The small cage in the Zoo of terminal Fano threefolds.

Kuzma Khrabrov

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Abstract

In this article we consider $\mathbb{Q}$-Fano varieties of dimension 3 of Fano index 2. We show how to construct new examples of families and give new steps in low-codimension classification.

Contents

1 Introduction.

Definition 1.1. $\mathbb{Q}$-Fano variety is a projective algebraic $\mathbb{Q}$-factorial variety whose anticanonical divisor is ample, Picard number equals 1 and which has at most terminal singularities.

Family of $\mathbb{Q}$-Fano varieties playes an important role in the classification of projective varieties, cause such varieties occur naturally in Mori minimal model program.

Definition 1.2. *Fano index of a Fano variety* X is a maximum integer $qW(X)$, such that there exists a Weil divisor A for which $-K_X \sim qW(X)A$.

In this text we will consider Fano index 2.

1.1 History

In the paper by K. Suzuki and G. Brown written in 2007 [1] all possible Hilbert series of index 2 $\mathbb{Q}$-Fano varieties were classified and the classification up to codimension 3 was given. Relatevely recenty M. Reid and Y. Prokhorov published the paper [5], in which the following theorem was proven:

Theorem 1.3. *Let X be a $\mathbb{Q} - Fano$ threefold of rank 1 such that $q\mathbb{Q}(X) = qW(X) = 2$, and assume K_X is not Cartier. Let A be a Weil divisor on X such that $-K_X = 2A$. Then $\dim|A| \leq 4$. Moreover, if $\dim|A| = 4$, then X belongs to the single irreducible family.*

Following this paper we look at the cases with $\dim|A| \leq 3$.
We start from the case $\dim|A| = 2$ and show how it is related with the case $\dim|A| = 3$ in several cases.

1.2 Statement of the results.

Theorem 1.4. *If X is a $\mathbb{Q}$-Fano variety of index 2 with $h^0(A) = 2$ and $\mathrm{codim}(X) = 4$, then X has the same Hilbert series as one of the following varieties:*

1. $X \subset \mathbb{P}(1,1,2,3,4,5,6,7)$ – *unprojection from a special* $Y_{6,7,8,9,10} \subset \mathbb{P}(1,1,2,3,4,5,6)$ *of* Tom_3 *or* $Jer_{2,4}$ *families.*

2. $X \subset \mathbb{P}(1,1,2,2,2,3,5,7)$ – *unprojection from a special* $Y_{4,5,6,6,7} \subset \mathbb{P}(1,1,2,2,2,3,5)$ *of* Tom_5 *family.*

3. $X \subset \mathbb{P}(1,1,2,2,3,3,4,7)$ – *unprojection from a special* $Y_{4,5,6,6,7} \subset \mathbb{P}(1,1,2,2,3,3,4)$ *of* Tom_1 *or* Tom_3 *families.*

4. $X \subset \mathbb{P}(1,1,2,2,3,3,4,5)$ – *unprojection from a special* $Y_{4,5,6,6,7} \subset \mathbb{P}(1,1,2,2,3,3,4)$ *of* Tom_4, Tom_3 *or* $Jer_{2,4}$ *families.*

5. $X \subset \mathbb{P}(1,1,2,2,2,3,3,5)$ – *unprojection from a special* $Y_{4,4,5,5,6} \subset \mathbb{P}(1,1,2,2,2,3,3)$ *of* Tom_4 *or* Tom_5 *families.*

6. $X \subset \mathbb{P}(1,1,2,2,2,3,3,3)$ – *unprojection from a special* $Y_{4,4,5,5,6} \subset \mathbb{P}(1,1,2,2,2,3,3)$ *of* Tom_2 *or* $Jer_{1,3}$ *families.*

Moreover all listed varieties are the examples of such X.
If X is $\mathbb{Q}$-Fano variety of index 2 with $h^0(A) = 3$ and $\operatorname{codim}(X) = 4$, *then X has the same Hilbert series as $X \subset \mathbb{P}(1,1,1,2,2,2,3,3)$ – non Gorenstein unprojection from a special $Y_{4,6} \subset \mathbb{P}(1,1,2,2,3,3)$ containing a plane $\mathbb{P}^2$. Moreover such X is an example.*

Remark 1.5. In the case of $\dim |A| = 1$ the list given here and the list of Hilbert series given in [1] coincide.

