

Algebraic Geometry

Exercises on graded rings

A. Elliptic curves

The first questions calculate the graded rings $R(E, kP)$, where E is an elliptic curve (that is, a curve of genus 1 with a marked point $P \in E$). You should use the RR formula

$$h^0(E, \mathcal{O}_E(D)) = \deg D \quad \text{for } D \text{ a divisor of } \deg > 0.$$

1. Let E be an elliptic curve and $P \in E$ a point. Consider the graded ring $R(E, P) = \bigoplus_{n \geq 0} H^0(nP)$. Prove that $R(E, P) = k[x, y, z]/f$, where $\text{wt } x, y, z = 1, 2, 3$, and the relation f of degree 6 is of the form $z^2 = y^3 + ax^4y + bx^6$. [Hint: Choose x, y, z so that

$$H^0(E, \mathcal{O}_E(P)) = \langle x \rangle, \quad H^0(E, \mathcal{O}_E(2P)) = \langle x^2, y \rangle$$

and $H^0(E, \mathcal{O}_E(3P)) = \langle x^3, xy, z \rangle$.

Prove that x^2, y defines a 2-to-1 map $E \rightarrow \mathbb{P}^1$, so that x, y are algebraically independent. Prove that y has a pole at P of order 2, and z a pole of order 3. Write out the monomials in x, y, z of weighted degree 4 and prove that they form a basis of $H^0(E, \mathcal{O}_E(4P))$, and ditto for degree 5. Prove that there must be a relation in degree 6, and that it must involve y^3 and z^2 . Finally, adjust $z \mapsto z + \alpha_3(x, y)$ and $y \mapsto y + \beta_2(x)$ to remove unwanted terms.]

2. The subring $R(E, 2P) \subset R(E, P)$ is generated by $u_1 = x^2, u_2 = y, t = xz$ with the relation $t^2 = u_1(u_2^3 + au_2u_1^2 + bu_1^3)$. This is the model of E as a double cover of \mathbb{P}^1 with 4 branch points, one at infinity ($u_1 = 0$).

3. $R(E, 3P)$ is generated by $v_1 = x^3, v_2 = xy, v_3 = z$ with the single relation $v_1v_3^2 = v_2^3 + av_1^2v_2 + bv_1^3$. This is the model of E as a plane cubic with an inflexion at infinity (the line $v_1 = 0$ has inflexional tangent at $(0, 0, 1)$).

4. $R(E, 4P)$ is generated by $w_1 = x^4, w_2 = x^2y, w_3 = y^2, w_4 = xz$ with two relations $w_1w_3 = w_2^2$ and $w_4^2 = w_2w_3 + aw_1w_2 + bw_1^2$. This is the model of E as an intersection of two quadrics in \mathbb{P}^3 with the plane $w_1 = 0$ having 4-fold intersection at $(0, 0, 1, 0)$.

5. $R(E, 5P)$ is generated by $\xi_1 = x^5, \xi_2 = x^3y, \xi_3 = xy^2, \xi_4 = x^2z, \xi_5 = yz$ with 5 quadratic relations

$$\text{rank} \begin{pmatrix} \xi_1 & \xi_2 & \xi_4 \\ \xi_2 & \xi_3 & \xi_5 \\ \xi_4 & \xi_3 & \xi_5 \end{pmatrix} \leq 1, \quad \text{and} \quad \begin{pmatrix} \xi_4^2 = \xi_2\xi_3 + a\xi_1\xi_2 + b\xi_1^2 \\ \xi_4\xi_5 = \xi_3^2 + a\xi_2^2 + b\xi_1\xi_2 \end{pmatrix}.$$

Show how to view these equations as the 5 Pfaffians of a 5×5 skewsymmetric matrix. This is the model of E as a curve of degree 5 in \mathbb{P}^4 obtained as a \mathbb{P}^4 section of the Grassmann variety $\text{Grass}(2, 5) \subset \mathbb{P}^9$.

6. Let C be a hyperelliptic curve of genus g ; we say that P is a *Weierstrass point* if P is a branch point of the double cover $C \rightarrow \mathbb{P}^1$, that is, $2P \in g_2^1$.

Show that $R(C, P) = k[x, y, z]/(F_{4g+2})$ with $\deg x = 1, \deg y = 2$ and $\deg z = 2g + 1$. Show that if z is chosen so that $F_{4g+2} = z^2 - f_{2g+1}(x^2, y)$ (by completing the square) then $\text{div } z$ is the sum of the remaining $2g + 1$ Weierstrass points.

Calculate $R(C, 2P)$ by eliminating x from $R(C, P) = k[x, y, z]/f_{4g+2}$. [Hint: Compare Q. 1, Q. 2 above.]

7. If C is a hyperelliptic curve of genus $g = 4$ and P a Weierstrass point, compute $R(C, 3P)$.

B. Subcanonical curves with $K_C = (2g - 2)P$

These examples are entertaining and have many applications to surfaces of general type. Consider a nonsingular curve C marked with a point P such that $K_C = (2g - 2)P$. The Riemann–Roch theorem for curves

$$h^0(C, nP) = 1 - g + n + h^0(C, K_C - nP)$$

and Clifford's theorem $h^0(nP) \leq \lfloor \frac{n}{2} \rfloor + 1$ determines the Hilbert function $P_n = h^0(nP)$ of $R(C, nP)$ up to a finite number of choices. Write $P(t) = \sum_{n=0}^{\infty} P_n t^n$.

