representations of S_3 give rise to the (unfaithful) representations of A_4 and S_4 that we use in deriving their character tables.

8.2 Products of 1-dimensional representation

A 1-dimensional representation of G is a homomorphism $\alpha\colon G \to \mathbb{C}^\times$. Taking the character of a 1×1 matrix identifies $\mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times$, so that we can simply identify a 1-dimensional representation with its character. We can take the product of 1-dimensional representations in an obvious way: if $\alpha,\beta\colon G \to \mathbb{C}^\times$ are homomorphisms then $\alpha\beta\colon G \to \mathbb{C}^\times$ given by $\alpha\beta(g) = \alpha(g)\beta(g)$ is again a 1-dimensional representation. Thus the 1-dimensional representations of G form a group $\mathrm{Hom}_{\mathrm{groups}}(G,\mathbb{C}^\times)$ in its own right. To repeat the material of 4.4–5, it follows from the universal property of the Abelianisation $A = G^{\mathrm{Ab}} = G/[G,G]$ (quotient by the commutator subgroup) that this is $\widehat{A} = \mathrm{Hom}_{\mathrm{groups}}(A,\mathbb{C}^\times)$ and is (noncanonically) isomorphic to A.

8.3 Product of a representation V by 1-dim representation

In the same way, if V is a $\mathbb{C}G$-module with representation ρ_V, we can take its product by a 1-dimensional representation α by taking $\alpha\rho_V\colon V \to GL(V)$ to be defined by $g \mapsto \alpha(g)\rho_V(g)$. Here we are just taking the product of homomorphism $\rho_V(g)$ by the scalar $\alpha(g) \in \mathbb{C}^\times$. Taking the product of a matrix aM by a scalar just multiplies each entry by a, so the sum of the diagonal entries is also multiplied by a, and we get

$$\chi_{\alpha\rho_V} = \alpha\chi_{\rho_V} \quad \text{or} \quad \chi_{\alpha\rho_V}(g) = \alpha(g)\chi_{\rho_V}(g) \quad \text{for } g \in G. \tag{8.1}$$

We use this as an automatic trick in working out character tables.

8.4 Tensor product $V_1 \otimes V_2$

Let V_1 and V_2 be vector spaces. The tensor product $V_1 \otimes V_2$ is most easily constructed using bases: Let $e_1,\dots,e_n$ base V_1 and $f_1,\dots,f_m$ base V_2. Then the tensor product $V_1 \otimes V_2$ is the vector space with basis the $n_1 n_2$ formal expressions $e_i \otimes f_j$.

From a theoretical point of view, the best way to deal with this is via its Universal Mapping Property. Recall from linear algebra that a bilinear map $s\colon V_1 \times V_2 \to W$ is a map that is linear separately in both variables.

Proposition *(i) There is a bilinear map $t\colon V_1 \times V_2 \to V_1 \otimes V_2$ that is uniquely determined by specifying that $(e_i, f_j) \mapsto e_i \otimes f_j$. It is given by*

$$\Big(\sum a_i e_i, \sum b_j f_j\Big) \mapsto \sum_{i,j} a_i b_j e_i \otimes e_j. \tag{8.2}$$

(ii) Moreover, t has the following UMP for bilinear maps: for any bilinear map $s\colon V_1 \times V_2 \to W$ there exists a unique $\mathbb{C}$-linear map $\varphi_s\colon V_1 \otimes V_2 \to W$ such that s factors as $s = \varphi_s \circ t$:

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "t"] \arrow[dr, "s"'] & V_1 \otimes V_2 \arrow[d, "\varphi_s"] \arrow[dl, phantom, "\copyright"] \\ & W \end{tikzcd} \tag{8.3}$$

The proof is straightforward: the $e_i \otimes f_j$ base V, so a linear map $\varphi\colon V \to W$ is uniquely determined by where it sends them. To achieve $s = \varphi_s \circ t$, we can and must send $e_i \otimes f_j \mapsto s(e_j, f_j)$. This proves the proposition.

The UMP clarifies what part of the discussion is the definition and what part the construction: we can state the UMP without saying basis, and think of the vector space with basis $e_i \otimes f_j$ as a construction proving that the universal object $V_1 \otimes V_2$ in (ii) actually exists. The UMP guarantees that the solution is unique: indeed for two solutions W_1 and W_2 of the UMP, there are $\mathbb{C}$-linear maps $W_1 \to W_2$ and $W_2 \to W_1$ between them compatible with the bilinear requirements, and unique with that property; then, again by uniqueness, the composite map $W_1 \to W_2 \to W_1$ equals Id_{W_1} and ditto for $W_2 \to W_1 \to W_2$. Uniqueness means in particular independence of the choice of basis.

The UMP also gives a clean way of making the tensor product $V_1 \otimes V_2$ of two $\mathbb{C}G$-modules into a $\mathbb{C}G$-module. In the proposition below, a map $\mathbb{C}$-bilinear map $s\colon V_1 \times V_2 \to W$ is *$\mathbb{C}G$-bilinear* if the maps

$$s(v,-)\colon V_2 \to V_1 \otimes V_2 \quad \text{and} \quad s(-,w)\colon V_1 \to V_1 \otimes V_2 \tag{8.4}$$

are $\mathbb{C}G$-module homorphisms for fixed $v \in V_1$ and $w \in V_2$.

Theorem *(I) The tensor product $V_1 \otimes V_2$ of two $\mathbb{C}G$ modules has a unique $\mathbb{C}G$-module structure with the property that $t\colon V_1 \times V_2 \to V_1 \otimes V_2$ is $\mathbb{C}G$-bilinear.*

(II) The character of $V_1 \otimes V_2$ is the product of those of V_1 and V_2:

$$\chi_{V_1 \otimes V_2} = \chi_{V_1} \chi_{V_2}. \tag{8.5}$$

(III) In the case $V_1 = V_2 = V$, the tensor product $V \otimes V$ decomposes as a direct sum of tensors that are symmetric and skew under the involution of $V \otimes V$ given by $v \otimes w \mapsto w \otimes v$, that is $V \otimes V = \operatorname{Sym}^2 V \oplus \bigwedge^2 V$, and each summand is a $\mathbb{C}G$-module.

(IV) The characters of $\operatorname{Sym}^2 V$ and $\bigwedge^2 V$ are given by

$$\begin{aligned} \chi_{\operatorname{Sym}^2 V}(g) &= \tfrac{1}{2}\big(\chi_V(g)^2 + \chi_V(g^2)\big) \\ \chi_{\bigwedge^2 V}(g) &= \tfrac{1}{2}\big(\chi_V(g)^2 - \chi_V(g^2)\big) \end{aligned} \tag{8.6}$$

for $g \in G$.

Proof (I) Given the actions of G on V_1 and V_2, make G act on the product $V_1 \times V_2$ by its diagonal action $\rho_1 \times \rho_2$, that is, $g(v, w) = (gv, gw)$ for $g \in G$.

Let $t\colon V_1 \times V_2 \to V_1 \otimes V_2$ be the universal bilinear map of the Proposition. The map $t(g(v, w))$ is also a bilinear map $V_1 \times V_2 \to V_1 \otimes V_2$, so by Proposition (ii), it factors uniquely via t. That is, for each $g \in G$ there is a $\mathbb{C}$-linear map $\rho_{12}(g)\colon V_1 \otimes V_2 \to V_1 \otimes V_2$ such that the diagram

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{t} & V_1 \otimes V_2 \\ {\scriptstyle \rho_1\times\rho_2}\big\downarrow & \copyright & \big\downarrow{\scriptstyle \rho_{12}} \\ V_1 \times V_2 & \xrightarrow{t} & V_1 \otimes V_2 \end{array} \tag{8.7}$$

commutes. Here the t and $\rho_1 \times \rho_2$ arrows are given, and the UMP provides the downarrow $\rho_{12} = \rho_{V_1 \otimes V_2}$. It is straightforward to check that $g \mapsto \rho_{12}(g)$ is a representation $\rho_{12}\colon G \to \mathrm{GL}(V_1 \otimes V_2)$.

(II) It follows from the definition that $\rho_{12}(g)(v \otimes w) = (gv \otimes gw)$ for $g \in G$ and for any $v \in V_1$, $w \in V_2$. When calculating a character, we are allowed to restrict to one element $g \in G$ at a time, and the trace $\chi(G)$ is independent of the choice of basis. So I can choose bases $e_1, \dots, e_{n_1}$ of V_1 and $f_1, \dots, f_{n_2}$ of V_2 to diagonalise g, so that $\rho_1(g) = \operatorname{diag}(\lambda_1, \dots, \lambda_{n_1})$ and $\rho_2(g) = \operatorname{diag}(\mu_1, \dots, \mu_{n_2})$. Then g acts diagonally on $V_1 \otimes V_2$, by

$$g(e_i \otimes f_j) = g(e_i) \otimes g(f_j) = \lambda_i \mu_j g(e_i \otimes f_j), \tag{8.8}$$

so that its trace is the sum of the eigenvalues

$$\textstyle\sum_{i,j} \lambda_i \mu_j = \big(\sum_i \lambda_i\big)\big(\sum_j \mu_j\big) = \chi_{V_1}(g)\chi_{V_2}(g). \quad \text{Q.E.D.} \tag{8.9}$$

(III) The direct sum decomposition $V \otimes V = \operatorname{Sym}^2 V \oplus \bigwedge^2 V$ is clear. The maps $V \otimes V \to \operatorname{Sym}^2 V$ and $V \otimes V \to \bigwedge^2 V$ have the same type of UMP for symmetric bilinear maps, respectively skew bilinear maps as described above for $V_1 \otimes V_2$. The G-action on the two summands follows from this, as does the fact that, given any basis $e_1, \dots, e_n$ of V, the symmetric product $\operatorname{Sym}^2 V$ is based by $(e_i \otimes e_j + e_j \otimes e_i)$ for $1 \le i \le j \le n$, and the skew product $\bigwedge^2 V$ by $(e_i \otimes e_j - e_j \otimes e_i)$ for $1 \le i < j \le n$.

(IV) As before, diagonalise the action of $g \in G$, so that $g = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Now g acts diagonally on $\operatorname{Sym}^2 V$ and $\bigwedge^2 V$, with eigenvalue $\lambda_i\lambda_j$ on each of $e_i \otimes e_j \pm e_j \otimes e_i$. The result (IV) comes by paying attention to the range of summation. On $V \otimes V$, we summed both i and j from 1 to n independently, giving $\sum_{i,j} \lambda_i\lambda_j = (\sum \lambda_i)^2 = \chi_V(g)^2$. On $\operatorname{Sym}^2 V$, we restrict the sum to $i \le j$, so count the two equal off-diagonal elements $\lambda_i\lambda_j$ and $\lambda_j\lambda_i$ once only, giving just half as much, except that we count the diagonal elements λ_i^2 once. Thus

$$\chi_{\operatorname{Sym}^2 V}(g) = \sum_{i \le j} \lambda_i\lambda_j = \tfrac{1}{2}\Big(\sum_{i,j} \lambda_i\lambda_j + \sum_i \lambda_i^2\Big) = \tfrac{1}{2}\big(\chi_V(g)^2 + \chi(g^2)\big), \quad (8.10)$$

where the second term $\sum \lambda_i^2 = \chi_V(g^2)$, since the eigenvalue λ_i^2 of g on $e_i \otimes e_i$ equals the eigenvalue of g^2 on e_i. Similarly,

$$\chi_{\bigwedge^2 V}(g) = \sum_{i \le j} \lambda_i\lambda_j = \tfrac{1}{2}\Big(\sum_{i,j} \lambda_i\lambda_j - \sum_i \lambda_i^2\Big) = \tfrac{1}{2}\big(\chi_V(g)^2 - \chi(g^2)\big). \quad (8.11)$$

As a mnemonic, it is useful to consider the character of $g = e_G$: the dimensions $\dim \operatorname{Sym}^2 V = \binom{n+1}{2} = \frac{n^2+n}{2}$ and $\dim \bigwedge^2 V = \binom{n}{2} = \frac{n^2-n}{2}$ are familiar. They are the result of giving each summand one half of $n^2 = \dim V \otimes V$ except that the diagonal terms are all given to Sym^2.

8.5 Additional notes

For more on tensor products, see [James and Liebeck, Chap. 19]. You need to be aware that they write group actions on the right, so $G \times V \to V$ by $(g, v) \mapsto vg$.

The dual representation $V^\vee$ is discussed in Ex. 8.5–6. Its character is given by $\chi_{V^\vee}(g) = \chi(g^{-1}) = \overline{\chi(g)}$.

The $\mathbb{C}$-linear maps $V_1 \to V_2$ form a $\mathbb{C}G$-module $\operatorname{Hom}_{\mathbb{C}}(V_1, V_2)$ with action $\varphi \mapsto g{\circ}\varphi{\circ}g^{-1}$. (See Ex. 8.3–4.) There is a natural isomorphism $\operatorname{Hom}_{\mathbb{C}}(V_1, V_2) = V_1^\vee \otimes V_2$ (see Ex. 9.1–3).

Applications, for example: the symmetric group S_5 has a 4-dimensional irreducible representation V_4 (obtained by taking a complement to the main diagonal $(1,1,1,1,1)$ in the natural permutation representation of S_5 on $\mathbb{C}^5$). Its exterior square $\bigwedge^2 V_4 = V_6$ is 6-dimensional, and evaluating $\langle \chi_{V_6}, \chi_{V_6} \rangle$ shows it is irreducible (see Ex. 9.8). Its symmetric square $\operatorname{Sym}^2 V_4 = V_{10}$ is 10-dimensional. Calculating the inner product of its character with χ_1 and χ_{V_4} shows that V_{10} is the direct sum of 1, V_4 and a new irreducible representation V_5 (see Ex. 9.9). Together with couple of the usual tricks, this allows us to complete the character table of S_5. For more on this, see Ex. 9.7–10, plus [James and Liebeck], around 19.14, pp. 199.

8.6 Homework to Chapter 8

8.1. Faithful representations A representation is *faithful* if the only $g \in G$ that acts trivially is the identity e. Define the kernel of a representation.

Prove from first principles that every finite group G has a faithful finite dimensional representation.

If G has a nontrivial normal subgroup $H \lhd G$, show how to construct a nonfaithful representation of G. Show how to construct every representation of G on which H acts trivially.

8.2. Kernel of a character If ρ is a representation of G, show how to use its character χ_ρ to determine whether ρ is faithful. Use your argument to give an appropriate definition of the kernel of a character.

Write out the character table of the dihedral group D_{12}. This is easy enough by hand, or you may prefer to use the Magma code below.

