Answers to Worksheet 3

P.1 $1 = \det(1_n) = \det({}^tAA) = (\det A)^2$, therefore $\det A = \pm 1$. Alternatively, use the normal form of Theorem 1.14. Composite of two motions is direct if both are direct or both are opposite, and opposite if one of each. In Theorem 2.5, a direct motion t is a composite of evenly many reflections, so, e.g. in $\mathbb{E}^2$, either 0 or 2.

P.2 Substitute $x = \sin\theta$, $\sqrt{1-x^2} = \cos\theta$, $\mathrm{d}x = \cos\theta \mathrm{d}\theta$ to give simply given integral $= \int_{\theta=0}^{a} \mathrm{d}\theta = [\theta]_0^a$.

The circle S^1 is parametrised by $x = \sin\theta$, $y = \cos\theta$, so the infinitesimal arc length $\mathrm{d}s$ is given by $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = (s^2 + c^2)\mathrm{d}\theta^2$, that is $\mathrm{d}s = \mathrm{d}\theta$. Thus θ is the arc length from $P_0 = (1,0)$ to $P = (\sin\theta, \cos\theta)$.

P.3 Interchanging two coordinates $x_i x_j$ is a reflection in the diagonal hyperplane $x_i = x_j$.

Let σ permute n elements. If it fixes m of them, I claim it is a composite of at most $n-m-1$ transpositions. Because, arguing as in the proof of Theorem 2.6, let $P \notin \mathrm{Fix}(t)$ and set $t(P) = Q$. Then clearly $Q \notin \mathrm{Fix}(t)$. Set $t_1 = (PQ) \circ t$, where (PQ) interchanges P, Q, but fixes everything else. Then obviously, $\mathrm{Fix}(t_1) \supset \mathrm{Fix}(t) \cup \{P\}$. By induction, t_1 fixes $\geq m+1$ points, so is a composite of $\leq n-m-2$ transpositions, and $t = (PQ) \circ t_1$ of at most $n-m-1$.

3.1 The answer is in terms of the rotation matrix

$$\begin{pmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}, \quad \text{where} \quad \frac{-1}{2} = \cos 120^\circ, \frac{\sqrt{3}}{2} = \sin 120^\circ$$

For example, take

$$\mathbf{e}_1 = \frac{1}{\sqrt{3}}(1,1,1), \quad \mathbf{e}_2 = \frac{1}{\sqrt{6}}(1,1,-2), \quad \mathbf{e}_3 = \frac{1}{\sqrt{2}}(-1,1,0).$$

Then $(x,y,z) \mapsto (z,x,y)$ fixes $\mathbf{e}_1$ and takes

$$\mathbf{e}_2 \mapsto \frac{1}{\sqrt{6}}(-2,1,1) = \frac{-1}{2}\mathbf{e}_2 + \frac{\sqrt{3}}{2}\mathbf{e}_3$$

and

$$\mathbf{e}_3 \mapsto \frac{1}{\sqrt{2}}(0,-1,1) = -\frac{\sqrt{3}}{2}\mathbf{e}_2 + \frac{-1}{2}\mathbf{e}_3.$$

If $O = (0,0,0)$ and $A = (1,1,1)$, the line $L = OA$ is obviously taken to itself, so that the answer is rotation through 120° about L, followed by translation through $(1,1,1)$.

3.2 By definition of great circle, $L = \Pi \cap S^2$, where $\Pi \subset \mathbb{R}^3$ is a 2-dim. vector subspace. Write $M = \Pi^\perp$, a perpendicular line. As a point in $\mathbb{R}^3$, $P = m + p$ with $m \in M$ and $p \in \Pi$. If $p \neq 0$ then $\langle M, p \rangle = \langle M, P \rangle$ is a plane of $\mathbb{R}^3$ meeting Π orthogonally. It cuts out a great circle through P meeting L orthogonally. The exception is when $P \in M$: then any great circle through P is perpendicular to L (e.g. $L =$ equator, $P =$ north pole, any meridian line).

3.3 Two distinct planes Π_1 and Π_2 in $\mathbb{R}^3$ meet along a line M at a dihedral angle θ (measured from Π_1 to Π_2). Then $\mathrm{Refl}(\Pi_2) \circ \mathrm{Refl}(\Pi_1) = \mathrm{Rot}(M, 2\theta)$ (because the points on M are fixed; all the motion take place in the plane orthogonal to M, so the composite is a rotation as in Figure 2.3 of the Book). For the last part, great circles L_1 and L_2 are cut out on S^2 by planes Π_1, Π_2 of $\mathbb{R}^3$, and the reflections are restrictions of the reflections of $\mathbb{R}^3$. Their composite is a rotation of $\mathbb{R}^3$, as just described, so the composite of the two reflections of S^2 is also the rotation through 2θ about the axis through the two points of intersection of L_1 and L_2

3.4 The mistake is that the top "line" BB' of the figure is a curve equidistant to the equator, but is not a great circle.