1. If $g = 2$ then C is always hyperelliptic, and $K_C = 2P$ for any branch point P . Check that

$$R(C, P) = k[x, y, z]/(z^2 = f_{10}(x, y)), \quad \text{where } \deg x, y, z = 1, 2, 5$$

and $f_{10}(x, y) = F_5(x^2, y)$ is the equation of the 5 other branch points (see Ex. A.6 above).

2. If $g = 3$, there are exactly two cases for $K_C = 4P$, giving $C_{14} \subset \mathbb{P}(1, 2, 7)$ if $2P = g_2^1$ and $C_{12} \subset \mathbb{P}(1, 3, 4)$ otherwise. In the latter case, show how to deduce the form of $R(C, P)$ from the model of C as a plane quartic touching a line to order 4 at P .

3. If $g = 4$, there are exactly 3 cases for $K_C = 6P$, of which two are hypersurfaces, and one is the codimension 2 complete intersection $C_{10,12} \subset \mathbb{P}(1, 4, 5, 6)$. Prove that every C, P with $K_C = 6P$ has ring given by these constructions. Count moduli.

4. If $K_C = (2g - 2)P$ then $(g - 1)P$ is a theta characteristic. It is known that the parity of $h^0((g - 1)P)$ (whether it is even or odd) is constant under deformation of C and P . Use this to determine the possible specialisations between the three families of curves in Ex. 3. [Hint: The parity argument does not exclude that trigonal curves $C_{15} \subset \mathbb{P}(1, 3, 5)$ could specialise to a hyperelliptic curve $C_{18} \subset \mathbb{P}(1, 2, 9)$. If this specialisation happens, you can see it by equations.]

5. If $g = 5$ and $K_C = 8P$ (and C is not hyperelliptic), prove that a priori, the possible Hilbert functions for $R(C, P)$ are the following 3 cases:

$$\begin{aligned} 1 + t + t^2 + t^3 + t^4 + 2t^5 + 3t^6 + 4t^7 + 5t^8 + \sum_{n \geq 9} (n - 4)t^n &= \\ = \frac{1 - t^{12} - t^{13} - t^{14} - t^{15} - t^{16} + t^{19} + t^{20} + t^{21} + t^{22} + t^{23} - t^{35}}{(1 - t)(1 - t^5)(1 - t^6)(1 - t^7)(1 - t^8)} \end{aligned}$$

or

$$\begin{aligned} 1 + t + t^2 + t^3 + 2t^4 + 2t^5 + 3t^6 + 4t^7 + 5t^8 + \sum_{n \geq 9} (n - 4)t^n &= \\ = \frac{1 - t^{12} - t^{14} + t^{26}}{(1 - t)(1 - t^4)(1 - t^6)(1 - t^7)} \end{aligned}$$

or

$$\begin{aligned} 1 + t + t^2 + 2t^3 + 2t^4 + 3t^5 + 3t^6 + 4t^7 + 5t^8 + \sum_{n \geq 9} (n - 4)t^n &= \\ = \frac{1 - t^6 - t^{10} - t^{14} - t^{18} - t^{22} + t^{13} + t^{17} + t^{21} + t^{25} + t^{29} - t^{35}}{(1 - t)(1 - t^5)(1 - t^6)(1 - t^7)(1 - t^8)} \end{aligned}$$

6. In question 5, show that the second case gives the codimension 2 complete intersection $C_{12,14} \subset \mathbb{P}(1, 4, 5, 7)$ and the first gives the Pfaffian variety in $\mathbb{P}(1, 5, 6, 7, 8)$ corresponding to a skewsymmetric matrix of weighted degrees

$$\begin{pmatrix} 3 & 4 & 5 & 6 & 7 \\ & 5 & 6 & 7 & 8 \\ & & 7 & 8 & 9 \\ -\text{sym} & & & 9 & 10 \\ & & & & 11 \end{pmatrix}$$

7. In question 5, show that the third case is not possible. [Hint: Let x, y, z be the generators of $R(C, P)$ in degrees 1, 3, 5. Determine the behaviour of x, y, z at P . There is a relation in degree 6, that we can write with unknown coefficients. Depending on which is zero derive a contradiction to the assumptions that x is a nonzero divisor or y has pole of order 3 at P .]