Show that D_{12} has a faithful irreducible representation. Show that $D_{12} \times \mathbb{Z}/2$ has no faithful irreducible representation.

```
K6<ep> := CyclotomicField(6); GL2 := GeneralLinearGroup(2,K6);
A := elt< GL2 | ep,0,0,ep^-1 >; B := elt< GL2 | 0,1,1,0 >;
G := sub < GL2 | A, B >; CharacterTable(G);
G2 := DirectProduct( quo< G | Id(G) >, CyclicGroup(2) );
CharacterTable(G2);
```

Better, give a theoretical reason why $D_{12} \times \mathbb{Z}/2$ cannot have a faithful irreducible representation. (The group is very close to being Abelian, and its irreducible representations are all 1 or 2-dimensional. So their characters take only a few possible values. Or you may like to consider group homomorphisms $\mathbb{Z}/2 \times \mathbb{Z}/2$ into $\boldsymbol{\mu}_2 = \{\pm 1\}$.)

8.3. Right action of G by g^{-1} Let V be a f.d. vector space over K, and $G = \mathrm{GL}(V)$. Think of $(g, v) \mapsto gv$ (with $g \in \mathrm{GL}(V)$ a matrix and v a column vector) as the *given* representation of $\mathrm{GL}(V)$.

Explain why $(w, g) \mapsto wg$ (with w a row vector) is *not* a representation of $\mathrm{GL}(V)$. Show that $(w, g) \mapsto wg^{-1}$ is a representation of $\mathrm{GL}(V)$.

8.4. Hom space For two f.d. vector spaces, $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ is the space of $\mathbb{C}$-linear maps $\varphi\colon V_1 \to V_2$. Choosing bases makes it into $\mathrm{Mat}(n_2 \times n_1, \mathbb{C})$.

Now suppose V_1, V_2 are $\mathbb{C}G$-modules, with representations $\rho_i\colon G \to \mathrm{GL}(V_i)$. Show that $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ becomes a $\mathbb{C}G$-module by $\varphi \mapsto \rho_2(g) \circ \varphi \circ \rho_1(g^{-1})$. Draw that commutative diagram again. Understand why writing $\rho_1(g)$ as the first factor in the composite doesn't work.

8.5. Dual representation $V^\vee$ For a f.d. vector space V, the dual vector space $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ consists of linear maps $\varphi\colon V \to \mathbb{C}$. If we view V as the column vectors in some basis, $V^\vee$ becomes the row vectors.

Now let V be a $\mathbb{C}G$-module. Let $\mathbb{C}$ be the trivial $\mathbb{C}G$-module: every $g \in G$ acts by the identity. Show that the dual vector space $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ has a natural structure of $\mathbb{C}G$-module defined by precomposing in V, that is

$$g(\varphi) = \varphi \circ g^{-1} \quad \text{for } \varphi\colon V \to \mathbb{C}. \tag{8.12}$$

8.6. Compatibility of $V^\vee$ With the above definition of the $\mathbb{C}G$-module $V^\vee$, verify that the evaluation map $V^\vee \times V \to \mathbb{C}$ defined by $(\varphi, v) \mapsto \varphi(v) \in \mathbb{C}$ is $\mathbb{C}G$-bilinear.

8.7. Is $V^\vee$ equivalent to V? The opportunity to introduce some confusion is too much to resist. $V^\vee$ is isomorphic to V as a vector space. A nondegenerate bilinear form (such as $\mathbf{x} \cdot \mathbf{y} = \sum x_i y_i$) defines an isomorphism $V = V^\vee$. Nevertheless, this isomorphism depends on a noncanonical choice of basis, and there is no reason why it should be compatible with a $\mathbb{C}G$-module structure.

The way that we define the $\mathbb{C}G$-action on $V^\vee$ makes the bilinear pairing $V^\vee \times V \to \mathbb{C}$ invariant under G. If we view V as column vectors and $V^\vee$ as row vectors, this means that the pairing goes

$$\text{row} \cdot \text{column} \mapsto \text{row} \cdot R(g)^{-1} \cdot R(g) \cdot \text{column} \tag{8.13}$$

where $R(g)$ is the matrix of $\rho(g)$. It does not make sense to multiply row $\times R(g)$ and still expect to get a G-action on row vectors, as explained in Ex. 8.3.

If $G = \mathbb{Z}/3$ and ρ_1 is the linear representation "multiply by ω" the dual $\rho_1^\vee$ is multiply by $\omega^{-1} = \omega^2 = \overline{\omega}$. These two representations are *not* isomorphic. (They are 1×1 matrices with unequal entries.) On the other hand, they are complex conjugates of one another.

8.8. Let $\rho\colon G \to \mathrm{GL}(n, \mathbb{R})$ be a representation of G on $\mathbb{R}^n$. Prove that $\mathbb{R}^n$ has a G-invariant $\mathbb{R}$-bilinear inner product Φ. [Hint: The usual argument; take any positive definite inner product such as $\sum x_i y_i$ in any basis, and average over G.] For an orthonormal basis of $\mathbb{R}^n$ w.r.t. this inner product, the matrices defining ρ are orthogonal with respect to Φ, so that the representation ρ takes values in the orthogonal group $O(n, \mathbb{R})$

Deduce that $V^\vee$ is isomorphic to V. A bilinear inner product on V provides an isomorphism $V^\vee \cong V$.

8.9. Prove that $V^{\vee\vee} \cong V$. If V is irreducible, prove that $V^\vee$ is also irreducible.

8.10. A *permutation representation* is the vector space $\mathbb{C}X$ with basis a finite set X, and G-action given by a permutation action of G on X. If V is a permutation representation of G, prove that $V^\vee \cong V$. This applies in particular to the regular representation $\mathbb{C}G$.

8.11. Let $g \in G$ be an element of order 2. For any representation V of G, show that $\chi_V(g)$ is an integer $n \equiv \dim V \bmod 2$. If moreover G does not have any subgroup of index 2, prove that $n \equiv \dim V \bmod 4$.

8.12. Tetrahedral group T_{12} in coordinates The alternating group $G = A_4$ acts by permutations on a set of 4 elements. Consider its permutation representation on $\mathbb{R}^4$ (or $\mathbb{C}^4$). The diagonal element $(1, 1, 1, 1)$ is invariant, so generates a trivial KG-submodule. Show how to find a complementary KG-submodule.

Write down in coordinates an action of A_4 on $\mathbb{R}^3$ that permutes the 4 vertices of a regular tetrahedron. [Hint. Let e_i be the standard basis of Euclidean $\mathbb{R}^4$. Their centroid is $d = 1/4 \sum e_i$. The 4 vectors $f_i = e_i - d$ are linearly dependent, so only span a subspace $\mathbb{R}^3$. Choose f_1, f_2, f_3 as basis of $\mathbb{R}^3$ (N.B. it is not orthonormal) and write f_4 as their linear combination.] Explain how the different elements of A_4 act by matrices in this basis.

8.13. More T_{12} Write Σ for a regular tetrahedron in $\mathbb{R}^3$, say with vertices P_1, P_2, P_3, P_4. Let X be the rotation by angle π around the median line joining the midpoint of P_1P_2 and P_3P_4, and R the rotation by $\frac{2\pi}{3}$ around the axis joining P_4 to the centroid of face $P_1P_2P_3$. Show that X and R generate A_4. Write $Y = XR$, and show that X, Y generate A_4 with relations $X^2 = Y^3 = (XY)^3 = e$. The group $T_{12} = A_4$ acts on the 4 vertices, 6 edges and 4 faces. Determine how each of the corresponding permutation representations break up into irreducibles.

8.14. Similar for O_{24} Prove $O_{24} \cong \mathrm{PGL}(2, \mathbb{F}_3)$. Prove that it has a normal subgroup of index 2 isomorphic to T_{12}. Find the conjugacy classes of O_{24}. The character table was given earlier in the course. O_{24} acts on the 6 faces, 8 vertices and 12 edges of the cube, and the corresponding set of 3, 4 and 6 antipodal pairs thereof. Show how the corresponding permutation representations break up into irreducibles.

8.15. Characterisation of an Abelian group Prove that a group G is Abelian if and only if every irreducible representation of G is 1-dimensional.

8.16. Another crossword puzzle Solve this puzzle (adapted from a past exam), quoting without proof whatever results of the course you need.

Given: A group G of order 12 has 4 conjugacy classes with representatives $\{e, g_1, g_2, g_3\}$, and a character χ_2 that scores $\chi_2(e) = 1$, $\chi_2(g_1) = 1$, $\chi_2(g_2) = \omega$, $\chi_2(g_3) = \omega^2$ on them (as usual, ω a primitive 3rd root of 1).

(I) Write out the full character table.

(II) Exhibit a group G having this character table.

9 Induced representations

9.1 Restricted representations

The material discussed in Chapter 8 concerns $\mathbb{C}G$-modules and homomorphisms between them for a fixed G. This is simpler material than the stuff on induced representations ($\mathbb{C}H$-modules to $\mathbb{C}G$), where the underlying group also varies.

Let $H \subset G$ be a subgroup. Give a representation V of G, we can restrict the map $\rho_V \colon G \to \mathrm{GL}(V)$ to the subset $H \subset G$ to get the *restricted representation*

$$\operatorname{Res}_H^G V = \text{same vector space } V, \text{ with } \rho_{V\big|H} \colon H \to \mathrm{GL}(V). \tag{9.1}$$

The character of $\operatorname{Res} V$ is simply the character of V restricted to $H \subset G$. Even if V is irreducible, there is no particular reason for $\operatorname{Res} V$ to remain so; likewise, two distinct representations of G may restrict to isomorphic representations of H. Both of these happen when restricting from S_5 to A_5. For example, S_5 has a 6-dimensional irreducible representation V_6, whereas $A_5 \lhd S_5$ does not. It is interesting (see Ex. 9.11) to calculate the restriction of the character to the smaller subgroup A_5, and deduce that it must split as the direct sum of two 3-dimensional representations, those with α, β in 6.5, (30). [James and Liebeck], Chapter 20 has lots more on this.

9.2 Induced representation

The operation in the other direction is called *induced representation.* It takes a representation L of a subgroup $H \subset G$ to a representation $V = \operatorname{Ind}_H^G L$ of G. Many representations of interest are constructed in this way, even starting from seemingly trivial subgroups H.

My approach in the first instance is to say what I want an induced representation to be. We can get a long way just assuming it exists: this includes deriving its UMP, proving the formula for its character, and proving the Frobenius reciprocity theorem. However, there is still just a bit missing: namely, given only $H \subset G$ and the H-module L, we still have to construct $V = \operatorname{Ind}_H^G L$ or prove it exists. Doing this properly involves a small difficulty, that I leave until last.

9.3 Examples

The *binary dihedral group* $\mathrm{BD}_{4m} \subset \mathrm{SL}(2, \mathbb{C})$ is the subgroup generated by the matrices $a = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ where $\varepsilon = \exp \frac{2\pi i}{2m}$, or can be viewed as the abstract group $\left\langle a, b \;\middle|\; a^{2m}, a^m = b^2, ba = a^{-1}b \right\rangle$. It has the cyclic group $A = \langle a \rangle \lhd \mathrm{BD}_{4m}$ as a normal subgroup of index 2, so is close to being Abelian. Now BD_{4m} has $(m-1)$ 2-dimensional irreducible representations V_i for $i = 1, \dots, m-1$, on which a and b act by

$$\rho_k(a) = \begin{pmatrix} \varepsilon^i & 0 \\ 0 & \varepsilon^{-i} \end{pmatrix} \quad \text{and} \quad \rho_k(a) = \begin{pmatrix} 0 & 1 \\ (-1)^i & 0 \end{pmatrix} \tag{9.2}$$

In each case, V_i is made up as a direct sum of 2 different representations L_i and L_{2m-i} of the subgroup A, that are interchanged by the action of b or of any element of the coset bA.

The family of trihedral groups T_{3r} is similar. Choose an integer $s \geq 1$ and suppose that r divides $1+s+s^2$, so that $s^3 \equiv 1$ modulo r. (The cases $s=2$, $r=7$ or $s=3$, $r=13$ are fairly typical.) Let $\varepsilon = \exp\frac{2\pi i}{r}$ be a primitive rth root of 1. Consider the matrices

$$a = \operatorname{diag}(\varepsilon, \varepsilon^s, \varepsilon^{s^2}) \quad \text{and} \quad t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{C}). \tag{9.3}$$

They generate a group $T_{3r} \subset \mathrm{SL}(3,\mathbb{C})$ of order $3r$, having a normal subgroup $A = \langle a \rangle \lhd T_{3r}$ with quotient $\mathbb{Z}/3$. It can be described as the abstract group $\left\langle a, t \;\middle|\; a^r = t^3 = e, ta = a^s t \right\rangle$. The given representation of T_{3r} on $\mathbb{C}^3$, when restricted to A, splits as a direct sum of 3 representations $L_\varepsilon \oplus L_{\varepsilon^s} \oplus L_{\varepsilon^{s^2}}$, and these 3 are permuted as a 3-cycle by the action of t or by any element of the coset tA.

9.4 First definition

Let $H \subset G$ be a subgroup (not necessarily normal). Given a representation V of G and a subspace $L \subset V$, we say that $V = \operatorname{Ind}_H^G L$ is *induced* from a representation of H on L if

(i) L is a H-invariant subspace of V;

(ii) $V = \bigoplus_{\gamma \in G/H} \gamma L$.

Here I write γL as the product of the whole coset γ with L, but in fact $\gamma L = gL$ for any element $g \in \gamma$, since $HL = L$; we will feel the need to fix coset representatives $g \in \gamma$ later in this chapter. The point of the definition is that V is a direct sum of copies γL of the fixed representations L of H, indexed by G/H, and G permutes these copies by its left action on G/H.

Please check that you understand how this applies to the binary dihedral groups BD_{4m} and the trihedral groups T_{3r} discussed in 9.3. As another key example, consider the permutation action of G on G/H by left multiplication, and the corresponding permutation representation, that has basis e_γ, and G acting by permuting the e_γ in the same way. This permutation representation is $\operatorname{Ind}_H^G L$ in Definition 9.4, with $L = \mathbb{C}$ the trivial $\mathbb{C}H$-module. The particular case $H = \{e\}$ gives the regular representation $\mathbb{C}G$.

9.5 The UMP of $V = \operatorname{Ind}_H^G L$

Proposition *Suppose that $V = \operatorname{Ind}_H^G L$ in the sense of Definition 9.4. Then it has the properties*

(1) There is $\mathbb{C}H$-module homomorphism $i\colon L \to \operatorname{Res}_H^G V$, namely the inclusion of L as the first summand of the direct sum $\bigoplus \gamma L$.