2 Going away from the Mori category

If $h^0(A) \leq 3$ than there are no possible Type I projections from it onto a $\mathbb{Q}$-Fano variety of less codimension.
One solution is to make a birational projection from a singular point.

Let X be a subvariety of a weighted projective space $\mathbb{P}^N(a_1, ..., a_N)$. Let X contain a singular point P corresponded to a_N, that is a $\frac{1}{a_N}$-singularity with some weights, say $(b_1, ..., b_n)$, where n is the dimension of X. We are interested in the result of a projection from the point P. In our cases the result of the birational projection is not an object from Mori category, it always contains a line of index 2 singularities. But still in several cases we can control its singularities.

Now start with a variety X in $\mathbb{P}^N(a_1, ..., a_{N-1})$. Impose a divisor D isomorphic to $\mathbb{P}^N(b_1, ..., b_n)$ to be contained in X. Then if all singularities of X are ODP singularities, than after blowing up all these singular points it is possible to contract D.

Definition 2.1. The process described before is called unprojection.

The result of contraction is a terminal Fano variety.

Example 2.2. Consider a $\mathbb{Q}$-Fano variety of index 2: $X \subset \mathbb{P}(1,1,2,3,4,5,6,7)$ with $h^0(A) = 2$ with one terminal singularity of type $\frac{1}{7}(1,2,6)$. Note that Hilbert series of X, $P(X)$ is uniquely determined by that information. Make a birational projection from the only singular point of X. The variety in the result is the Fano threefold $Y \subset \mathbb{P}(1,1,2,3,4,5,6)$ given by 5 equations of degrees 4, 5, 6, 6, 7 correspondingly which are pffafians of a 5×5 skewsymmeric matrix M. Where M is of the form:

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & 3 & 4 & 5 \\ & & 5 & 6 \\ & & & 7 \end{pmatrix} \tag{2.2.1}$$

So by the construction Y contains the plane $D = \mathbb{P}(1,2,6)$.
Now start with such variety $Y \subset \mathbb{P}(1,1,2,3,4,5,6)$. By a computer algebra package (i.e. Magma) it is possible to check that over a finite field a general Y given by pffafians of matrix M of Tom_3 or $Jer_{2,4}$ formats with an ideal $I = I(D)$, has only ODP singularities all on the plane D. Hence it is true for the field of characteristic 0.
So it is possible to unproject from Y, contracting D to the point of index 7. The result is the terminal Fano variety with Hilbert series $P(X)$.
Note that in this case varieties given by pffafians of different formats belong to different families, cause they have different number of ODP singularities, so different Betti numbers.
Finally, let's write an exact equations of the unprojection variety X in the case Tom_3.
Denote $\mathbb{K}[\mathbb{P}(1,1,2,3,4,5,6)] = \mathbb{K}[u,v,x,y,z,t,w]$, so the ideal of $\mathbb{P}^2(1,2,6)$ is $I =< v,y,z,t >$
Up to the change of coordinates matrix M of type Tom_3 can be written as

$$M = \begin{pmatrix} v & x & y & z \\ & u^3 & z & t \\ & & \psi_5 & w \\ & & & \theta_6 v \end{pmatrix} \tag{2.2.2}$$

where θ and *psi* are homogenius polynomials of degrees 6 and 5 respectively.