C. Regular surface of general type

The following questions treat the canonical ring $R(X, K_X)$ of a regular surface of general type. You need to use the RR formula

$$P_n = \begin{cases} 1 & \text{if } n = 0, \\ p_g & \text{if } n = 1, \\ 1 + p_g + \binom{n}{2} & \text{if } n \geq 2. \end{cases}$$

1. Show that $R(X, K_X)$ can be a hypersurface or a complete intersection in projective space or weighted projective space in the following cases:

1. $p_g = 1, K^2 = 1$;
2. $p_g = 2, K^2 = 1$;
3. $p_g = 2, K^2 = 2$;
4. $p_g = 3, K^2 = 2$;
5. $p_g = 3, K^2 = 3$;
6. $p_g = 3, K^2 = 4$;
7. $p_g = 4, K^2 = 5$;

8. $p_g = 4, K^2 = 6;$
9. $p_g = 5, K^2 = 8;$
10. $p_g = 5, K^2 = 9;$
11. $p_g = 6, K^2 = 12;$
12. $p_g = 7, K^2 = 16;$

2. Show that $R(X, K_X)$ can be Pfaffian in the following cases:

1. $p_g = 3, K^2 = 5;$
2. $p_g = 4, K^2 = 7;$
3. $p_g = 5, K^2 = 10;$
4. $p_g = 6, K^2 = 11;$
5. $p_g = 6, K^2 = 13;$
6. $p_g = 6, K^2 = 14$ (note: 7×7);

D. Rings and geometry

1. Suppose that $f_{2g+2}(x)$ has degree equal to $2g + 2$ and distinct roots, and let $C_0 : y^2 = f_{2g+2}$. Show how to resolve the singularity at $(0, 1, 0)$ of the $z^{2g}y^2 = f_{2g+2}(x, z)$ by successive birational changes of \mathbb{P}^2 , leading to a nonsingular model $C \subset \mathbb{F}(0, g + 1)$ in the scroll \mathbb{F}_{g+1} .

2. Let C be a nonsingular hyperelliptic curve of genus g , with $D = g_2^1$. We know that $R(C, D) = k[x_1, x_2, y]/(y^2 = f_{2g+2}(x, x_2))$, and we have an interpretation of the image of the projective embedding of C into \mathbb{P}^{g+2} by $\varphi_{(g+1)D}$ (as the intersection of the cone $\overline{\mathbb{F}}_{g+1}$ with a quadric).

Give an analogous interpretation of the image of C by φ_{aD} for $a \geq g + 1$.

3. Consider surfaces of general type with $p_g = 4, K^2 = 6$ for which the canonical ring can be a complete intersections $X_{3,4} : (f_3 = g_4 = 0) \subset \mathbb{P}(1^4, 2)$. Show that this model can be nonsingular even if f_3 does not involve y ; in this case the map $X \rightarrow \mathbb{P}^3$ is a double cover of a cubic surface.

Show how to construct a 1-parameter family X_λ so that $\lambda = 0$ gives a double cover of a cubic and $\lambda \neq 0$ a sextic with a singular curve.

4. Let C be a curve of genus 4 and D a divisor such that $2D = K_C$ (so that in particular, $\deg D = 3$) and $h^0(D) = 2$. Show that one of the following holds:

1. $|D|$ is a g_3^1 with no base point;
2. C is hyperelliptic and $|D| = P + g_2^1$ where the fixed point P is a Weierstrass point.

In case (1), prove that $R(C, D) = k[x_1, x_2, y]/f_6$ where $\deg x_i = 1$, $\deg y = 2$, and $f = y^3 + a_2y^2 + b_4y + c_6$. Show that the canonical ring $R(C, K_C) = R(C, 2D)$ is a complete intersection of type $(2, 3)$ where the quadric is a cone.

In case (2), prove that $R(C, D) = k[x_1, x_2, y, z]/(g_3, f_6)$ where $\deg z = 3$ and $g_3 = x_1y + d_3$. [Hint: use Worksheet 3, Q. 7.]

5. Use the previous question to construct a 1-parameter family C_λ, D_λ of curves of genus 4 with a marked g_3^1 such that $\lambda = 0$ corresponds to a hyperelliptic curve and $\lambda \neq 0$ a nonhyperelliptic curve.

6. Consider surfaces of general type with $p_g = 3, q = 0$ and $K^2 = 3$. It is known that the general curve $C \in K_C$ is a nonsingular curve, and that $|K_X|_C$ is a divisor with $2D = K_C$ and $h^0(D) = 2$. Prove that $R(X, K_X)$ is either (1) $k[x_0, x_1, x_2, y]/f_6$ or (2) $k[x_0, x_1, x_2, y, z]/(g_3, f_6)$. [Hint: use Q. 4 and the hyperplane section principle.]

7. Consider surfaces of general type with $p_g = 3, q = 0$ and $K^2 = 3$. Using the form $R(X, K_X) = k[x_0, x_1, x_2, y, z]/(g_3, f_6)$, prove the following:

1. there exist surfaces X for which $|K_X|$ has a base point;
2. these surfaces deform to surfaces for which $|K_X|$ has no base point.
3. (harder) the 1-canonical map $\varphi_X: X \dashrightarrow \mathbb{P}^2$ of such a surface is a double cover with branch locus consisting of a line L and a curve C_9 of degree 9 having a triple tacnode tangent to L . [Hint: eliminate y from $x_1y = d_3, z^2 = f_6(x_i, y)$.]

(This question is based on work of Horikawa and Iliev around 1980.)