(2) Moreover, the $\mathbb{C}G$-module V is universal w.r.t. (1), in the sense that for a $\mathbb{C}G$-module W and a $\mathbb{C}H$-module homomorphism $\alpha\colon L \to \operatorname{Res}_H^G W$, there is a unique $\mathbb{C}G$-module homomorphism $\varphi_\alpha\colon V \to W$ such that $\alpha = \varphi_\alpha \circ i$.

The property can be stated in the alternative form

$$\operatorname{Hom}_{\mathbb{C}H}(L, \operatorname{Res}_H^G W) = \operatorname{Hom}_{\mathbb{C}G}(\operatorname{Ind}_H^G L, W). \tag{9.4}$$

The left-hand side consists of maps $\alpha\colon L \to W$ that are $\mathbb{C}H$-module homomorphisms (with W viewed only as the restricted representation $\operatorname{Res}_H^G W$ of H), and on the right-hand side we have maps from $V = \bigoplus \gamma L$ to W that are $\mathbb{C}G$-module homomorphisms. The equal sign in (9.4) *means that there is a canonical identification between the two sides (no choices involved).*

The statement is a bit complicated to state, but the proof is very easy. Indeed, V is a $\mathbb{C}G$-module and $V = \bigoplus \gamma L$; if a map $\alpha\colon L \to W$ is given, a $\mathbb{C}G$-module homomorphism φ_α must map each summand γL to W by $gv \mapsto g\alpha(v)$ for any $g \in \gamma$. This is the only possible image in order for φ_α to be a $\mathbb{C}G$-module homomorphism, and it does define one.

9.6 Simple analogy

For $L = \mathbb{R}^n$ a vector space over $\mathbb{R}$, set $V = \mathbb{C}^n = L \otimes_{\mathbb{R}} \mathbb{C}$ for L extended to $\mathbb{C}$. We can also view V as an $\mathbb{R}$ vector space of twice the dimension ("restriction of scalars", an analog of restriction). Then V has the two properties

(1) There is an $\mathbb{R}$-linear map $i\colon L \to V$, namely the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$. (It is $\mathbb{R}$-linear, so V is only viewed as V with restricted scalars.)

(2) Moreover, V is universal w.r.t. (1): for any $\mathbb{C}$-vector space W and $\mathbb{R}$-linear map $\alpha\colon L \to W$, there is a unique $\mathbb{C}$-linear map $\varphi_\alpha\colon V \to W$ with $\alpha = \varphi_\alpha \circ i$.

If you restrict scalars in V, you get $\mathbb{C}^n$ as $\mathbb{R}$-vector space, which is of course $\mathbb{R}^{2n}$. So induction followed by restriction is a bigger object.

9.7 Character of an induced representation

If $V = \operatorname{Ind}_H^G L = \bigoplus \gamma L$ is an induced representation as in (9.4), H acts on L and on each factor γL by the same action. On the other hand $g \in G$ acts on V by permuting the summands according to the left action of G on the cosets G/H. The following sections analyse the character of V as a $\mathbb{C}G$-module. The basic formula is

$$\chi_V(g) = \frac{1}{|H|} \sum_{x \in G} (\chi_L)^0(xgx^{-1}). \tag{9.5}$$

This will be expanded and manipulated in several ways in what follows, but the rough picture is perfectly clear. In the sum (9.5), for $g \in G$, if $g\gamma \neq \gamma$ then g maps the summand γL of V to a different summand, so that it contributes

0 to the trace $\chi_V(g) = \mathrm{Tr}(\rho_V(g))$: only the diagonal blocks contribute to the trace of a block matrix. On the other hand, to say that $g\gamma = \gamma$ means that some conjugate of g is in H; then for each occurrence of $xgx^{-1} \in H$ we take $\chi_L(xgx^{-1})$. The more subtle point in the case $g\gamma = \gamma$ is to ask how often the conjugate xgx^{-1} belongs to H, and to which H-conjugacy classes it belongs. The next section treats this in a slightly wider context.

9.8 Induced class function

Write $\mathcal{C}(H)$ for the space of class functions $H \to \mathbb{C}$ of H and $\mathcal{C}(G)$ for that of G. The restriction from G to H of a class function $\psi \in \mathcal{C}(G)$ is simply $\mathrm{Res}_H^G \psi = \psi_{|H} \in \mathcal{C}(H)$, or if you prefer, the composite $\psi \circ i$ of the inclusion $i\colon H \hookrightarrow G$ with ψ.

To go the other way, a function $\varphi\colon H \to \mathbb{C}$ is not defined on the bigger set G, so we first *extend it by* 0: write φ^0 for the function that is φ on H but just scores 0 for $g \in G \setminus H$ (that is, $g \notin \operatorname{dom}\varphi$). Induction of class functions is a map $\mathrm{Ind}_H^G\colon \mathcal{C}(H) \to \mathcal{C}(G)$ obtained by composing two operations: extend by 0, then average over the conjugacy action of G. The formal definition is as follows: for $\varphi \in \mathcal{C}(H)$, set

$$\mathrm{Ind}_H^G(\varphi)(g) = \frac{1}{|H|}\sum_{x\in G} \varphi^0(x^{-1}gx), \quad \text{where } \varphi^0(g) = \begin{cases} \varphi(g) & \text{if } g \in H, \\ 0 & \text{else.} \end{cases} \tag{9.6}$$

This is now defined on the whole of G, and is a class function for G because we average over G. The sum in (9.6) runs over all $x \in G$, but the elements of one coset xH all contribute the same term: in fact,

$$(xh)^{-1}gxh = h^{-1}(x^{-1}gx)h \in H \iff x^{-1}gx \in H \quad \text{for } h \in H, \tag{9.7}$$

and when this holds, $\varphi \in \mathcal{C}(H)$ takes the same value on the two. This proves the next result.

Lemma

$$\mathrm{Ind}_H^G(\varphi)(g) = \sum_{x\in G/H} \varphi^0(x^{-1}gx) \tag{9.8}$$

where now the sum involves just one representative of each coset.

The definition simplifies by analysing how the conjugacy class $C_G(g)$ of $g \in G$ intersects H. The subset $C_G(g) \cap H \subset H$ may be empty, but it is invariant under conjugacy by H, and so is a disjoint union of conjugacy classes inside H:

$$C_G(g) \cap H = \bigsqcup C_H(g_i) \quad \text{for some } g_1, \dots, g_m, \text{ with } m \geq 0. \tag{9.9}$$

For example, let $H = A_5 \subset G = S_5$. Then $H \cap C_G(g) = \emptyset$ is empty if g is odd; but as we have seen, the single conjugacy class of 5-cycles such as $g = (12345)$ (of size 24) in S_5 breaks up into two conjugacy classes of g and $g^2 = (13524)$ in H (of size 12 each).

In the sum (9.8), $\varphi \in \mathcal{C}(H)$ scores the same value $\varphi(g_i)$ on all the elements of its conjugacy class $C_H(g_i)$ in H, so that the value of $\operatorname{Ind}_H^G(g)$ is a weighted average of $\varphi(g_i)$ weighted by the size $|C_H(g_i)|$ of $C_H(g_i)$, its conjugacy class in H. To state the result more formally, recall that I write $Z_G(g)$ for the centraliser[2] of g in G, which is the number of times $x^{-1}gx$ hits the same value g. The Main Formula is as follows:

Proposition *In the notation of* (9.9),

$$\begin{aligned} \operatorname{Ind}_H^G \varphi(g) &= \frac{1}{|H|} \times |Z_G(g)| \times \sum_{i=1}^{m} |C_H(g_i)| \varphi(g_i) \\ &= [G:H] \sum_{i=1}^{m} \frac{|C_H(g_i)|}{|C_G(g)|} \varphi(g_i). \end{aligned} \tag{9.10}$$

Proof When we evaluate the sum over $x \in G$ in (9.6), we only score when $x^{-1}gx$ is in $C_H(g_i)$ for one of the g_i. The number of times this happens is $|Z_G(g)|$ (the number of times that $x^{-1}gx$ stays at g) times $|C_H(g_i)|$ (the number of successful hits). The second line simply replaces $\frac{|Z_G(g)|}{|H|}$ by $\frac{|G|}{H \times |C_G(g)|}$. Q.E.D.

9.9 Induced character

The definition of induced class function in 9.4 was fixed up so as to include the character of an induced representation.

Proposition *Let $H \subset G$ and $V = \bigoplus \gamma L$ be as in Definition 9.4, and write χ_L for the character of L as $\mathbb{C}H$ and χ_V for that of V as G-module.*

Then $\chi_V = \operatorname{Ind}_H^G \chi_L$.

Proof $\chi_V(g)$ is the trace of $\rho_V(g) \colon V \to V$. But the action of g is subordinate to the direct sum decomposition $V = \bigoplus \gamma L$, taking each summand γL to another summand $g\gamma L$. Which summand it goes to is determined by the left action of G on G/H: if x represents a coset xH then gx represents gxH. When we calculate the trace of g acting on V in this block form, any off-diagonal block scores 0. The diagonal blocks correspond to $gxH = xH$, that is $x^{-1}gx \in H$.

In this case the component $\rho_V(g)$ does $g \colon xL \to xL$ and fits into a commutative diagram with $\rho(x^{-1}gx) \colon L \to L$, so has the same trace. (Write out the commutative diagram as Ex. 9.17.)

This makes $\operatorname{Tr}(\rho_V(g)) = \sum_{x \in G/H} \chi_L^0(x^{-1}gx)$, with the sum taken over cosets (that is, include just one representative of each coset xH when summing). If we write instead $\sum_{x \in G}$ the effect is to include all $|H|$ elements of the coset xH so we compensate by writing $\frac{1}{|H|}$ as in (9.6). Q.E.D.

[2] To compare with [James and Liebeck], Proposition 21.23, please bear in mind that they write x^G for the conjugacy class of x, my $C_G(g)$, and $C_G(g)$ for the centraliser of g, my $Z_G(g)$.

9.10 Induced representation, the construction

Theorem *Given $H \subset G$ and a $\mathbb{C}H$-module L, there exists a $\mathbb{C}G$-module $V = \operatorname{Ind}_H^G L$ containing L as an H-invariant subspace with the properties of Definition 9.4.*

In particular, $V = \operatorname{Ind}_H^G L$ has the UMP described in 9.5 and (9.4), and is uniquely characterised by it. That is, $V = \operatorname{Ind}_H^G$ is a $\mathbb{C}G$-module that is universal with the property that its restriction has a $\mathbb{C}H$-morphism $i\colon L \to \operatorname{Res}_H^G V$. This UMP should be viewed as the genuine definition of $\operatorname{Ind}_H^G L$, and the construction below as the proof that the UMP problem has a solution.

Proof We intend to set V equal to a direct sum of copies of L indexed by G/H. We think of the copies as gL for coset representatives, with the G action permuting these copies by its left action on G/H. However, we need some more notation and a little care to define the G action consistently.

The notation: write $\gamma \in G/H$ for the cosets, and choose once and for all a particular representative g_γ of each coset; choose $e = e_G$ as representative for the identity coset H. Now define $V = \bigoplus_{\gamma \in G/H} L_\gamma$ where the γ are indices labelling copies of L. This is V as a $\mathbb{C}$-vector space and as a $\mathbb{C}H$-module. I still have to say what its G-action is.

For $x \in G$, the product xg_γ is in some new coset βH, so $xg_\gamma = g_\beta h$ where $h = g_\beta^{-1} x g_\gamma \in H$. It is this element h that tells us how to multiply by x as a map $x\colon L_\gamma \to L_\beta$; namely, it sends $w \mapsto hw = g_\beta^{-1} x g_\gamma w$. This is defined, because L is an H-module, L_γ, L_β are copies of L, and $h \in H$. In terms of the motivating Definition 9.4, we think of the abstract summand L_γ as $g_\gamma L$, and define the action of $x \in G$ on it as fitting in the commutative diagram

$$\begin{tikzcd} g_\gamma L \arrow[r, "w"] & g_\beta L \\ L \arrow[u, "g_\gamma"] \arrow[r, "h"] \arrow[ur, phantom, "©"] & L \arrow[u, "g_\beta"'] \end{tikzcd} \tag{9.11}$$

For each $x \in G$ and each coset γ, this specifies $x\colon L_\gamma \to L_\beta$. Summing over all the L_γ gives a map $x\colon V \to V$, and defines a G action on $V = \bigoplus_{\gamma \in G/H} L_\gamma$. Q.E.D.

9.11 Frobenius reciprocity

The pair of operations Res_H^G and Ind_H^G are defined for representations in 9.4 and 9.10 and on class functions and characters in 9.7–9.9. On representations, 9.5 discussed the relation between them as the Universal Mapping Property of (9.4). That equality of vector spaces gives

$$\dim \operatorname{Hom}_{\mathbb{C}H}(L, \operatorname{Res}_H^G V) = \dim \operatorname{Hom}_{\mathbb{C}G}(\operatorname{Ind}_H^G L, V). \tag{9.12}$$

This numerical result implies the following adjunction property between the restriction $\operatorname{Res}_H^G\colon \mathcal{C}(G) \to \mathcal{C}(H)$ and induction $\operatorname{Ind}_H^G\colon \mathcal{C}(H) \to \mathcal{C}(G)$ of class functions.

Corollary (Frobenius reciprocity) *(I) Suppose that L is an irreducible H-module and V an irreducible G-module. Then the number of copies of L occurring in the direct sum decomposition of $\operatorname{Res}_H^G U$ equals the number of copies of U occurring in the direct sum decomposition of $\operatorname{Ind}_H^G L$.*

(II) Let $\varphi \in \mathcal{C}(H)$ be a class function on H and $\psi \in \mathcal{C}(G)$ a class function on G. Then

$$\left\langle \varphi, \operatorname{Res}_H^G \psi \right\rangle_H = \left\langle \operatorname{Ind}_H^G \varphi, \psi \right\rangle_G. \tag{9.13}$$

Proof (I) Write $\operatorname{Res}_H^G V = \sum a_i L_i$ for the irreducible decomposition of the restricted $\mathbb{C}H$-module and $\operatorname{Ind}_H^G = \sum b_j U_i$ for that of the induced $\mathbb{C}G$-module. Then $\dim \operatorname{Hom}_{\mathbb{C}H}(L, \operatorname{Res}_H^G V)$ equals the number a_1 of occurrences on $L = L_1$ in $\operatorname{Res}_H^G V$ by Schur's lemma on $\mathbb{C}H$-modules, whereas $\dim \operatorname{Hom}_{\mathbb{C}G}(\operatorname{Ind}_H^G L, V)$ equals the number b_1 of occurrences of $V = V_1$ in $\operatorname{Ind}_H^G L$. These two numbers are equal by (9.4), which proves (I).