Now choose t as the unprojection variable.
We can write the 2 by 2 pffafians of M not containing t in a form:

$$\begin{pmatrix} x\theta & -w & \psi \\ \psi & u^3 & -x \end{pmatrix} \begin{pmatrix} v \\ y \\ z \end{pmatrix} = 0 \tag{2.2.3}$$

So the equations of X which not involve t are given by 2 by 2 pffafians of the following matrix:

$$N = \begin{pmatrix} s & x\theta & -w & \psi \\ & \psi & u^3 & -x \\ & & z & -y \\ & & & v \end{pmatrix} \tag{2.2.4}$$

where s is a new variable of degree 7.
The last equation involving t can be found from the following calculation:

$$st = stzz^{-1} = z^{-1}(-x\theta u^3 t - wt\psi) = z^{-1}(-tx\theta u^3 - zw^2 + u^3\theta vw) =$$
$$= z^{-1}(-zw^2 - tx\theta u^3 + u^3\theta xt - zu^3u^3\theta) = -w^2 - u^6\theta$$

3 Non Gorenstein unprojection.

In this section we will discribe the construction of a codimension 4 Q-Fano threefold with index 2 and $h^0(A) = 3$.

We start from the variety $Y_{4,6} \subset \mathbb{P}(1,1,2,2,3,3)$ containing the plane $\mathbb{P}^2$. First of all in that case the word contain means that there is an embedding of $\mathbb{P}^2$ into $Y_{4,6}$. The image D of that embedding can not be a smooth surface, but as we will see further it may still have only mild singularities, which are ordinary double points.
As in previous construction we first blow up the singular points and continue with the contraction to the terminal Fano variety X of index 2 with $h^0(A_X) = 3$.
Let's show how this construction is related with commutative algebra.
Denote by A the coordinate ring of $Y_{4,6} \subset \mathbb{P}(1,1,2,2,3,3)$
$A = \mathbb{K}[x_1, x_2, y_1, y_2, z_1, z_3]$
. Write the standard resolution of the coordinate ring of Y $\mathbb{K}[Y]$ as the Gorenstein A-module. Compose it with the resolution of $\mathbb{K}[D]$ and the resolution of the normalisation of $\mathbb{K}[D]$. Where the latter is the Gorenstein A-module as well as $\mathbb{K}[Y]$.

$$\begin{array}{ccccccccccc}
\mathbb{K}[Y] & \leftarrow & A & \leftarrow & A(-4) \oplus A(-6) & \leftarrow & A(-10) & \leftarrow & 0 & & \\
\downarrow & & \| & & \downarrow & & \downarrow & & & & \\
\mathbb{K}[D] & \leftarrow & A & \leftarrow & K_1 & \leftarrow & K_2 & \leftarrow & \cdots & & \\
\cap & & \cap & & \downarrow & & \downarrow & & & & \\
\mathbb{K}[u,v,h] & \leftarrow & L_0 & \leftarrow & L_1 & \leftarrow & L_2 & \leftarrow & L_3 & \leftarrow & 0
\end{array} \tag{3.0.5}$$

The map from $\mathbb{K}[Y]$ to $\mathbb{K}[D]$ is just a natural surjection given by embedding of D into Y and the map from $\mathbb{K}[D]$ to $\mathbb{K}[u,v,h] = \mathbb{K}[\mathbb{P}^2]$ is the normalization map. Other maps between the complexes are given automaticaly. It suffices to construct the maps between the resolutions of $\mathbb{K}[Y]$ and $\mathbb{K}[\mathbb{P}^2]$ in order to write down unprojection equations.

First of all let's specify the embedding of $\mathbb{P}^2$ into the projective space. One can check that if the map

$$\mathbb{P}^2 \to \mathbb{P}(1,1,2,2,,3,3) \quad \text{is given by} \quad (u,v,h) \mapsto (u,v,h^2,hu,h^3,v^2h) \tag{3.0.6}$$

than it is actually an embedding and there exists a terminal $Y_{4,6}$ containing D. In addition there exist such $Y_{4,6}$ with singularities only on D.
Moreover $\mathbb{K}[\mathbb{P}^2]$ is generated as A-module by 2 elements $(1,h)$. The relations between these generators are

$$(1,h)M = 0, \tag{3.0.7}$$

where

$$M = \begin{pmatrix} y_2 & -y_1u & z_1 & -y_1^2 & z_2 & -y_1v^2 \\ -u & y_2 & -y_1 & z_1 & -v^2 & z_2 \end{pmatrix} \tag{3.0.8}$$