The Main Theorem tell us that the characters χ_{L_i} of irreducible $\mathbb{C}H$-modules form a basis of $\mathcal{C}(H)$ and the characters χ_{V_j} of irreducible $\mathbb{C}G$-modules form a basis of $\mathcal{C}(G)$. (II) thus follows from (I) by the bilinearity property of the Hermitian pairings $\langle -,- \rangle_H$ and $\langle -,- \rangle_G$: if we set $\varphi = \sum \lambda_i \chi_{L_i}$ and $\psi = \sum \mu_j \chi_{V_j}$, both sides of (9.13) equal $\sum_{i,j} \overline{\lambda_i} \mu_j$.

9.12 Homework to Chapter 9

9.1. Tensor product The tensor product $V_1 \otimes V_2$ of two vector spaces can be defined as the vector space with basis $\{e_i \otimes f_j\}$ where $\{e_i\}$ is a basis of V_1 and $\{f_j\}$ a basis of V_2. If $g_1 \in \mathrm{GL}(V_1)$ and $g_2 \in \mathrm{GL}(V_2)$ are both diagonalisable, describe the action of $g_1 \otimes g_2 \in \mathrm{GL}(V_1 \otimes V_2)$.

Deduce that $\operatorname{Tr}(g_1 \otimes g_2) = \operatorname{Tr}(g_1) \operatorname{Tr}(g_2)$.

9.2. Character of $V_1 \otimes V_2$ If V_1, V_2 are representations of a finite group G, show that $\chi_{V_1 \otimes V_2} = \chi_{V_1} \chi_{V_2}$.

9.3. Character of $\operatorname{Sym}^2 V$ and $\bigwedge^2 V$ If V is a representation of G, the tensor product $V \otimes V$ splits as a direct sum

$$V \otimes V = \operatorname{Sym}^2 V \oplus \bigwedge^2 V. \tag{9.14}$$

Here the space of symmetric tensors Sym^2 is based by $e_i \otimes e_j + e_j \otimes e_i$ for $1 \le i \le j \le n$ whereas $\bigwedge^2$ is based by $e_i \otimes e_j - e_j \otimes e_i$ for $1 \le i < j \le n$. Prove that the characters are given by

$$\chi_{\mathrm{Sym}^2 V}(g) = \tfrac{1}{2}\big(\chi_V^2(g) + \chi_V(g^2)\big) \quad \text{and} \quad \chi_{\bigwedge^2 V}(g) = \tfrac{1}{2}\big(\chi_V^2(g) - \chi_V(g^2)\big). \tag{9.15}$$

[Hint: work in a basis in which $\rho_V(g)$ is diagonal, separately for each $g \in G$.]

9.4 Recall dual $\mathbb{C}G$-module $V^\vee = \mathrm{Hom}_C(V, \mathbb{C})$, and how G acts on it. Calculate its character $\chi_{V^\vee}$.

9.5 Run through the definitions then give a proof of the following assertion: the Hom space $\mathrm{Hom}_{\mathbb{C}}(V_1^\vee, V_2)$ is canonically isomorphic to $V_1 \otimes_C V_2$ as $\mathbb{C}G$-modules

9.6 Show that a linear map $\varphi\colon V^\vee \to V$ can be interpreted as a bilinear form $V \times V \to \mathbb{C}$ and state and prove an assertion on the Hom space $\mathrm{Hom}_{\mathbb{C}}(V^\vee, V)$ analogous to the splitting of $V \otimes V$ as the direct sum of $\mathrm{Sym}^2 V$ and $\bigwedge^2 V$.

9.7 The symmetric group S_5 has a 4-dimensional irreducible representation V_4 (obtained by taking a complement to the main diagonal $(1,1,1,1,1)$ in the natural permutation representation of S_5 on $\mathbb{C}^5$).

Show that its character is

S_5	e	$(12)(34)$	(12)	(123)	(1234)	(12345)	$(123)(45)$
size	1	15	10	20	30	24	20
χ_{V_4}	4	0	2	1	0	-1	-1

(9.16)

9.8 With S_5 and V_4 as in Ex. 9.7, determine the character of $\bigwedge^2 V_4$. Use the formula $\chi_{\wedge^2 V} = \frac{1}{2}\big(\chi(g)^2 - \chi(g^2)\big)$.

By calculating its inner product $\langle \chi, \chi \rangle$, prove that it is irreducible.

9.9 With S_5 and V_4 as in Ex. 9.7, determine the character of $V_{10} = \mathrm{Sym}^2 V_4$ using the formula $\chi_{\wedge^2 V} = \frac{1}{2}\big(\chi(g)^2 - \chi(g^2)\big)$.

Calculate its inner product $\langle \chi, \chi \rangle$, and prove that it is the sum of 3 irreducibles. Calculate the inner product of its character with χ_1 and χ_{V_4}, and use this to show that V_{10} is the direct sum of $\mathbb{C}$, V_4 and a new irreducible representation V_5, and write out the character of V_5.

9.10 Use the sign representation and all the above to assemble the whole character table of S_5.

9.11 Restrict each of the representations of S_5 to A_5 and determine how they decompose into irreducibles. For examples the restrictions of V_4 and $\wedge^2 V_4$ have characters

A_5	e	(12)(34)	(123)	(12345)	(13524)
size	1	15	20	12	12
χ_{V_4}	4	0	1	-1	-1
$\bigwedge^2$	6	-2	0	1	1

(9.17)

so that one remains irreducible, and the other splits as a sum of two irreducibles.

9.12 Each of the dihedral groups D_{2m} and binary dihedral groups BD_{4m} has a cyclic normal subgroup $A \lhd G$ of index 2. Show how to find their 2-dimensional irreducible representations by inducing up from A. Specifically, consider D_{14} and the 1-dimensional representations of $A = \mathbb{Z}/7$ given by multiplication by $\varepsilon, \varepsilon^2, \varepsilon^3$ where $\varepsilon = \exp \frac{2\pi}{7}$. Write out the character table of D_{14}.

9.13 Consider the group $G_{39} = \langle x, y \mid x^3, y^7, xy = y^3x \rangle$. Show that $xy^kx^{-1} = y^{3k}$, and deduce that $H \lhd G$ and that conjugacy by x acts as a product of 4 3-cycles on H, that is the permutation

$$(y, y^3, y^9)(y^2, y^6, y^5)(y^4, y^{12}, y^{10})(y^7, y^8, y^{11}). \tag{9.18}$$

Calculate the effect of conjugacy on the elements xy^k and x^2y^k for $k = 0, \dots, 12$, and deduce that G_{39} has exactly 7 conjugacy classes.

9.14 G_{39} has $H = \langle y \rangle$ as a normal subgroup, with quotient $G_{39}/H = \mathbb{Z}/3$ generated by the class of x. As a cyclic group, H has 13 representations L_k (for $k = 0, \dots, 12$) consisting of $\mathbb{C}$ on which y acts by ε^k (where $\varepsilon = \exp \frac{2\pi i}{13}$ is the usual 13th root of 1. Calculating the induced characters $\mathrm{Ind}_H^G L_k$ for $k = 0, \dots, 12$, show that G_{39} has 4 nonisomorphic irreducible 3-dimensional representation.

9.15 Write out the whole character table of G_{39}.

9.16 Prove Proposition 9.4. [Hints: Start from the sum in (9.8). We need to know how many of the conjugates $x^{-1}gx$ of $g \in G$ are in H, and then how many are conjugate to each of the g_i.]

9.17 The construction of the induced representation in Theorem 9.6 involved the choice of coset representatives g_γ. The action of x on $\bigoplus W^\gamma$ is made up of components $h \colon W^\gamma \to W^\beta$, where the element $h \in H$ is deduced from the formula $xg_\gamma = g_\beta h$. If $W \subset V$ is given by the more simple-minded definition of Section 9.3, draw the corresponding commutative diagram and deduce the contribution that each component makes in the formula for the induced character in 9.5.

10 The simple group G_{168}

10.1 Introduction

There are 4 separate descriptions of G_{168} in geometry.

(1) $G_{168} \cong \mathrm{PSL}(2, \mathbb{F}_7)$. Its action on $\mathbb{P}^1_{\mathbb{F}_7}$ makes it into a permutation group on 8 elements $\{0, 1, 2, 3, 4, 5, 6, \infty\}$.

(2) $G_{168} \cong \mathrm{GL}(3, \mathbb{F}_2)$. This acts on the projective plane $\mathbb{P}^2_{\mathbb{F}_2}$, a configuration of 7 points and 7 lines with 3 points on each line and 3 lines through each point (also called the Fano plane, see Ex. 10.8). This action makes G_{168} into a permutation group on the 7 column vectors:

$$\begin{array}{ccccccc} P_1 & P_2 & P_4 & P_3 & P_5 & P_6 & P_7 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array} \tag{10.1}$$

(3) G_{168} is constructed from the infinite reflection group $R_{2,3,7}$ generated by reflections in the sides of a hyperbolic triangle $\Delta_{2,3,7}$ with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$. (The point is that $\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{7} = \frac{41}{42}\pi < \pi$.) This is the symmetry group of the tesselation of the hyperbolic plane $\mathcal{H}^2$ by triangles congruent to $\Delta_{2,3,7}$, or equivalently, the tesselation by regular heptagons of angle $\frac{2\pi}{3}$. The index 2 subgroup $R^+_{2,3,7} \lhd R_{2,3,7}$ of direct motions includes the rotations of order $2, 3, 7$ about the vertices of $\Delta_{2,3,7}$, but no reflections. In turn, $R^+_{2,3,7}$ has a unique normal subgroup π_1 of index 168 that acts with no fixed points (that is, avoids every rotation), and $G_{168} = R^+_{2,3,7}/\pi_1$.

This description makes G_{168} into the symmetry group of a certain polyhedron drawn on the Riemann surface $\mathcal{H}^2/\pi_1$ of genus 3, that has many beautiful descriptions. My treatment here is intended to advertise and complement John Baez's webpage "Klein's quartic curve"

`http://math.ucr.edu/home/baez/klein.html`

(4) G_{168} is the group of automorphisms of the Klein quartic curve

$$K_4 : (xy^3 + yz^3 + zx^3 = 0) \subset \mathbb{P}^2_{\mathbb{C},\langle x,y,z\rangle} \tag{10.2}$$

As you see from the equation, this curve K_4 has the remarkable property that each of the 3 coordinate lines is the tangent line at a flex (inflexion point) and cuts it at a second such flex (for example $z = 0$ intersects K_4 at $(1, 0, 0)$ with multiplicity 3 and $(0, 1, 0)$ with multiplicity 1. In fact there are 8 such triangles, the other 7 defined over the cyclotomic field $\mathbb{Q}[\varepsilon]$ of degree 7. This description makes G_{168} a subgroup of $\mathrm{SL}(3, \mathbb{C})$.

(5) We will see by computer algebra that G_{168} has the presentation

$$G_{168} = \left\langle x, y \mid x^2, y^3, (xy)^7, (yxyxy)^4 \right\rangle. \tag{10.3}$$

Status This chapter is not examinable, and is basically a self-indulgent work-out of personal obsessions. However, some of the material (esp. Ex. 10.1–5) provide useful practice in handling groups in various contexts, roots of unity, representations and character tables, all of which are examinable.

With each of the models, a basic aim should be to express everything so that all the calculations can be done in a convincing way, preferably without excessive reliance on computer algebra. I have fallen short of this aim to some extent, but maybe I can do better next year.

10.2 Linear groups over finite fields

Groups such as $\mathrm{PSL}(n, \mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ of order $q = p^n$ are an important source of finite simple groups. We know a lot about them, based on the analogy with the theory of algebraic groups. The reason for considering $\mathrm{PSL}(n, \mathbb{F}_q)$ is that a nontrivial determinant map $G \to K^\times$ or central element $\lambda\,\mathrm{Id}$ prevents a group G from being simple. Setting determinant equal to 1 and dividing by the centre gets around this trivial disqualification. I give a brief round-up of notation and basic facts. (Similar ideas apply to other simple algebraic groups.)

The *general linear group* $\mathrm{GL}(n, K)$ has rows giving every possible basis of the vector space K^n. In particular, over $K = \mathbb{F}_q$ there are $q^n - 1$ choices for the first row (any nonzero vector), then $q^n - q$ for the second row (anything other than a multiple of the first row), and $q^n - q^{i-1}$ for the ith row, giving order $|\mathrm{GL}(n, \mathbb{F}_q)| = \prod_{i=1}^n (q^n - q^{i-1})$.

The *special linear group* $\mathrm{SL}(n, K)$ is the kernel of the surjective group homomorphism $\det\colon \mathrm{GL}(n, K) \to K^\times$. Its order over $K = \mathbb{F}_q$ is thus $|\mathrm{SL}(n, \mathbb{F}_q)| = |\mathrm{GL}(n, \mathbb{F}_q)|/(q - 1)$.

The *projective general linear group* $\mathrm{PGL}(n, K)$ is the quotient of $\mathrm{GL}(n, K)$ by its centre, the scalar matrices $\lambda\,\mathrm{Id}$. In other words it is the quotient by the equivalence relation $M \sim \lambda M$ with $\lambda \in K^\times$. Over $K = \mathbb{F}_q$ there are $q - 1$ elements in every equivalence class, so its order is also $|\mathrm{GL}(n, K)|/(q - 1)$. We can also define $\mathrm{PGL}(n, K)$ as the group of all projective linear transformations of $\mathbb{P}^{n-1}_K$, so its order is the number of projective frames of reference in $\mathbb{P}^{n-1}_K$.

Finally, $\mathrm{PSL}(n, K)$ is the *projective special linear group.* The kernel of $\mathrm{SL}(n, K) \to \mathrm{PSL}(n, K)$ consists of the scalar matrices $\lambda\,\mathrm{Id}$ of determinant 1. This means $\lambda \in K^\times$ has $\lambda^n = 1$, so λ is an nth root of 1 in K; the number of these depends on n and on K. For our present purposes the most important case is $n = 2$. In this case, $\mathbb{F}_q$ has two distinct square roots of 1 if and only if K has odd characteristic, $\mathrm{char}\, K \neq 2$.

Another way of counting the order of $\mathrm{PSL}(n, K)$ is via the following claim: an element $g \in \mathrm{PGL}(n, K)$ has a well defined determinant in the quotient group $K^\times/(K^\times)^n$, and $g \in \mathrm{PSL}(n, K)$ if and only if this is 1. Indeed, g is a class of matrices M up to scalar multiplication, and $\det(\lambda M) = \lambda^n \det M$. For a finite field, the nth power map

$$K^\times \xrightarrow{n} K^\times \quad \text{given by} \quad \lambda \mapsto \lambda^n \tag{10.4}$$

has kernel $\boldsymbol{\mu}_n(K)$ and cokernel $K^\times/(K^\times)^n$ of the same order, so the index $[\mathrm{PGL}(n,K):\mathrm{PSL}(n,K)]$ equals the order of $\boldsymbol{\mu}_n(K)$, and the two ways of calculating $|\,\mathrm{PSL}(n,K)|$ give the same answer.

10.3 $\mathrm{PSL}(2,\mathbb{F}_7)$

Over $\mathbb{F}_7$ the matrices

$$X=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad Y=\begin{pmatrix}0&1\\-1&-1\end{pmatrix},\quad XY=\begin{pmatrix}1&1\\0&1\end{pmatrix} \tag{10.5}$$

satisfy $X^2=-\mathrm{Id}$, $Y^3=\mathrm{Id}$, $(XY)^7=\mathrm{Id}$. Passing to the projective linear group $\mathrm{PSL}(2,\mathbb{F}_7)$ gives $X^2\mapsto\mathrm{Id}$, since we quotient by scalar matrices. X acts on $\mathbb{P}^1$ by $z\mapsto -1/z$, and XY acts by $(z\mapsto z+1)$.

Over $\mathbb{F}_7$ the projective line has 8 points. It is convenient to label $z=j$ as P_j for $j=1,\dots,6$, with the point $z=0$ as P_7, and $z=\infty$ as P_8. Then X,Y,XY act as

$$x=(1,6)(2,3)(4,5)(7,8),\quad y=(1,3,5)(6,8,7),\quad xy=(1,2,3,4,5,6,7). \tag{10.6}$$

The context of permutations in S_8 is a convenient playground to experiment with $\mathrm{PSL}(2,\mathbb{F}_7)$. For example, we know that $G_7:=\langle x,y \mid x^2,y^3,(xy)^7\rangle$ is infinite, whereas permutation groups and linear groups over finite fields are finite. We can easily find other relations satisfied by the generators x,y of (10.6): for example, composing permutations gives $(xy)^3=(1,4,7,3,6,2,5)$, hence $x(xy)^3=(1,5,6,3)(2,4,8,7)$. This is an element of order 4, which implies that $(x(xy)^3)^4=(yxyxy)^4=\mathrm{Id}$. At the level of matrices $(YXYXY)^4=-\mathrm{Id}$ is a relation between the matrices (10.5). Magma says that adding $(yxyxy)^4$ as a relation cuts G_7 down to a group of order 168.

```
Gbar<x,y> := Group< x,y | x^2, y^3, (x*y)^7, (y*x*y*x*y)^4 >;
Order(Gbar);
```

I've not seriously attempted to prove this by hand. Two of the relations are words of order 14 and 20 in x,y, making hand calculations from first principles with this presentation cumbersome and extremely error-prone.

At the same time, we can use the permutation group playground to list the conjugacy classes in $\mathrm{PSL}(2,\mathbb{F}_7)$. Experimenting a bit gives the 6 classes:

- e
- 21 elements conjugate to $x=(1,6)(2,3)(4,5)(7,8)$
- 56 elements conjugate to $y=(1,3,5)(6,8,7)$
- 42 elements conjugate to $yxyxy=(1,5,6,3)(2,4,8,7)$
- 24 elements conjugate to $xy=(1,2,3,4,5,6,7)$
- 24 elements conjugate to $(xy)^3=(1,4,7,3,6,2,5)$

There is an interesting point about the 48 elements of order 7. Namely xy, $(xy)^2$, $(xy)^4$ are in one conjugacy class, while their inverses $(xy)^6, (xy)^5, (xy)^3$ are in another (compare Ex. 10.6). Here $1, 2, 4$ are the quadratic residues mod 7, and $3, 5, 6$ the nonresidues. This is analogous to the 5-cycles $(1, 2, 3, 4, 5)$ and $(1, 3, 5, 2, 4) = (1, 2, 3, 4, 5)^2$ in A_5, that are conjugate in the slightly bigger group S_5. The bigger group $\mathrm{PGL}(2, \mathbb{F}_7)$ of order 336 plays the same role for G_{168}.

10.4 $\mathrm{GL}(2, \mathbb{F}_2)$

Over $\mathbb{F}_2$ the matrices

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad XY = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{10.7}$$

satisfy the relations $X^2 = Y^3 = (XY)^7 = (YXYXY)^4 = \mathrm{Id}$.

Acting by these matrices on the 7 columns $P_{1\dots7}$ of (10.1) permutes them as

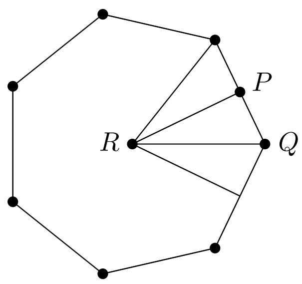
$$x = (4, 5)(6, 7), \quad y = (1, 2, 4)(3, 6, 5), \quad xy = (1, 2, 5, 3, 7, 6, 4),$$

that satisfy $x^2 = y^3 = (xy)^7 = \mathrm{Id}$ and $(yxyxy) = (1, 3, 5, 7)(2, 6)$, so that $(yxyxy)^4 = \mathrm{Id}$.

Much the same games apply as in the case of permutations on 8 elements. It is an interesting (if somewhat elaborate) exercise to arrange all the elements of order 3 and 7 of $\mathrm{GL}(2, \mathbb{F}_2)$ into 8 conjugate subgroups $H_{21} \subset \mathrm{GL}(2, \mathbb{F}_2)$ of order 21, and thus recover the action of $\mathrm{GL}(2, \mathbb{F}_2)$ as a permutation group on a set of 8 elements. It is a straightforward puzzle to derive the whole character table of G_{168} using only the conjugacy classes described in 10.3, and the two permutation representations on $\mathbb{P}^1_{\mathbb{F}_7}$ and $\mathbb{P}^2_{\mathbb{F}_2}$ (see Ex. 10.12).

10.5 Tesselation of $\mathcal{H}^2$ by $2, 3, 7$ triangles

Let $PQR = \Delta_{2,3,7}$ be a triangle in $\mathcal{H}^2$ with $\angle P = \frac{\pi}{2}$, $\angle Q = \frac{\pi}{3}$, $\angle R = \frac{\pi}{7}$.



(10.8)

Write a, b, c for the reflections $a = \mathrm{Refl}(QR)$, $b = \mathrm{Refl}(PR)$, $c = \mathrm{Refl}(PQ)$ and $R_{2,3,7}$ for the group of hyperbolic motions they generate. Since both a and b fix R, one sees that ba is the rotation by $\frac{2\pi}{7}$ anticlockwise about R, and $(ba)^7 = \mathrm{Id}$.

The successive images of $\triangle_{2,3,7}$ under a, b exactly cover the regular heptagon. In the same way, ac is rotation by $\frac{2\pi}{3}$ about Q, and $bc = cb$ is half-turn or rotation by π about P. These rotations generate a subgroup $R^+_{2,3,7} \lhd R_{2,3,7}$ of index 2 with $R^+_{2,3,7} \cong G_7$. One sees that performing the reflections and rotations of $R_{2,3,7}$ iteratively leads to a tesellation of $\mathcal{H}^2$ by triangles congruent to $\triangle_{2,3,7}$. More precisely the last picture on John Baez's webpage depicts this tesselation, with the images of $\triangle PQR$ by rotations drawn as coloured triangles, and their reflections in grey. The triangles correspond 1-to-1 with the elements of the group $R_{2,3,7}$, and the coloured triangles to the subgroup $R^+_{2,3,7}$ so this picture is its Cayley graph drawn out as a geometric model in $\mathcal{H}^2$.

It follows from this that the topological quotient space $\mathcal{H}^2/R_{2,3,7}$ is a single triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{7}$. The quotient $\mathcal{H}^2/R^+_{2,3,7}$ by the rotation group is made of two copies of the same $(2,3,7)$ triangle glued along their edges. It is a figure like a samosa, topologically S^2 but with 3 cone points as corners, with total angle around P, Q, R respectively π, $\frac{2\pi}{3}$, $\frac{2\pi}{7}$. Outside the corners, the surface $S^2\setminus\{P,Q,R\}$ inherits the structure of hyperbolic 2-manifold from the covering space $\mathcal{H}^2 \to S^2$.

10.6 The torsion free subgroup π_1

The Magma argument in 10.3 tells us that the 4th power of the word $w = yxyxy$ generates a normal subgroup in $G_7 = R^+_{2,3,7}$ of index 168. We can verify this explicitly in Magma by writing out 6 of the first few conjugates of w, and considering the subgroup of G_7 generated by their 4th powers. After some trial and error, I choose the following 6 conjugates of $w = yxyxy$:

$$w,\ xwx,\ y^2wy,\ ywy^2,\ yxwxy^2,\ xyxwxy^2x \tag{10.9}$$

and let their 4th powers generate $\pi_1 \subset G$. Magma asserts that π_1 is normal of index 168. Then $x, y \in G/\pi_1$ satisfy the relations x^2, y^3, $(xy)^7$, $(yxyxy)^4$ so that G/π_1 is isomorphic to G_{168}.

```
G<x,y> := Group< x,y | x^2,y^3,(x*y)^7 >; // G7 as f.p. group
w := y * x * y * x * y;
L:=[w, x*w*x, y^2*w*y, y*w*y^2, y*x*w*x*y^2, x*y*x*w*x*y^2*x];
pi1 := sub< G | [w^4 : w in L] >;
Index(G, pi1); IsNormal(G, pi1);
```

Choosing a fundamental domain for $\pi_1 \subset G$ is closely related to choosing the 24 heptagons described on John Baez's website. (Sorry, no time to do this in detail.) The elements $w \in R^+_{2,3,7}$ in (10.9) are hyperbolic translations, and their 4th powers identify the heptagons with the same number $1, \dots, 24$.

10.7 The group $R^+_{2,3,7}$ as Möbius transformations

To write out the reflection group $R_{2,3,7}$ and its rotation subgroup $R^+_{2,3,7}$, the most convenient model of $\mathcal{H}^2$ is the unit ball $|z| < 1$ in $\mathbb{C}$ with hyperbolic motions given by fractional-linear transformations $z \mapsto \frac{az+b}{cz+d}$ preserving the boundary $S^1 : |z| = 1$. For s in the real interval $(0,1)$, the transformation $X\colon z \mapsto \frac{z-s}{sz-1}$ does $0 \leftrightarrow s$ and $1 \leftrightarrow -1$. The generators I want are X, the 7-fold rotation $R\colon z \mapsto \varepsilon z$ and their composite $Y = XR$. The angle at P_s is a monotonically decreasing function of s (going from $\frac{2\pi}{7}$ down to 0), and there is a unique value of s for which it is $\frac{\pi}{3}$, so that Y is a rotation by $\frac{2\pi}{3}$. Writing $Y^3 = \mathrm{Id}$ gives rise to a quadratic equation having the root $s = \sqrt{2\cos\frac{2\pi}{7} - 1} \doteq 0.496970$, with minimal polynomial $s^6 + 4s^4 + 3s^2 - 1 = 0$.

With this choice of matrices

$$X = \begin{pmatrix} 1 & -s \\ s & -1 \end{pmatrix}, \quad R = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = XR \tag{10.10}$$

generate a copy of $G_7 = \left\langle x, y \mid x^2, y^3, (xy)^7 \right\rangle$ acting on the ball $|z| < 1$ in $\mathbb{C}$, a model of hyperbolic space $\mathcal{H}^2$. Compare Ex. 10.14.

10.8 The Klein quartic

The Klein quartic is the plane curve

$$K_4 : (xy^3 + yz^3 + zx^3 = 0) \subset \mathbb{P}^2_{\mathbb{C},\langle x,y,z\rangle}. \tag{10.11}$$

A nonsingular plane quartic has genus $g = 3$. The curve K_4 is remarkable as the algebraic curve having the maximal number $84(g-1)$ of symmetries of any curve of genus $g \geq 2$. The order 3 symmetry $b\colon x \mapsto y \mapsto z \mapsto x$ is clear.

Assuming that we own $\varepsilon = \varepsilon_7 = \exp\frac{2\pi i}{7}$, we also get the order 7 symmetry of $\mathbb{P}^2$ given by $\frac{1}{7}(1,2,4)$, meaning $a\colon (x,y,z) \mapsto (\varepsilon x, \varepsilon^2 y, \varepsilon^4 z)$. Since xy^3 is multiplied by $\varepsilon \cdot (\varepsilon^2)^3 = 1$ and similarly for yz^3 and zx^3, each monomial in the equation of K_4 is invariant under this action, so that a is a symmetry of K_4 of order 7.

We see that $ab = ba^2$, which means that a, b generate a group H_{21} of order 21 with $A = \langle a \rangle = \mathbb{Z}/7$ as a normal subgroup of index 3:

$$\mathbb{Z}/7 \lhd H_{21} \twoheadrightarrow \mathbb{Z}/3. \tag{10.12}$$

It turns out that there are 8 other coordinate systems on $\mathbb{P}^2$ in which K_4 has the same symmetry.

10.9 Flexes of K_4 and $G_{168} \subset \mathrm{SL}(3,\mathbb{C})$

I express everything about the action of G_{168} on K_4 in terms of the 24 flexes of K_4 given in (I) below. For example, each row of each matrix of G_{168} is proportional to one of the flexes. Finding the flexes requires a little algebraic geometry, that I explain in Section 10.10 to avoid interrupting the algebra.