In addition the ideal $\mathcal{I}(D)$ is generated by the 2 by 2 minors of M:

$$< \Lambda^2 M >= \mathcal{I}(D) \tag{3.0.9}$$

So the equations of $Y_{4,6}$ containing D are combinations of $\Lambda^2 M$ with coefficiens from A.
One can check that $Y = Z(f,g)$ satisfies all needed conditions, where

$$\begin{gathered} f = M_{1,2} + M_{1,3} - M_{1,5} \\ g = M_{5,6} + M_{3,4} + y_1 M_{1,2} + y_2 M_{1,3} + uv M_{1,5} \end{gathered} \tag{3.0.10}$$

Finally write $M = (\alpha\beta\gamma)$, where α, β, γ are 2 by 2 matrices. Now we are able to discribe L_i from 3.0.5 and maps between them.

$$\mathbb{C}[u,v,h] \longleftarrow L_0 \xleftarrow{M} L_1 \xleftarrow{M_1} L_2 \xleftarrow{M_2} L_3 \leftarrow 0, \tag{3.0.11}$$

where
$L_0 = A \oplus A(-1)$
$L_1 = A(-2) \oplus A(-3) \oplus A(-3) \oplus A(-4) \oplus A(-3) \oplus A(-4)$
$L_2 = A(-5) \oplus A(-6) \oplus A(-5) \oplus A(-6) \oplus A(-6) \oplus A(-7)$
$L_3 = A(-8) \oplus A(-9)$

$$M_1 = \begin{pmatrix} \beta & \gamma & 0 \\ -\alpha & 0 & \gamma \\ 0 & -\alpha & -\beta \end{pmatrix} \tag{3.0.12}$$

and

$$M_2 = \begin{pmatrix} \gamma \\ -\beta \\ \alpha \end{pmatrix} \tag{3.0.13}$$

Denote by N_2 the vertical map between $A(-10)$ and L_2. One checks that one possible form of N_2 is:

$$N_2 = \begin{pmatrix} -u^2v^3 - uy_1^2 - uv^2y_2 + uy_1y_2 - uy_2^2 - v^2z_1 + y_1z_1 - y2z_1 - v^2z_2 - y_1z_2 \\ v^4 + u^2y_1 + y_1^2 - u^2y_2 - y_1y_2 - uz_1 \\ u^2vy_1 - uv^2y_1 - u^2vy_2 + v^2z_1 + y_1z_1 - v^2z_2 + y_1z_2 - y_2z_2 \\ -u^3v - v^4 - u^2y_1 - y_1^2 - v^2y_2 - uz_2 \\ -u^3v - u^2y_2 - v^2y_2 - y_1y_2 - uz_1 - uz_2 \\ 0 \end{pmatrix} \tag{3.0.14}$$

Finally we follow [2], [4], [3].
Denote unprojections variables by s_0, s_1, then the linear relations $(S_1, S_2, S_3, S_4, S_5, S_6)$ between them can be given in Kustin-Miller form: $M_2\ {}^t(s_0, s_1) = N_2$.
The quaratic relation

$$S_0 : s_1u^2v + s_0^2 - s_1^2y_1 = -s_1uy_1 - 2s_0y_1 - s_1uy_2 + s_0y_2 - 2s_1z_2 \tag{3.0.15}$$

can be found from the fact that y_2S_0 is an element of the ideal
$I =< f, g, S_1, \cdots, S_6 >$.
The uprojection variety $X \subset \mathbb{P}(1,1,1,2,2,2,3,3)$, where $\mathbb{K}[\mathbb{P}] = \mathbb{K}[u, v, s_1, s_0, y_1, y_2, z_1, z_2]$ is given by the set of equations: (f, g, S_i).

Remark 3.1. Unfortunately we can not currently check the singularities of X by computer algebra, due to the computational complexity.

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