Proposition *(I) The flex points of K_4 are the three coordinate points*

$$P_1 = (1,0,0), \quad P_2 = (0,1,0), \quad P_3 = (0,0,1), \tag{10.13}$$

the 18 *points*

$$R_i = (1-\varepsilon^i, \varepsilon^i - \varepsilon^{5i}, 1-\varepsilon^{2i}), \quad S_i = (1-\varepsilon^{2i}, 1-\varepsilon^i, \varepsilon^i - \varepsilon^{5i}),$$
$$T_i = (\varepsilon^i - \varepsilon^{5i}, 1-\varepsilon^{2i}, 1-\varepsilon^i) \quad \textit{for } i = 1, \dots, 6, \tag{10.14}$$

and the 3 *points*

$$Q_1 = (\varepsilon - \varepsilon^6, \varepsilon^4 - \varepsilon^3, \varepsilon^2 - \varepsilon^5), \quad Q_2 = (\varepsilon^2 - \varepsilon^5, \varepsilon - \varepsilon^6, \varepsilon^4 - \varepsilon^3),$$
$$Q_3 = (\varepsilon^4 - \varepsilon^3, \varepsilon^2 - \varepsilon^5, \varepsilon - \varepsilon^6) \tag{10.15}$$

(or as real ratios, $Q_1 = \left(\sin\frac{2\pi}{7} : \sin\frac{8\pi}{7} : \sin\frac{4\pi}{7}\right)$ *and similar).*

(II) The flex lines to K_4 *are*

$$\begin{aligned}
&T_{K_4,P_1} : z = 0, \quad T_{K_4,P_2} : x = 0, \quad T_{K_4,P_3} : y = 0,\\
&T_{K_4,R_i} : (-\varepsilon^i + \varepsilon^{3i})x + (1-\varepsilon^{6i})y + (1-\varepsilon^{3i})z = 0,\\
&T_{K_4,S_i} : (1-\varepsilon^{3i})x + (-\varepsilon^i + \varepsilon^{3i})y + (1-\varepsilon^{6i})z = 0,\\
&T_{K_4,T_i} : (1-\varepsilon^{6i})x + (1-\varepsilon^{3i})y + (-\varepsilon^i + \varepsilon^{3i})z = 0,\\
&T_{K_4,Q_i} : (\varepsilon^{2i} - \varepsilon^{-2i})x + (\varepsilon^i - \varepsilon^{-i})y + (\varepsilon^{4i} - \varepsilon^{-4i})z = 0.
\end{aligned} \tag{10.16}$$

Each flex line meets K_4 *at a further flex point. Thus*

$$T_{K_4,R_i} \cap K_4 = 3R_i + S_{2i}, \quad T_{K_4,S_{2i}} \cap K_4 = 3S_{2i} + T_{4i},$$
$$T_{K_4,T_{4i}} \cap K_4 = 3S_{2i} + R_i \quad \textit{for } i = 1, \dots, 6. \tag{10.17}$$

This gives the 8 *flexing triangles referred to above.*

$$P_1P_2P_3, \quad Q_1Q_2Q_3, \quad R_iS_{2i}T_{4i} \quad \textit{for } i = 1, \dots, 6. \tag{10.18}$$

Curiously, the flex lines and points are given by the same array of numbers: the line T_{K_4,R_1} *has the coordinates of* T_3 *as its coefficients, and so on.*

(III) Consider the subgroup $G = \langle I, A\rangle \subset \mathrm{SL}(3,\mathbb{C})$ *generated by*

$$I = \frac{-1}{\sqrt{-7}} \begin{pmatrix} \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 \\ \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 \\ \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}. \tag{10.19}$$

Then $|G| = 168$*. There are* 336 *triples of matrices* (X, Y, XY) *in* G *that satisfy the relations* $X^2 = Y^3 = (XY)^7 = (YXYXY)^4 = e$*, so that* $G \cong G_{168}$*.*

(IV) Every $g \in G$ *has rows and columns made up of triples proportional to the flex points of* K_4*.*

(V) For every $g \in G$, if I write x_g, y_g, z_g for the linear forms

$$\begin{pmatrix} x_g \\ y_g \\ z_g \end{pmatrix} = g \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{10.20}$$

then $x_g y_g^3 + y_g z_g^3 + z_g x_g^3 = K_4$. That is, the equation of K_4 is invariant under G_{168}.

10.10 Flexes and Hessian

For a nonsingular plane curve $C : (F = 0) \subset \mathbb{P}^2_{\mathbb{C}}$ and $P \in C$, the tangent line $T_P C$ is the unique line L having intersection multiplicity ≥ 2 with C at P. This means simply that if x is a local coordinate on the line at P, the equation F restricted to L written as a polynomial in x has a zero of multiplicity ≥ 0 at P, or is divisible by x^2. The tangent line is given by the linear equation

$$T_P C : \left(\tfrac{\partial F}{\partial x}(P)x + \tfrac{\partial F}{\partial y}(P)y + \tfrac{\partial F}{\partial z}(P)z = 0 \right) \subset \mathbb{P}^2. \tag{10.21}$$

For example, the tangent line to K_4 at $(1, 0, 0)$ is $z = 0$.

It sometimes happens that the tangent line $T_P C$ has higher order contact with C (that is, the equation of C restricted to $T_P C$ has a zero of order ≥ 3 at P). In this case, $P \in C$ is called a *flex* (or *inflection point*). For example, K_4 has the 3 lines $xyz = 0$ as flex lines, with $z = 0$ intersecting K_4 in $xy^3 = 0$, that is, the point $(1, 0, 0)$ (given by $y = 0$) with multiplicity 3, and $(0, 1, 0)$ with multiplicity 1.

This behaviour is controlled by second derivatives. A familiar analogy from school calculus is that a function $y = f(x)$ has a maximum where $f'(x) = 0$ and $f''(x) < 0$, and a minimum where $f'(x) = 0$ and $f''(x) > 0$. If $f'(x) = f''(x) = 0$ then the line $y = f(x)$ f restricted to the line $y = f(x)$ has a zero of order ≥ 3.

It is known that the flex points of a plane curve $F = 0$ in $\mathbb{P}^2$ are determined by the Hessian, or matrix of second derivatives $\left| \frac{\partial^2 F}{\partial x_i} \right|$. In our case $K_4 = xy^3 + yz^3 + zx^3$, (taking out common factors of 3 and 2) we get

$$H = \frac{1}{2} \times \det \begin{vmatrix} 2xz & y^2 & x^2 \\ y^2 & 2xy & z^2 \\ x^2 & z^2 & 2yz \end{vmatrix} = 5x^2y^2z^2 - x^5 - y^5 - z^5x. \tag{10.22}$$

Part (I) of Proposition 10.9 states that the curves K_4 and H intersect in the 24 points listed. This Magma routine performs the calculations that prove (I).

```
K7<ep> := CyclotomicField(7);
Rxyz<x,y,z> := PolynomialRing(K7,3); K4 := x*y^3+y*z^3+z*x^3;
HessMat := 1/3*Matrix(3,
  [Derivative(Derivative(K4,Rxyz.i),Rxyz.j) : i,j in [1..3]]);
HessMat; H := 1/2*Determinant(HessMat); H;
PP := Proj(Rxyz); Flex := Points(Scheme(PP,[K4,H]));
```

```
#Flex; // Flex[1..4];
Flex[2]; Flex[2] eq PP![1-ep^3, ep-ep^6, -1+ep];
Flex[6]; Flex[6] eq PP![ep-ep^6, ep^4-ep^3, ep^2-ep^5];
```

10.11 Homework to Chapter 10

10.1 Let $\varepsilon = \exp \frac{2\pi}{7}$ be the standard primitive root of 1. Find the quadratic equation with roots $\alpha = \varepsilon + \varepsilon^2 + \varepsilon^4$ and $\beta = \varepsilon^6 + \varepsilon^5 + \varepsilon^3$. Hence find a surd expression for α and β. Or if you prefer, find an expression for $\sqrt{-7}$ as a linear combination of the ε^i.

10.2 Using the fact that $\cos \frac{2\pi}{7} = \varepsilon + \varepsilon^6$, find the cubic equation with roots $\cos \frac{2\pi}{7}$, $\cos \frac{4\pi}{7}$, $\cos \frac{6\pi}{7}$.

10.3 Check that $z \mapsto -1/z$ acts on $\mathbb{P}^1_{\mathbb{F}_7}$ (with the points labelled as in 10.3) by $(1,6)(2,3)(4,5)(7,8)$.

10.4 Over any field K, any 3 distinct points of $\mathbb{P}^1_K$ form a projective frame of reference; it is a theorem of projective geometry that there is a unique projective linear transformation $\mathbb{P}^1_K \to \mathbb{P}^1_K$ taking any projective frame of reference to any other. Therefore, for a finite field $\mathbb{F}_q$, the group $\mathrm{PGL}(2, K)$ has order $q \cdot (q-1) \cdot (q-2)$; in particular $|\,\mathrm{PGL}(2, \mathbb{F}_7)| = 8.7.6 = 336$.

10.5 Label the points of $\mathbb{P}^1_{\mathbb{F}_7}$ as in 10.3, so that $P_7 = (0:1)$, $P_8 = (1:0)$ and $P_1 = (1,1)$ is the standard projective frame of reference. Show that for $i = 1, \dots, 7$ the projective linear transformation taking

$$(P_7, P_8, P_1) \mapsto (P_7, P_8, P_i) \tag{10.23}$$

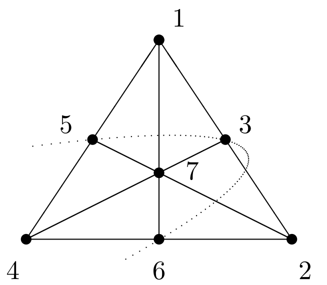
is in $\mathrm{PSL}(2, F_7)$ if and only if $i = 1, 2, 4$, that is, i is a quadratic residue mod 7.

10.6 $G_{168} = \mathrm{PSL}(2, \mathbb{F}_7)$ is a subgroup of index 2 of $\mathrm{PGL}(2, \mathbb{F}_7)$ (see the end of 10.3). This is in complete analogy with the inclusion $A_5 \subset S_5$ that we have seen several times in earlier chapters. In the notation of 10.3, verify that xy and $(xy)^3$ (that are not conjugate in $\mathrm{PSL}(2, \mathbb{F}_7)$, as I said in the last sentence of 10.3) are conjugate in the bigger group $\mathrm{PGL}(2, \mathbb{F}_7)$.

10.7 The following Magma code verifies the conjugacy classes of $\mathrm{PSL}(2, \mathbb{F}_7)$ as listed in 10.3:

```
F7 := FiniteField(7); G := ProjectiveSpecialLinearGroup(2,F7);
ConjugacyClasses(G);
```

10.8. The Fano plane The finite projective plane $\mathbb{P}^2_{\mathbb{F}_2}$ with its 7 points and 7 lines is traditionally pictured as follows:



P_1	P_2	P_4	P_3	P_5	P_6	P_7
1	0	0	1	1	0	1
0	1	0	1	0	1	1
0	0	1	0	1	1	1

(10.24)

The 7 lines (whose equations are given by the same array) are

$$\begin{gathered} L_1 = P_2P_4P_6, \quad L_2 = P_1P_4P_5, \quad L_4 = P_1P_2P_3, \\ L_3 = P_3P_4P_7, \quad L_5 = P_2P_5P_7, \quad L_6 = P_1P_6P_7 \\ \text{and} \quad L_7 = P_3P_5P_6. \end{gathered} \tag{10.25}$$

10.9 In the same coordinates as (10.24), write down the matrix $Y \in \mathrm{GL}(3, \mathbb{F}_2)$ that permutes the points as $(1, 2, 4)(3, 6, 5)$ (that is, rotates the triangle clockwise) and the matrix X that does $(4, 5)(6, 7)$. Check that $X^2 = Y^3 = 1$, that $XY = \left(\begin{smallmatrix} 0&1&1 \\ 1&0&0 \\ 0&1&0 \end{smallmatrix}\right)$ does $(1, 2, 5, 3, 7, 6, 4)$ and that $(YXYXY) = \left(\begin{smallmatrix} 1&0&0 \\ 1&1&0 \\ 1&1&1 \end{smallmatrix}\right)$ does $(1, 3, 5, 7)(2, 6)$, so $(YXYXY)^4 = e$.

10.10 Check that the matrices X, Y, XY of 10.4 act as claimed on the 7 column vectors (10.24). Deduce that they satisfy the stated relations (10.3). (This is quite a lot easier than multiplying the matrices.)

10.11 Argue on the collinearity properties of the Fano plane (Figure (10.24)) to prove than any symmetry x of order 2 is conjugate to $(4, 5)(6, 7)$. [Hints: First assume that x fixes 3 collinear points, for example, P_1, P_2, P_3. Prove that it then swaps two pairs of points, and that all three cases are possible. Next, prove that there are 21 possible sets of 3 ordered collinear points, and any x of order 2 must fix one of these.]

10.12 The character table of G_{168} is an interesting extended puzzle. The conjugacy classes were discussed in 10.3–4, which also exhibited faithful permutation representations of G_{128} on sets of 8 and 7 elements. Derive 7- and 6-dimensional irreducible representations from them in the usual way, and calculate their characters. Calculate the character of $\bigwedge^2$ of the 6-dimensional representation by the methods of 8.4, and show that it splits as a sum of 2 irreducible representations whose dimensions add to 15. Finish the character table from these hints. G_{168} must have a 3-dimensional representation over $\mathbb{C}$, since it acts on the Klein quartic curve $K_4 \subset \mathbb{P}^2_{\mathbb{C}}$.

10.13 I leave it as a fun exercise to calculate the character table of $\mathrm{PGL}(2,\mathbb{F}_7)$ (no more hints, this is a challenge question!).

10.14 Generators for G_7 acting on $\mathcal{H}^2$. Take the ball $|z| < 1$ in $\mathbb{C}$ as the model of $\mathcal{H}^2$. Write its isometries as fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$ for suitable $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{C})$.

The rotation $R\colon z \mapsto \varepsilon z$ of order 7 corresponds to the matrix $\left(\begin{smallmatrix} \varepsilon^4 & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)$.

For s in the real interval $(0,1)$, the map $X\colon z \mapsto \frac{z-s}{sz-1}$ is an order 2 map exchanging $(0 \leftrightarrow s)$ and $(1 \leftrightarrow -1)$. It can be represented by the matrix $\frac{1}{\sqrt{s^2-1}} \left(\begin{smallmatrix} 1 & -s \\ s & -1 \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{C})$.

Show that $Y = XR$ satisfies $Y^3 = \mathrm{Id}$ if an only if $s^2 = 2\cos\frac{2\pi}{7} - 1 = -1 + \varepsilon + \varepsilon^{-1}$.

This gives generators for the group of the tesselation of 10.4. Programming these in floating point complex numbers, allowing us to draw the pictures of the tesselation of $\mathcal{H}^2$ by $2,3,7$ triangles or by regular heptagons.

```
K7<ep> := CyclotomicField(7);
s2 := -1+ep+ep^6; s2; // s2 is 2*cos 2pi/7 - 1
MinimalPolynomial(s2); IsSquare(s2);
RT<T> := PolynomialRing(K7); Ks<s> := NumberField(T^2-s2);
s^2; // Ks is the field K7[s] with s^2 = s2.
X := Matrix(2, [1,-s, s,-1]); X^2/((X^2)[1,1]) eq 1;
R := Matrix(2, [ep, 0, 0, 1]); R^7 eq 1;
Y := X*R; Y^3/((Y^3)[1,1]) eq 1;
```

11 Appendix: Final notes, scrap

11.1 Cyclic groups and roots of unity

I recall for emphasis what I said in Chapter 1. The cyclic group $\mathbb{Z}/n$ and its representation theory is the key case of the whole theory. A 1-dimensional representation of $\mathbb{Z}/n$ over $\mathbb{C}$ consists of the generator $1 \in \mathbb{Z}/n$ acting on $\mathbb{C}$ by $v \to \varepsilon^i v$ for some $i \in [0, \dots, n-1]$, where $\varepsilon = \exp \frac{2\pi i}{n}$. Every finite dimensional representation of $\mathbb{Z}/n$ is a direct sum of 1-dimensional representations.

Ex. 1.2 asked the easiest possible exam question in the subject (a fraction of students did poorly on it).

11.2 Strength and weakness of the theory

A group representation is a priori a sophisticated and complicated object. An n-dimensional representation ρ of a group G is by definition the data of a map of G into $n \times n$ matrices. If $|G|$ and n are reasonably large, this amounts to a huge pile of data – no-one will ever write out a hundred 8×8 matrices by hand and get them right. The character of the representation records only the trace of the matrices $\rho(g)$, and we only need to specify it for one element of each conjugacy class. The main theorems say that characters know everything about the representations up to isomorphism. Thus it is the perfect kind of mathematical invariant. It maps the complicated thing we want to study faithfully onto a small and easily manipulated set of numbers.

Thus characters give you everything. Everything that is, except how to write out the actual matrices. So if you're happy with an existence result you are perfectly satisfied, but you might feel that there is an element of ineffectivity about the knowledge achieved.

11.3 Characters

A square matrix M over $\mathbb{C}$ of finite order is determined up to conjugacy by its eigenvalues λ_i. The elementary symmetric functions σ_j of the eigenvalues are the coefficients of the characteristic polynomial $\prod(T - \lambda_i) = \sum(-1)^j \sigma_j T^{n-j}$. The character is just $\operatorname{Tr} M = \sigma_1 = \sum \lambda_i$. However, $\rho(G)$ also contains all the powers M, $M^2, \dots$, and if you include these, the character also knows the sums of the powers of the eigenvalues $\operatorname{Tr} M^j = \Sigma_j = \sum \lambda_i^j$. Newton's rule recovers the elementary symmetric functions σ_j from the power sums Σ_i. So the character gives the matrix M up to conjugacy, and it is pretty reasonable to say that it tells you a lot about the representation.

11.4 Problems with set-theoretic foundations

In the preamble to the Main Theorem 5.6, we had to choose a "complete set of nonisomorphic irreducible representations, that is a list $\{U_i\}$ of irreducible modules containing exactly one copy of every irreducible module up to isomorphism. There are set-theoretical and categorical constraints on what we have to say there. For example, it is illegat to talk of the "set" of irreducible modules, because it is not a set, but a proper class (by doing so, we would fall into a version of Russel's paradox about the set of all sets).

Around 1900 various researchers following in the footsteps of Georg Cantor were trying to write textbooks that defined a number as an equivalence class of sets, which is messed up by Russel's paradox. You can't get away from this obstruction by restricting to finite sets: sets with only 1 element already form a proper class.

Every irreducible representation of a finite group G is isomorphic to a subrepresentation of the regular representation. At least assuming that we take for granted the notion of a field K, and of a group G, the regular representation is a single definite object, namely $V^{\text{reg}} = \bigoplus_{g\in G} K \cdot g$, and we can restrict attention only to subsets of V^{reg}, which is a perfectly respectable set. This does not really solve the set-theoretic problem, but it does shift it somewhere else.

11.5 Isotypical decomposition as a canonical expression

When we write a representation as a direct sum of irreducibles, we get something like ... ahem ... once we have made a some choice of irreducible representations U_i that are not isomorphic but contain a complete list of irreducibles up to isomorphisms. Once we have made a choice of U_i (one particular module in each isomorphism class of irreducibles) every finite dimensional expression V can be written as a direct sum of irreducibles $V \cong \bigoplus n_i U_i$ (that is, $\bigoplus U_i^{\oplus n_i}$); moreover, the multiplicities n_i are uniquely determined. The submodule $n_i U_i \subset V$ is determined as the image of all possible KG-homomorphisms $U_i \to V$, or as the kernel of all possible KG-homomorphisms $V \to U_j$ for $j \neq i$. It is sometimes called the *isotypical summand* of U_i in V.

It is also natural to write this summand as $K^{n_i} \otimes_K U_i$ or $E_i \otimes_K U_i$, where E_i an n_i-dimensional vector space (with no nontrivial G-action). In fact (after the above choices), there is a canonical equality

$$V = \sum E_i \otimes_K U_i, \quad \text{with } E_i = \operatorname{Hom}_{KG}(U_i, V). \tag{11.1}$$

11.6 Free group $F(m)$

You're supposed to think of the graph of 4.2, (5) as a discrete variant of the hyperbolic plane. It is homogeneous (looks the same from every vertex), connected, and simply connected (has a unique shortest path between any two vertices).

11.7 Group algebra KG

Recall that 2.4 described concisely the algebra structure of KG, but then promptly added the disclaimer that the expression KG-module just means a K-vector space V together with a representation of G on V, viewed as a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$. I arranged the material of the course to be independent of the algebra structure of KG insofar as possible, so that for example, I supressed the action $KG \times V \to V$ making V into a module over the algebra KG. Several of the many textbooks adopt a similar strategy, which has the advantage of avoiding a longwinded detour into noncommutative ring theory.

However, several results of the course are most naturally stated in terms of KG, here with $K = \mathbb{C}$. The notation $\mathbb{C}G$ reappeared as the regular representation $\mathbb{C}G = V^{\mathrm{reg}}$ in 6.3, and the preferred element $1_{\mathbb{C}G} = 1 \cdot e_G \in \mathbb{C}G$ of 6.4 is of course the identity element of the algebra $\mathbb{C}G$.

At a deeper level, results of ring theory apply to the noncommutative algebra $\mathbb{C}G$. Wedderburn's theorem implies that $\mathbb{C}G$ splits up to isomorphism as a direct product of full matrix rings $\mathbb{C}G \cong R_1 \times R_2 \times \cdots \times R_k$ where each $R_i = \mathrm{Mat}(d_i \times d_i, \mathbb{C})$. Then the irreducible representations of G and the isotypical decomposition of $\mathbb{C}G$-module correspond 1-to-1 to irreducible idempotents of $\mathbb{C}G$. This is a strategy to an alternative treatment of the Main Theorem and the equality $\sum d_i^2 = |G|$.

11.8 What's hard about the induced representation?

The group algebra is relevant to the tricky part of the definition of induced representation in 9.4 and Theorem 9.10. If $A \to B$ is a homomorphism of commutative algebras, we pass automatically from a B-module to an A-module by "restriction of scalars" (forget the action of B, just remember A). In the other direction, the tensor product $M \mapsto B \otimes_A M$ provides an automatic way of taking an A-module to a B-module. It has the following UMP, as in 9.6: for any A-module M and B-module N we have a canonical isomorphism

$$\mathrm{Hom}_A(M, N_{|A}) = \mathrm{Hom}_B(B \otimes_A M, N) \tag{11.2}$$

(with various functorial compatibilities). For a subgroup $H \subset G$, if we treat KG as an algebra, then $KH \subset KG$ is of course a subalgebra, and induced representation can be handled in a similar style. However, there is still a little issue involved in doing this properly. Namely, KG is a KG-bimodule (with multiplication on the left and on the right), and to write $\mathrm{Ind}_H^G L = KG \otimes_{KH} L$ involves treating KG as a right module when making the tensor product, and as a left module when making the whole construction into a KG-module. This explains what is going on in the proof of Theorem 9.10, and why we need the nasty $h = g_\beta^{-1} x g_\gamma$ trick.

11.9 Abelian category

It may seem paradoxical that at the start, I found the theory of KG-modules (or representations of G) easier to describe than the theory of the group G itself. The point is that KG-modules form an Abelian category, and it is semisimple by Maschke's theorem. The word category just means the class of all modules and A-linear maps between modules. Saying that it is Abelian just means that we have subobjects and quotient objects by subobjects, satisfying the same compatibilities as for vector spaces. If you ever say "exact sequence", you must be working in an Abelian category (whether you realise it or not). The category of groups has no such useful structure.

More generally, when discussing a module M over a ring A, the study of M given A is a problem analogous to linear algebra. The structure of the ring A itself is a harder object to study, and is a nonlinear problem, with linear algebra providing little or no help. Similarly, the study of sheaves over a space is amenable to methods of linear algebra, whereas the are usually no such explicit methods of studying the space itself.

11.10 Finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(3,\mathbb{C})$ and their role in algebraic geometry

This is a research topic that I would like to have developed more in the course.

Felix Klein classified the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ around 1860; there are two infinite families corresponding to regular polygons in the plane, together with three exceptional groups of order 24, 48 and 120 that are binary or "spinor" double covers of the symmetry groups of the regular polyhedra the tetrahedron, octahedron and icosahedron (see footnote **??**). The finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ are also classified (and also $\mathrm{SL}(n,\mathbb{C})$ for higher n), although the problem gets harder and it is not clear how to view the assortment of solutions with any pretence to elegance.

The quotient spaces $X = \mathbb{C}^2/G$ by Klein's finite subgroups $G \subset \mathrm{SL}(2,\mathbb{C})$ form a very remarkable family of isolated surface singularities, that were studied by Du Val during the 1930s (aided by Coxeter). Du Val's work was central to the study of algebraic surfaces during the 1970s and 1980s, and played a foundational role in the study of algebraic 3-folds from the 1980s onwards. In the 1980s McKay observed that the representation theory of the group G is reflected in the geometry of the resolution of singularities of X. This correspondence has been generalised to 3-dimensions, with the same proviso concerning the nature of the problem and its solutions.

11.11 Unsorted draft homework and exam questions

11.1 The condition that two $n \times n$ matrices A and B commute implies compatibilities between their eigenspaces. For example, if A is diagonalisable with n distinct eigenvalues, prove that $AB = BA$ implies that B is also diagonal (in the eigenbasis of A).

More generally, prove that $AB = BA$ implies that B takes the eigenspaces of A to themselves.

Consider a finite dimensional representation $\rho\colon \mathbb{Z}/n \times \mathbb{Z}/m \to \mathrm{GL}(V)$ of the product of two cyclic groups. Show that V decomposes as the sum of 1-dim irreducible summands.

11.2 Write $\varepsilon = \exp\frac{2\pi i}{n}$ for a primitive nth root of 1. Show how to write down n different irreducible representations L_i of the cyclic group $\mathbb{Z}/n$. Prove that every representation of $\mathbb{Z}/n$ is a direct sum of copies of these.

For $\varepsilon = \exp\frac{2\pi i}{5}$, set

$$A = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon^2 & 0 & 0 \\ 0 & 0 & \varepsilon^4 & 0 \\ 0 & 0 & 0 & \varepsilon^3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and write $G_{20} \subset \mathrm{GL}(4,\mathbb{C})$ for the subgroup generated by A and B.

(i) Prove that G_{20} contains a copy of $\mathbb{Z}/5$ as a normal subgroup, with quotient group cyclic of order 4. **[4]**

(ii) Prove that the given representation of G_{20} is irreducible. **[4]**

(iii) Prove that AB, A^2B, A^3B and A^4B are all conjugate to B. Deduce that the conjugacy classes of G_{20} are
e, $\{A, A^2, A^3, A^4\}$, $\{A^iB\}$, $\{A^iB^2\}$, $\{A^iB^3\}$ **[8]**

(iv) Use the above to write out the character table of G_{20} with a brief justification. **[9]**

11.3 The same exercise with $5 \mapsto 7$, and the primitive 7th roots of 1 organised as the 6×6 diagonal matrix $\mathrm{diag}(\varepsilon, \varepsilon^3, \varepsilon^2, \varepsilon^6, \varepsilon^4, \varepsilon^5)$.

11.4 For all the groups that you know about of order ≤ 24, the character table can be written out somehow or other by the methods we have discussed.

Do this for the quaternion group H_8.

Likewise the alternative group A_4.

Likewise the binary dihedral groups BD_{16} and BD_{18}.

11.5 Let $\mathbb{F}_3$ be the field with 3 elements and $G = \mathrm{SL}(2,\mathbb{F}_3)$ the group of 2×2 matrices with entries in $\mathbb{F}_3$ and determinant 1. Write $a = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Prove that $a^2 = bab$ and $b^2 = aba$.

Deduce that G has a 2-dim representation $\rho\colon G \to \mathrm{SL}(2,\mathbb{C})$ with $\rho(a) = \frac{1}{2}\left(\begin{smallmatrix} 1+i & 1+i \\ -1+i & 1-1 \end{smallmatrix}\right)$ and $\rho(b) = \frac{1}{2}\left(\begin{smallmatrix} 1+i & -1+i \\ 1+i & 1-1 \end{smallmatrix}\right)$.

Prove the sporadic isomorphism $\mathrm{SL}(2,\mathbb{F}_3) \cong \mathrm{BT}_{24}$, where BT_{24} the binary tetrahedral group of Chapters 2–3.

11.6. Conjugate 5-cycles in A_5 Recall that $a = (01234)$ and $a_2 = (01243)$ are conjugate in S_5 but not in A_5. Find $x \in A_5$ such that $xa^2x^{-1} = a_2$. [Hint: To spell this out, it means to find an even permutation of $\{0,1,2,3,4\}$ which applied to the entries of $a^2 = (02413)$ gives (01243) as a 5-cycle up to cyclic permutation. There are several solutions, of which one is a product of two 2-cycles.]

11.7. Outer automorphism of A_5 Construct an automorphism of A_5 that take the 5-cycle $a = (01234)$ to its square $a^2 = (02413)$. [Hint: Assume as given that A_5 is generated (say) by $a = (01234)$ and $b = (031)$ with defining relations $a^5 = b^3 = (ab)^2 = e$. Find a suitable element $c \in A_5$ to map it to, so that $a \mapsto a^2$, $b \mapsto c$ defines a group homomorphism.]

11.8 Let $\mathbb{F}_5$ be the field with 5 elements and $\mathrm{SL}(2, \mathbb{F}_5)$ the group of 2×2 matrices with entries in $\mathbb{F}_5$ and determinant 1.

Let $A = \left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$; calculate the order of A, B and AB.

11.9 Prove that the centre of $\mathrm{SL}(2, \mathbb{F}_5)$ equals the scalar diagonal matrices $\{\pm 1\}$.

Prove that the quotient $G = \mathrm{PSL}(2, \mathbb{F}_5) = \mathrm{SL}(2, \mathbb{F}_5) / \langle \pm 1 \rangle$ is isomorphic to A_5. [Hint: Use the images in G of the matrices of (11.8).]

11.10 By considering the action of $\mathrm{PSL}(2, \mathbb{F}_5)$ on the projective line $\mathbb{P}^1_{\mathbb{F}_5} = \{0, 1, 2, 3, 4, \infty\}$ prove that A_5 has a permutation action on a set of 6 elements, such that any 3-cycle of A_5 acts as the product of two disjoint 3-cycles.

11.11. Fifth roots of 1 Write $\varepsilon = \exp \frac{2\pi i}{5}$ for a primitive 5th root of 1, so that $1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \varepsilon^4$. Calculate the sum and product of $2\cos\frac{2\pi}{5} = \varepsilon + \varepsilon^4$ and $2\cos\frac{4\pi}{5} = \varepsilon^2 + \varepsilon^3$, and deduce that $\varepsilon + \varepsilon^4$ and $\varepsilon^2 + \varepsilon^3$ are the two roots of $x^2 + x - 1 = 0$.

It follows that $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$ and $\cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$. Although not essential, we note for completeness that $\sin\frac{2\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and $\sin\frac{\pi}{5} = \sin\frac{4\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}$.

11.12. More preparation for the icosahedron

(i) Prove that any rotation matrix $M \in \mathrm{SO}(3)$ of order 2 is conjugate to $\mathrm{diag}(1, -1, -1)$ and therefore has trace $= -1$.

(ii) Prove that any rotation matrix $M \in \mathrm{SO}(3)$ of order 3 is conjugate to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ 0 & \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \tag{11.3}$$

and conjugate over $\mathbb{C}$ to $\mathrm{diag}(1, \omega, \omega^2)$, where $\omega = \exp\frac{2\pi i}{3} = \frac{-1+\sqrt{-3}}{2}$, and so has trace $= 0$. [Hint. Determine the eigenvalues.]

(iii) A rotation of $\mathbb{R}^3$ of order 5 has angle either $\pm\frac{2\pi}{5}$ or $\pm\frac{4\pi}{5}$. Prove that in the first case the rotation is given by a matrix conjugate to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2\pi}{5} & -\sin\frac{2\pi}{5} \\ 0 & \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-1+\sqrt{5}}{4} & -\sin\frac{2\pi}{5} \\ 0 & \sin\frac{2\pi}{5} & \frac{-1+\sqrt{5}}{4} \end{pmatrix} \tag{11.4}$$

or over $\mathbb{C}$ to $\operatorname{diag}(1, \varepsilon, \varepsilon^4)$, so has trace the Golden Ratio $-\varepsilon^2 - \varepsilon^3 = \frac{1+\sqrt{5}}{2}$. The other case is similar, with trace $-\varepsilon - \varepsilon^5 = \frac{1-\sqrt{5}}{2}$

11.13 Consider the alternating group A_5 on 5 elements $\{0,1,2,3,4\}$ and its permutation representation $W_5 = \mathbb{C}^5_{\{e_0,\dots,e_4\}}$. Write out the conjugacy classes of A_5 and their sizes (number of elements) (repeating Chapter 4, (4.2)). Calculate the character χ_{W_5} and verify that $(\chi_{W_5}, \chi_{W_5}) = 2$. [Hint: The trace of a permutation matrix is the number of elements fixed by the permutation.]

11.14 Write L_0 for the trivial representation $\mathbb{C}$. The permutation representation W_5 contains a copy of L_0 based by $\sum e_i$, with A_5-invariant complement

$$V_4 = \left\{ \sum x_i e_i \mid \sum x_i = 0 \right\}. \tag{11.5}$$

Calculate the character of V_4 and prove that V_4 is irreducible.

11.15 Write A_6 for the symmetric group on $\{0,1,2,3,4,5\}$ and consider the permutations

$$A = (01234) \quad \text{and} \quad B = (015)(243). \tag{11.6}$$

Show that A, B generate a subgroup of A_6 isomorphic to A_5. [Hint: Assume known that A_5 is generated by $a = (01234)$, $b = (031)$ with $ab = (04)(23)$ and the defining relations $a^5 = b^3 = (ab)^2 = e$.]

11.16 Write W_6 for the permutation representation of A_5 corresponding to A and B of (11.14). Calculate its character χ_{W_6} and verify that $(\chi_{W_6}, \chi_{W_6}) = 2$.

11.17 As in (11.14), deduce that A_5 has a 5-dim irreducible representation and calculate its character.

11.18 For this question, assume as given that A_5 is isomorphic to the subgroup $I_{60} \subset \mathrm{SO}(3)$ of rotations of the icosahedron in $\mathbb{R}^3$. Its elements must be rotation matrices of order 2, 3 or 5. Use (11.11–12) to prove that there are exactly two possible cases for the character of this representation. Both occur. Use this to complete the character table of A_5.

11.19 A_5 contains 20 3-cycles falling into 10 subgroups of order 3,

$$G_{123},\ G_{124},\ G_{125},\ G_{134},\ G_{135},\ G_{145},\ G_{234}, G_{235},\ G_{245},\ G_{345}. \tag{11.7}$$

Conjugacy by A_5 defines a permutation action on these 10 subgroups. Write W_{10} for the corresponding permutation representation. Calculate its character $\chi_{W_{10}}$. [Hint. For example, conjugacy by $(12)(34)$ takes G_{125} and G_{345} to themselves, and no other. Therefore $\chi_{W_{10}}((12)(34)) = 2$.]

Calculate (χ, χ), and use the character table (given on p. 5 below) to prove that $W_{10} \cong L_0 \oplus V_4 \oplus V_5$.

11.20 Continuing the game of (11.19), let U_{20} be the permutation representation of A_5 acting by conjugacy on the 20 3-cycles $(1,2,3)$, $(1,3,2)$, etc. This space has an involution $\iota\colon U_{20} \to U_{20}$ defined by $(i,j,k) \mapsto (i,k,j)$ (taking the generator of each G_{ijk} to its inverse).

The involution ι splits U_{20} as a sum of two eigenspaces. The $\iota = 1$ eigenspace of U_{20}, based by all possible $(i,j,k) + (i,k,j)$, is the representation W_{10} of (11.19). The $\iota = -1$ eigenspace U_{20}^- is based by all possible $(i,j,k) - (i,k,j)$.

As a representation of A_5 it is formed of matrices with a single entry ± 1 in each row and column. (A kind of decorated permutation matrix.) Calculate its character $\chi_{U_{20}^-}$ and its norm-squared (χ,χ). [Hint: Similar to (11.19); the main point is to calculate (12)(34)(125)(12)(34), etc.]

Use the character table to prove that $U_{20}^- = V_4 \oplus U_3 \oplus U_3'$.

11.21 Use the notation $N, P_{1\dots 5}, Q_{1\dots 5}, S$ for the vertices of the icosahedron as in the lectures (see below), and let α be the 3-fold rotation in the face $\triangle P_1P_2N$. Prove the following:

(i) α permutes the vertices as the product of 4 disjoint 3-cycles

$$(P_1,P_2,N)(P_3,P_5,Q_4)(Q_3,Q_5,P_4)(Q_1,Q_2,S). \tag{11.8}$$

(ii) α permutes the edges as

$$P_1P_2 \mapsto P_2N \mapsto NP_1 \mapsto P_1P_2, \quad P_2P_3 \mapsto NP_5 \mapsto P_1Q_4 \mapsto P_2P_3, \tag{11.9}$$

etc., and therefore permutes the 5 orthogonal frames

$$\Sigma_k : P_kN \,||\, SQ_k, \quad Q_{k-1}P_{k+1} \,||\, Q_{k+1}P_{k-1}, \quad P_{k-2}P_{k+2} \,||\, Q_{k+2}Q_{k-2} \tag{11.10}$$

as the 3-cycle $(\Sigma_1,\Sigma_4,\Sigma_2)$. (Each frames is an unordered triple of pairs of orthogonal vectors $\pm e_0, \pm e_1, \pm e_2$.)

(iii) $a = (12345)$ and $b = (142)$ with $ab = (15)(34)$ generate A_5, with the defining relations $a^5 = b^3 = (ab)^2 = e$.

(iv) If we write $L_i : P_iQ_i$ and $L_0 : NS$ for the 6 axes of 5-fold rotation then α permutes them as the product of disjoint 3-cycles $(L_1,L_2,L_0)(L_3,L_5,L_4)$.

(v) $A = (12345)$ and $B = (126)(354)$ with $AB = (13)(26)$ (satisfying $A^5 = B^3 = (AB)^2 = e$) generate a subgroup of A_6 isomorphic to A_5.

11.22 Each orthogonal frame Σ_k is taken to itself by a tetrahedral group $T_{12} \cong A_4$ (generated in that basis by $\mathrm{diag}(1,-1,-1)$ and a 3-fold rotation). The permutation action of I_{60} on the Σ_k thus corresponds to the action of A_5 by conjugacy on its 5 subgroups isomorphism to A_4.

11.23 Calculate the character table of S_5 by the following scheme.

(a) Compute the 7 conjugacy classes of S_5 and their number of elements.

(b) Compute the character of the permutation representation, and its twist by the sign character $g \mapsto \mathrm{sign}(g)$.

This gives you 4 irreducible characters.

(c) Let φ be an irreducible character of A_5 of degree 3. Compute the induced character $\varphi_{A_5}^{S_5}$ and prove that it remains irreducible.

(d) Let ψ be the irreducible character of A_5 of degree 5. Compute the induced character $\psi_{A_5}^{S_5}$ and prove that it is the sum of the two remaining irreducible characters of S_5.

(e) Use $\psi_{A_5}^{S_5}$ and the orthogonality to fill in the rest of the character table.

11.24 Let χ be a character of a group G and $g \in G$ an element of order 2.

(i) Prove that $\chi(g)$ is an integer, with the same parity as $\chi(1)$.

(ii) Suppose that G has an irreducible character of degree 2. Prove that G has an element of degree 2. [Hint: The degree of an irreducible character divides the order of the group.]

Assume moreover that G is simple. We wish to prove that G has no irreducible character χ of degree 2. As above, write g for an element of order 2.

(iii) Prove that $\chi(g) \neq 0$.

(iv) Prove that g is in the centre of G. Show that this contradicts the assumption that G is simple.

11.25 A finite group G acts on each of its conjugacy classes by the action is $g \mapsto xgx^{-1}$. Show that the stabiliser of g is its *centraliser* $C_G(g)$ and that the conjugacy class of g is the set of right cosets $C_G(g)\backslash G$.

11.26 If V_i are $\mathbb{C}G$-modules with characters χ_i, calculate the character of the direct sum $\bigoplus_i n_i V_i$.

11.27 For a finite group G, write $U_1, \dots, U_k$ for a complete set of nonisomorphic irreducible representations and χ_i for their characters (hereinafter *irreducible characters*.) We know that the χ_i form an orthonormal basis for the class functions $\mathcal{C}(G)$ under the inner product (χ, ψ). Deduce that a representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$ and is a direct sum of two irreducibles if and only if $(\chi_V, \chi_V) = 2$.

11.28 Let G be a group. Show that the formula

$$x \in G \quad \text{acts by} \quad g \mapsto xgx^{-1}. \tag{11.11}$$

defines an action of G on itself. The orbit of $g \in G$ under this action is the conjugacy class $C_G(g)$.

Prove that the stabiliser of g under this action is the centraliser $Z_G(g)$. If G is finite, use the orbit-stabiliser formula to deduce that $C_G(g)$ consists of $|G|/|Z_G(g)|$ elements.

Describe and count the conjugacy classes of the permutation $(1,2)(3,4)$ in the alternating group A_5 and in the symmetric group S_5.

Describe and count the conjugacy classes of the permutation $(1,2,3,4,5)$ in A_5 and in S_5.

11.29. Characters Define the character of a finite dimensional $\mathbb{C}G$-module V. Prove that it is a class function (that is, conjugacy invariant). If V_i are $\mathbb{C}G$-modules with characters χ_i, calculate the character of $\bigoplus V_i$.

Let G be finite, and $U_1, \dots, U_r$ a complete set of nonisomorphic irreducible $\mathbb{C}G$-modules. Assume known that their characters χ_i form an orthonormal basis of the class functions $\mathcal{C}(G)$ under the Hermitian inner product $\langle -, - \rangle$.

(I) Prove that two finite dimensional representations are isomorphic if and only if they have equal characters.

(II) State and prove a criterion for a representation to be irreducible in terms of its character and the pairing $\langle f, g \rangle$.

Consider the natural permutation representation of A_5 on $\mathbb{C}^5$ (that is, an element $\sigma \in A_5$ permutes the coordinates). Calculate its character. Find an invariant subspace of $\mathbb{C}^5$. Find a complementary invariant subspace and calculate its character. Use the Hermitian inner product $\langle f, g \rangle$ to determine the irreducible decomposition of $\mathbb{C}^5$.

11.30. Binary dihedral group BD_{4m} For $m > 0$ write $\varepsilon_m = \exp \frac{2\pi i}{m}$ for a chosen primitive mth root of 1. Choose any $m \geq 2$ and set $\varepsilon = \varepsilon_{2m}$. Consider the two matrices

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{11.12}$$

Prove that $A^m = B^2 = -1$ and $BAB^{-1} = A^{-1}$ so that A and B generate a matrix group in $\mathrm{GL}(2, \mathbb{C})$ of order $4m$, isomorphic to the abstract group

$$\mathrm{BD}_{4m} = \left\langle A, B \mid A^{2m} = B^4 = e,\ A^m = B^2,\ BAB^{-1} = A^{-1} \right\rangle. \tag{11.13}$$

It contains the subgroup $\langle A \rangle \cong \mathbb{Z}/2m$ as a subgroup of index 2. Prove that the quotient by the central element $A^m = B^2$ is the usual dihedral group of order $2m$:

$$\mathrm{BD}_{4m} / \left\langle B^2 \right\rangle = D_{2m}. \tag{11.14}$$

11.31. Alternative construction of BT_{24} **and** $T_{12} \cong A_4$ Let $\mathbb{F}_3 = \mathbb{Z}/3$ be the field with 3 elements, and consider the group $G_0 = \mathrm{GL}(2, \mathbb{F}_3)$ of 2×2 nondegenerate matrices over $\mathbb{F}_3$. Calculate $|G|$. [Hint: the first row e_1 is any nonzero vector in $\mathbb{F}_3^2$; once that is chosen, the second row is any vector in $\mathbb{F}_3^2 \setminus \mathbb{F}_3 \cdot e_1$.]

The special linear group $G = \mathrm{SL}(2, \mathbb{F}_3)$ is the kernel of $\det\colon G_0 \to \mathbb{F}_3^\times$. Calculate $|G|$.

Set

$$a = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{F}_3). \tag{11.15}$$

Prove that $a^2 = bab$ and $b^2 = aba$.

By considering the action of a and b on the four 1-dimensional subspaces of $\mathbb{F}_3^2$ based by $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$, construct a surjective homomorphism $G \to A_4$.

Calculate enough words in a, b to deduce that subgroup of G they generate has order divisible by 4 and by 6, and deduce that $\langle a, b \rangle = G$.

Prove that G has three different 1-dimensional representations. [Hint: try $a \mapsto \omega$ and $b \mapsto \omega^?$.]

Prove that G has a 2-dimensional representation $G \to \mathrm{GL}(2, \mathbb{C})$ taking

$$a \mapsto \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} \quad \text{and} \quad b \mapsto \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}. \tag{11.16}$$